

# **A comparative study of ordinary cross-validation, $v$ -fold cross-validation and the repeated learning-testing methods**

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## SUMMARY

Concepts of  $v$ -fold cross-validation and repeated learning-testing methods have been introduced here. In many problems, these methods are computationally much less expensive than ordinary cross-validation and can be used in its place. A comparative study of these three methods has been carried out in detail.

*Some key words:* Ordinary cross-validation; Repeated learning-testing method;  $v$ -fold cross-validation.

## 1. INTRODUCTION

Model selection has attracted the attention of many researchers. One of the most well-known methods is ordinary cross-validation. Much work has been done on this, for example Stone (1974, 1977), Bowman (1984) and Härdle & Marron (1985). However, the difficulty with ordinary cross-validation is that it can be computationally very expensive in a number of practical problems, for example  $L^1$  regression, tree structured methods for classification and regression (Breiman et al., 1984), estimation of the optimal transformations of variables for correlation and regression (Breiman & Friedman, 1985). We have therefore recently introduced two techniques (Burman, 1989) in a study of the optimal transformations of variables. We call them corrected  $v$ -fold cross-validation and repeated learning-testing methods. These methods can be used in the place of ordinary cross-validation whenever the latter is computationally very expensive. In this present paper we study these methods in detail.

The organization of the paper is as follows. In § 2 we introduce the concepts of  $v$ -fold cross-validation and the repeated learning-testing methods. In § 3 we discuss the stability of our estimates. In § 4 we present a few simulation results on various aspects of our proposed estimates. We then summarize our findings in § 5. Section 6 presents a few technical results to support our simulation study, and finally we give proofs in § 7.

## 2. $v$ -FOLD CROSS-VALIDATION AND REPEATED LEARNING-TESTING METHODS

### 2.1. *General description*

In this section we describe  $v$ -fold cross-validation and repeated learning-testing methods, demonstrate the need for bias correction and introduce the proper correction terms. The concepts of  $v$ -fold cross-validation and repeated learning-testing methods are not new. They are discussed by Breiman et al. (1984, Ch. 3, 8). Based on their simulation studies they concluded that these methods do not always work. To rectify that we have introduced certain correction terms. Burman (1989) showed that with these correction terms  $v$ -fold cross-validation and repeated learning-testing methods work very well for

model selection purposes. Here we study various aspects of these methods in detail. Is  $v$ -fold cross-validation a better method than repeated learning-testing? For what values of  $v$  does  $v$ -fold cross-validation work best?

Let  $Z_1, \dots, Z_n$  be independent and identically distributed observations with distribution function  $F$  and let  $F_n$  be the empirical estimate of  $F$ . Our goal is to estimate a quantity of the form

$$s_n = \int T(z, F_n) dF(z). \quad (2.1)$$

We may also wish to estimate  $E(s_n)$ , which could be estimated by ordinary cross-validation. We introduce three other methods for estimating  $s_n$ . We first give a few examples of  $T$  and  $s_n$ .

*Example 1.* Let  $Z_i = (Y_i, X_i)$  ( $i = 1, \dots, n$ ) be independent and identically distributed random variables and let  $\mu(x) = E(Y_1 | X_1 = x)$  be the regression function. If we approximate the regression function by a polynomial of degree  $k$ , estimate the parameters by least-squares and denote the estimated regression function by  $\hat{\mu}_k(x)$ , then  $z = (y, x)$ ,

$$T(z, F_n) = \{y - \hat{\mu}_k(x)\}^2, \quad s_n = \int \{y - \hat{\mu}_k(x)\}^2 dF(x, y).$$

Here  $s_n$  is the predictive error. In model selection problems we estimate  $s_n$  for various values of  $k$  and choose that value of  $k$  for which the estimated value of  $s_n$  is the smallest.

*Example 2.* Let  $Z = (Y, X_1, \dots, X_d)$  be a vector of univariate random variables. Breiman & Friedman (1985) considered transformations  $h(Y), \phi_1(X_1), \dots, \phi_d(X_d)$  of the variables  $Y, X_1, \dots, X_d$  such that

$$E\{h(Y)\} = E\{\phi_1(X_1)\} = \dots = E\{\phi_d(X_d)\} = 0, \quad E\{h^2(Y)\} = 1, \quad E\{\phi_j^2(X_j)\} < \infty.$$

Transformations  $h^*, \phi_1^*, \dots, \phi_d^*$  are called optimal if they minimize

$$e^2(h, \phi_1, \dots, \phi_d) = E\{h(Y) - \phi_1(X_1) - \dots - \phi_d(X_d)\}^2.$$

Assume that we have a data set of size  $n$  and estimate the optimal transformations using splines with  $k_0, k_1, \dots, k_d$  knots respectively. If  $\hat{h}, \hat{\phi}_1, \dots, \hat{\phi}_d$  are the sample estimates of the optimal transformations, then the predictive error is

$$e^2(\hat{h}, \hat{\phi}_1, \dots, \hat{\phi}_d) = E[\{\hat{h}(Y) - \hat{\phi}_1(X_1) - \dots - \hat{\phi}_d(X_d)\}^2 | Y_i, X_i, i = 1, \dots, n].$$

In this example,  $s_n$  is the quantity above and we are interested in estimating it for various values  $k_0, k_1, \dots, k_d$ .

*Example 3:  $L^1$  regression.* Consider the regression problem in Example 1. Let  $\hat{\theta}_0, \dots, \hat{\theta}_k$  be the estimates obtained by minimizing

$$\sum_{i=1}^n |Y_i - \theta_0 - \theta_1 X_i - \theta_2 X_i^2 - \dots - \theta_k X_i^k|.$$

Let  $(X, Y)$  be independent of  $(X_i, Y_i)$  ( $i = 1, \dots, n$ ). Then the prediction error is

$$E(|Y - \hat{\theta}_0 - \hat{\theta}_1 X - \dots - \hat{\theta}_k X^k| | Y_i, X_i, i = 1, \dots, n).$$

Here  $s_n$  is the quantity given above. The  $L^1$  method is typically used to get a robust estimate of the regression function. It is known that a good deal of computing is needed to calculate the estimates  $\hat{\theta}_0, \dots, \hat{\theta}_k$ .

In the last two examples ordinary cross-validation could be computationally very expensive and there thus arises the need for alternative methods. Let us now formally introduce the concepts of  $v$ -fold cross-validation, the repeated learning testing and the repeated  $v$ -fold cross-validation methods.

2.2.  $v$ -fold cross-validation

We first divide the data randomly in  $v$  groups so that their sizes are as nearly equal as possible. Let the size of the  $\alpha$ th group be  $m_\alpha$  and assume that  $[n/v] \leq m_\alpha \leq [n/v] + 1$  for all  $\alpha$ . Let  $F_{n\alpha}$  be the empirical estimate of  $F$  based on all the  $n_\alpha = n - m_\alpha$  observations outside group  $\alpha$  and let  $\bar{F}_{n\alpha}$  be the estimate of  $F$  based on the  $m_\alpha$  observations in group  $\alpha$ . Then a  $v$ -fold cross-validated estimate of  $s_n$  is given by

$$CV_{nv} = \sum_{\alpha=1}^v p_\alpha \int T(z, F_{n\alpha}) d\bar{F}_{n\alpha}(z),$$

where  $p_\alpha = m_\alpha/n$ . Note that  $p_\alpha \simeq v^{-1}$  for all  $\alpha$ .

Let  $s = \int T(z, F) dF(z)$ . If we assume that  $T(\cdot, F_n)$  can be expanded about  $T(\cdot, F)$  and note that  $n_\alpha \simeq (v-1)v^{-1}n$  for all  $\alpha$ , then

$$E(s_n - s) = E \int \{T(z, F_n) - T(z, F)\} dF(z) \simeq c_0 n^{-1},$$

$$E(CV_{nv} - s) = \sum p_\alpha E \int \{T(z, F_{n\alpha}) - T(z, F)\} dF(z) \simeq v(v-1)^{-1} c_0 n^{-1},$$

where  $c_0$  is a constant depending on  $T$  and  $F$ . Consequently,

$$E(CV_{nv} - s_n) \simeq (v-1)^{-1} c_0 n^{-1}. \tag{2.2}$$

For ordinary cross-validation, that is  $v = n$ , the right-hand side of the last expression is  $O(n^{-2})$ , but when  $v$  is small, say  $v = 3$  or  $4$ , this term is not necessarily very small. Simulation results presented in § 4 clearly bring this out. Also the term  $c_0$  is of the order of  $k$ , the number of parameters being estimated. Indeed, for the regression case in Example 1, if we fit a polynomial of degree  $k$ , then  $c_0$  is a constant times  $k + 1$ . So  $CV_{nv}$  may turn out to be a poor estimate of  $s_n$  if the number of parameters is not small. This observation is consistent with Stone's response to Geisser's comment (Stone, 1974).

Thus a correction term for  $CV_{nv}$  is needed. We show later that the following works well. Let

$$CV_{nv}^* = \sum p_\alpha \int T(z, F_{n\alpha}) d\bar{F}_{n\alpha}(z) + \int T(z, F_n) dF_n(z) - \sum p_\alpha \int T(z, F_{n\alpha}) dF_n(z). \tag{2.3}$$

As shown in § 6,

$$E(CV_{nv}^* - s_n) = E \sum p_\alpha \int \{T(z, F_{n\alpha}) - T(z, F_n)\} d(F_n - F)(z) \simeq (v-1)^{-1} c_1 n^{-2} \tag{2.4}$$

for some constant  $c_1$  depending on  $T$  and  $F$ . Expressions (2.2) and (2.4) show that the corrected  $v$ -fold cross-validated estimate  $CV_{nv}^*$  works much better than  $CV_{nv}$  asymptotically as  $n$  increases; the correction term is negligible when  $v \simeq n$ .

In the regression case, Example 1, let  $\hat{\mu}_k$  be as in Example 1. Let  $\hat{\mu}_{k\alpha}$  be the regression estimate based on the  $n_\alpha = n - m_\alpha$  observations not in group  $\alpha$ . Then the uncorrected  $v$ -fold cross-validated estimate of  $s_n$  is

$$CV_{nv} = \sum_{\alpha=1}^v (m_\alpha/n) [\sum \{Y_i - \hat{\mu}_{k\alpha}(X_i)\}^2 / m_\alpha],$$

where the sum inside the bracket is over all the  $m_\alpha$  observations in group  $\alpha$ . We have already discussed the need for a correction term as given in (2.3). The corrected  $v$ -fold cross-validated estimate of  $s_n$  is

$$CV_{nv}^* = CV_{nv} + \sum_{i=1}^n \{Y_i - \hat{\mu}_k(X_i)\}^2 / n - \sum_{\alpha=1}^v (m_\alpha/n) \left[ \sum_{i=1}^n \{Y_i - \hat{\mu}_{k\alpha}(X_i)\}^2 / n \right].$$

### 2.3. Repeated learning-testing method

In this method, we repeatedly split the data randomly in two parts, a learning set of size  $n_0$  and a test set of size  $m_0$  ( $m_0 + n_0 = n$ ). Typically  $n_0 \geq m_0$ . For each split, estimates are developed based on the data in the learning set and then these are tested on the data in the test set. On the  $\alpha$ th split of the data let  $F_{n\alpha}$  and  $\bar{F}_{n\alpha}$  be the estimates of  $F$  based on the observations in the learning set and the test set respectively. If the data is split  $v$  times, then a repeated learning-testing estimate of  $s_n$  is given by

$$LT_{nv} = v^{-1} \sum_{\alpha=1}^v \int T(z, F_{n\alpha}) d\bar{F}_{n\alpha}(z).$$

As for  $v$ -fold cross-validation, this does not provide an adequate estimate of  $s_n$  and a bias correction term is needed here too. The following gives a corrected repeated learning-testing estimate of  $s_n$ .

$$LT_{nv}^* = LT_{nv} + \int T(z, F_n) dF_n(z) - v^{-1} \sum \int T(z, F_{n\alpha}) dF_n(z).$$

The corrected  $v$ -fold cross-validated and repeated learning-testing estimates have exactly the same form. It can be shown that

$$E(LT_{nv} - s_n) \simeq (m_0/n_0)c_0n^{-1}, \quad E(LT_{nv}^* - s_n) \simeq (m_0/n_0)c_1n^{-2}, \quad (2.5)$$

where  $c_1$  is the same as in (2.4). The bias does not depend on  $v$ , the number of repeats, and is reduced if the ratio of the size of the test set to the size of the learning set is small.

The predictive sample reuse method of Geisser (1975) could be regarded as a version of the repeated learning-testing method. However, he did not introduce a correction term. Since he considered all possible partitions of the data into learning and test sets of sizes  $n_0$  and  $m_0$ , the number,  $v$ , of repeats is  $n!/\{m_0!(n - m_0)!\}$ , and consequently his method involves a lot more computing than ordinary cross-validation.

### 2.4. Repeated $v$ -fold cross-validation

This method is a combination of the  $v$ -fold cross-validation and the repeated learning-testing methods described in this section. This method is quite simple. We repeat the method of  $v$ -fold cross-validation  $t$  times. On the  $\beta$ th repeat we randomly split the data in  $v$  groups as described in § 2.2, get a bias corrected estimate  $CV_{nv\beta}^*$  of  $s_n$ , and finally we take a simple average of these  $t$  estimates. Let us call this estimate  $RCV_{nvt}^*$ , so that

$$RCV_{nvt}^* = t^{-1} \sum CV_{nv\beta}^*;$$

$RCV_{nv}$  has about the same bias as  $CV_{nv}^*$ :

$$E(RCV_{nv}^* - s_n) \approx (v - 1)^{-1} c_1 n^{-2}.$$

However, a variance calculation shows that it may be better to use a  $vt$ -fold corrected cross-validated estimate rather than  $RCV_{nv}^*$ ; see § 3.

### 3. STABILITY OF THE ESTIMATES

We now compare the stability of the ordinary cross-validated estimate of  $s_n$  with those defined in § 2.

Retaining only the terms of order  $n^{-2}$  and higher, we get, see § 6,

$$\begin{aligned} \text{var}(CV_{nn} - s_n) &\approx \gamma_0 n^{-1} + \gamma_1 n^{-2} + 2\gamma_2 n^{-2}, \\ \text{var}(CV_{nv}^* - s_n) &\approx \gamma_0 n^{-1} + v(v - 1)^{-1} \gamma_1 n^{-2} + 2\gamma_2 n^{-2}. \end{aligned} \tag{3.1}$$

Here  $\gamma_0 > 0$ ,  $\gamma_1 > 0$  and  $\gamma_2$  are constants depending on  $T$  and  $F$ . Thus

$$\text{var}(CV_{nv}^* - s_n) > \text{var}(CV_{nn} - s_n).$$

However, the difference is  $(v - 1)^{-1} \gamma_1 n^{-2}$  and this decreases as  $v$  increases. So we can expect the difference to be small for  $v > 2$  and this is supported by simulation. The expression for  $\text{var}(CV_{nv} - s_n)$  is not pretty, but it can be shown that

$$\text{var}(CV_{nv} - s_n) = \text{var}(CV_{nv}^* - s_n) + O((v - 1)^{-1} n^{-2});$$

see § 6. In the regression case it can be shown that it is larger than  $\text{var}(CV_{nv}^* - s_n)$  and  $\text{var}(CV_{nn} - s_n)$ . We believe this to be true always, but do not have a proof. However, as expected, the difference between  $\text{var}(CV_{nv}^* - s_n)$  and  $\text{var}(CV_{nv} - s_n)$  is negligible as  $v$  becomes large.

Now consider the repeated learning-testing estimate. Even though  $LT_{nv}^*$  has a small bias as an estimate of  $s_n$ , its variance is not as small as we want it to be. Retaining terms of order  $n^{-1}$  we get

$$\begin{aligned} \text{var}(LT_{nv}^* - s_n) &\approx \{1 + n_0 / (m_0 v)\} \gamma_0 n^{-1}, \\ \text{var}(LT_{nv}^* - s_n) - \text{var}(CV_{nn} - s_n) &\approx n_0 / (m_0 v) \gamma_0 n^{-1}. \end{aligned}$$

Recall from § 2 that

$$E(LT_{nv}^* - s_n) \approx (m_0 / n_0) c_1 n^{-2}.$$

As the ratio  $m_0 / n_0$  decreases the bias becomes smaller but the variance increases. The only way we can reduce variance in the learning-testing case is by increasing  $v$ , the number of repeats. From a computational point of view, it seems that  $v$ -fold cross-validation is better than the repeated learning-testing method. As in the case of  $v$ -fold cross-validation, the expression for  $\text{var}(LT_{nv} - s_n)$  is not a nice one; see § 6.

Finally, consider the repeated  $v$ -fold cross-validated estimate as given in § 2.4. It turns out that

$$\text{var}(RCV_{nv}^* - s_n) \approx \gamma_0 n^{-1} + t(t - 1)^{-1} (v - 1)^{-1} \gamma_1 n^{-2} + 2\gamma_2 n^{-2}, \tag{3.2}$$

where  $\gamma_0$ ,  $\gamma_1$  and  $\gamma_2$  are as in (3.1). An easy calculation shows that the expression in (3.2) is larger than the variance of the corrected  $vt$ -fold cross-validated estimate, though the difference  $t^{-1} (v - 1)^{-1} \gamma_1 n^{-2}$  is rather small. In repeated  $v$ -fold cross-validation, the estimate  $T(\cdot, \cdot)$  has to be calculated  $vt$  times, as it has to be in the case of  $vt$ -fold cross-validation, so that we may be better off using  $vt$ -fold cross-validation than repeating  $v$ -fold cross-validation  $t$  times.

## 4. SOME SIMULATION RESULTS

In this section we present some simulation results for a regression model. The correct regression is a quadratic model but we fit a linear model to it. Let

$$Y = \mu(X) + \varepsilon,$$

where

$$X \sim N(0, 1), \quad \varepsilon \sim N(0, 2), \quad \mu(x) = x|x| = \text{sgn}(x)x^2,$$

$X$  and  $\varepsilon$  being independent. Based on a data set of size  $n$  we estimate the linear regression  $\hat{a} + \hat{b}x$  by least-squares. We want to estimate

$$s_n = E\{(Y - \hat{a} - \hat{b}X)^2 | (X_i, Y_i), i = 1, \dots, n\}.$$

We consider two sample sizes,  $n = 12$  and  $n = 24$ . For  $v$ -fold cross-validation we randomly split the data in  $v$  groups and calculate the linear regression  $\hat{a}_\alpha + \hat{b}_\alpha x$  by least-squares based on all the observations not in group  $\alpha$ . The uncorrected  $v$ -fold cross-validated estimate of  $s_n$  is

$$CV_{nv} = \sum_{\alpha=1}^v (m_\alpha/n) \left\{ \sum (Y_i - \hat{a}_\alpha - \hat{b}_\alpha X_i)^2 / m_\alpha \right\},$$

where the sum inside the second bracket is over all the  $m_\alpha$  observations in group  $\alpha$ . The corrected  $v$ -fold cross-validated estimate of  $s_n$  is

$$CV_{nv}^* = CV_{nv} + \sum_{i=1}^n (Y_i - \hat{a} - \hat{b}X_i)^2 / n - \sum_{\alpha=1}^v (m_\alpha/n) \left\{ \sum_{i=1}^n (Y_i - \hat{a}_\alpha - \hat{b}_\alpha X_i)^2 / n \right\}.$$

For the repeated learning-testing method,  $n_0$  and  $m_0$  are the sizes of the learning and the test sets and  $v$  is the number of times the data has been split. The uncorrected repeated learning-testing estimate is

$$LT_{nv} = v^{-1} \sum_{\alpha=1}^v \left\{ \sum (Y_i - \hat{a}_\alpha - \hat{b}_\alpha X_i)^2 / m_0 \right\},$$

where the second sum inside the second bracket is over all the  $m_0$  observations in the test set on the  $\alpha$ th split of the data.

The corrected repeated learning-testing estimate is

$$LT_{nv}^* = LT_{nv} + \sum_{i=1}^n (Y_i - \hat{a} - \hat{b}X_i)^2 / n - v^{-1} \sum_{\alpha=1}^v \left\{ \sum_{i=1}^n (Y_i - \hat{a}_\alpha - \hat{b}_\alpha X_i)^2 / n \right\}.$$

The simulations in Tables 1, 2 and 3 are based on 40 000 repeats. In Table 1 we have calculated  $E(CV_{nv}^* - s_n)$ ,  $E(CV_{nv} - s_n)$ , and the standard deviations of  $CV_{nv}^* - s_n$  and  $CV_{nv} - s_n$  for various values of  $v$ . The case  $v = n$  is evaluated only for the uncorrected case;  $CV_{nn}$  is the ordinary cross-validated estimate of  $s_n$ .

Table 1 shows that the bias of  $CV_{nv}^*$  is always smaller than that of  $CV_{nv}$  and  $CV_{nv}^*$  has a smaller variance than  $CV_{nv}$  even though the difference is quite small when  $n = 24$  and  $v > 3$ . The simulations support the observations in §§ 2 and 3 that the performance of  $CV_{nv}^*$  is worst when  $v = 2$ . Though it seems that the bias for  $CV_{nv}^*$  initially decreases and then increases, there is no reason to support that simply because the standard error is about 0.01 when  $n = 12$  and is about 0.004 when  $n = 24$ .

Table 2 gives results for the repeated learning-testing method for various values of  $p_0 = m_0/n$ , the proportion of observations in the test set. For each  $p_0$ , we consider two values of  $v$ ,  $v = [1/p_0]$  and  $v = [2/p_0]$ , where  $[.]$  is the integer part of a number. The reason for these two choices is as follows. If  $p_0 = \frac{1}{3}$  and  $v = [1/p_0] = 3$  then we are calculating

Table 1. Biases and variances of uncorrected and corrected  $v$ -fold cross-validated estimates

$n$	$E(s_n)$		$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$	$v = n^\dagger$
12	2.566	$E(CV_{nv} - s_n)$	0.78	0.31	0.20	0.14	0.12	0.06
		$E(CV_{nv}^* - s_n)$	0.16	0.05	0.04	0.01	0.03	
		st. dev. $(CV_{nv} - s_n)$	2.66	1.79	1.62	1.54	1.49	1.41
		st. dev. $(CV_{nv}^* - s_n)$	1.79	1.54	1.48	1.44	1.42	
24	2.321	$E(CV_{nv} - s_n)$	0.25	0.11	0.07	0.05	0.04	0.02
		$E(CV_{nv}^* - s_n)$	0.02	0.002	0.0001	0.0004	0.002	
		st. dev. $(CV_{nv} - s_n)$	1.05	0.88	0.84	0.82	0.80	0.78
		st. dev. $(CV_{nv}^* - s_n)$	0.86	0.81	0.80	0.79	0.78	

† The case for  $v = n$  corresponds to ordinary cross-validation.

Table 2. Biases and variances of uncorrected and corrected repeated learning-testing estimates;  $v$ , number of repeats

$n$	$E(s_n)$		$p_0 = \frac{1}{2}$		$p_0 = \frac{1}{3}$		$p_0 = \frac{1}{4}$		$p_0 = \frac{1}{5}$		$p_0 = \frac{1}{6}$	
			$v = 2$	$v = 4$	$v = 3$	$v = 6$	$v = 4$	$v = 8$	$v = 5$	$v = 10$	$v = 6$	$v = 12$
12	2.566	$E(LT_{nv} - s_n)$	0.74	0.75	0.31	0.32	0.19	0.20	0.11	0.12	0.13	0.12
		$E(LT_{nv}^* - s_n)$	0.14	0.14	0.05	0.06	0.03	0.04	0.02	0.03	0.04	0.02
		st. dev. $(LT_{nv} - s_n)$	2.59	2.19	1.93	1.74	1.87	1.65	1.85	1.65	1.82	1.61
		st. dev. $(LT_{nv}^* - s_n)$	1.78	1.59	1.68	1.53	1.71	1.53	1.76	1.58	1.73	1.54
24	2.321	$E(LT_{nv} - s_n)$	0.26	0.24	0.12	0.12	0.08	0.08	0.05	0.04	0.04	0.04
		$E(LT_{nv}^* - s_n)$	0.02	0.01	0.01	0.01	0.01	0.01	0.01	-0.003	-0.001	-0.001
		st. dev. $(LT_{nv} - s_n)$	1.09	0.98	1.01	0.92	1.02	0.92	1.06	0.94	1.02	0.90
		st. dev. $(LT_{nv}^* - s_n)$	0.93	0.85	0.95	0.87	0.98	0.88	1.03	0.91	0.99	0.88

the regression line three times for three learning sets. Three-fold cross-validation corresponds to the case  $p_0 = \frac{1}{3}$  and  $v = 3$  in the sense that test set sizes are about one-third of the data and the regression line has been calculated three times. When  $p_0 = \frac{1}{3}$  and  $v = 6$  we have 6 learning sets and consequently we have to estimate the regression line 6 times.

As we have discussed earlier, the variances of the repeated learning-testing estimates, corrected and uncorrected, decrease as  $v$ , the number of repeats, increases. However, as shown in (2.5), the biases do not depend on the number of repeats. The simulation results support these facts and also show that the bias of  $LT_{nv}$  is always larger than that of  $LT_{nv}^*$ . As for  $v$ -fold cross-validation, it appears that  $LT_{nv}^*$  has a smaller variance than  $LT_{nv}$ , but the difference seems to be quite small when  $n = 24$  and  $v > 3$ . Comparison of Tables 1 and 2 shows that  $CV_{nv}^*$  and  $LT_{nv}^*$  have about the same biases, but  $\text{var}(CV_{nv}^*) < \text{var}(LT_{nv}^*)$ .

For a fixed data set the value of the corrected and the uncorrected  $v$ -fold cross-validated estimates vary from partition to partition. Table 3 is based on a simulation study of this variability. For the quantities  $E\{\text{var}(CV_{nv}|\mathcal{Z}_n)\}$ ,  $E\{\text{var}(CV_{nv}^*|\mathcal{Z}_n)\}$  given in Table 3, the variance is first taken over random partitions for a given data set  $\mathcal{Z}_n = \{Z_i = (Y_i, X_i): i = 1, \dots, n\}$  and then the expectation is taken over  $\mathcal{Z}_n$ . Tables 1 and 3 clearly show that  $E\{\text{var}(CV_{nv}|\mathcal{Z}_n)\}$  and  $E\{\text{var}(CV_{nv}^*|\mathcal{Z}_n)\}$  are quite small compared to the variances of  $CV_{nv} - s_n$  and  $CV_{nv}^* - s_n$ .

Table 3. Variability of the estimates over the partitions

	$n = 12$					$n = 24$				
	$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$	$v = 2$	$v = 3$	$v = 4$	$v = 5$	$v = 6$
$E\{\text{var}(CV_{nv} \mathcal{Z}_n)\}$	0.046	0.037	0.029	0.025	0.032	0.014	0.011	0.008	0.008	0.006
$E\{\text{var}(CV_{nv}^* \mathcal{Z}_n)\}$	0.023	0.023	0.021	0.019	0.025	0.008	0.008	0.006	0.007	0.005

5. CONCLUSIONS

In summary whenever ordinary cross-validation is computationally very expensive, we can use  $v$ -fold cross-validation or the repeated learning-testing methods. If  $v$  is not large the uncorrected  $v$ -fold cross-validated estimate or the uncorrected repeated learning-testing estimate may have large biases. Thus we recommend that the correction terms given in § 2 should be routinely used.

The corrected  $v$ -fold cross-validated estimate is preferable to the corrected repeated learning-testing estimate because the former has a smaller variance.

The bias and the variance of the  $v$ -fold cross-validated estimate decrease as  $v$  increases. From a practical point of view, 2-fold corrected cross-validation should be avoided if possible.

For the corrected repeated learning-testing estimate, the bias depends on the ratio  $m_0/n_0$ , the ratio of the size of the test set to the learning set, and not on the number of repeats  $v$ . The smaller the ratio  $m_0/n_0$ , the smaller the bias. However, the smaller the ratio  $n_0/(m_0v)$ , the smaller the variance.

6. SOME TECHNICAL RESULTS

This section gives a few technical results on the biases and the variances. More details appear in an unpublished technical report. We assume that for any  $z$ ,  $T(z, \cdot)$  is von Mises differentiable with derivatives  $T_1$  and  $T_2$  (Serfling, 1980, Ch. 6). This condition can be relaxed a little. For all the results in this section, we keep only the dominant terms. All proofs we present here can be made rigorous with careful analysis (Beran, 1984; Serfling, 1984). Let  $D_n = F_n - F$ ,  $D_{n\alpha} = F_{n\alpha} - F$ ,  $\bar{D}_{n\alpha} = \bar{F}_{n\alpha} - F$ ,

$$nE \int \{T(z, F_n) - T(z, F)\} dF(z) \simeq c_0,$$

$$- E \int \{T_1(z_1, z, F_n) - T_1(z_1, z, F)\} dD_n(z_1) dD_n(z) = c_1 n^{-2} + O(n^{-3}).$$

Note that the constants  $c_0$  and  $c_1$  depend on  $T$  and  $F$ .

**THEOREM 6.1.** *We have that*

- (a)  $E(CV_{nv} - s_n) \simeq (v - 1)^{-1} c_0/n$ ,
- (b)  $E(LT_{nv} - s_n) \simeq (m_0/n_0) c_0/n$ ,
- (c)  $E(CV_{nv}^* - s_n) \simeq (v - 1)^{-1} c_1/n^2$ ,
- (d)  $E(LT_{nv}^* - s_n) \simeq (m_0/n_0) c_1/n^2$ ,
- (e)  $E(RCV_{nvt}^* - s_n) \simeq (v - 1)^{-1} c_1/n^2$ .

To find  $c_0$  and  $c_1$  in the regression case of Example 1, let us assume that the regression function  $\mu$  is linear,

$$Y = a + bX + \varepsilon,$$

where  $\varepsilon$  and  $X$  are independent normal random variables,  $E(\varepsilon) = 0$  and  $\text{var}(\varepsilon) = \sigma^2$ . Then  $c_0 = 2\sigma^2$  and  $c_1 = 4\sigma^2$ . Now

$$E(CV_{nv} - s_n) \simeq 2\sigma^2(v - 1)^{-1} n^{-1}, \quad E(CV_{nn} - s_n) \simeq 2\sigma^2 n^{-2},$$

$$E(CV_{nv}^* - s_n) \simeq 4\sigma^2(v - 1)^{-1} n^{-2}.$$

If the regression function  $\mu$  is not linear, but we fit a linear regression, then the expressions for  $c_0$  and  $c_1$  are not so simple because they include terms involving the bias



function  $b(x) = \mu(x) - \mu_1(x)$ , where  $\mu_1(x) = a + bx$  is the least-squares linear model closest to  $\mu$ .

Next, for any  $\alpha$ , let

$$nE \left\{ \int T(z) dD_n(z) \right\}^2 = \gamma_0.$$

$$\frac{1}{2}(v-1)v^{-2}n^2E \left\{ \int T_1(z_1, z) dD_{n\alpha}(z_1) d\bar{D}_{n\alpha}(z) \right. \\ \left. + \int T_1(z_1, z) d\bar{D}_{n\alpha}(z_1) dD_{n\alpha}(z) \right\}^2 = \gamma_1 + O(n^{-1}).$$

$$(v-1)v^{-2}n^2E \left\{ \int T(z) d\bar{D}_{n\alpha}(z) \right\} \\ \times \left\{ \int T_2(z_2, z_1, z) dD_{n\alpha}(z_2) dD_{n\alpha}(z_1) d\bar{D}_{n\alpha}(z) \right\} = \gamma_2 + O(n^{-1}).$$

Then  $\gamma_0, \gamma_1$  and  $\gamma_2$  are constants depending on  $T$  and  $F$  and  $\gamma_1 > 0$ . Since  $m_\alpha \approx n/v$  for all  $\alpha$ , two quantities described above are meaningful.

**THEOREM 6.2.** *We have*

- (a)  $\text{var}(CV_{nv}^* - s_n) \approx \gamma_0/n + v(v-1)^{-1}\gamma_1/n^2 + 2\gamma_2/n^2,$
- (b)  $\text{var}(LT_{nv}^* - s_n) \approx \{1 + n_0/(m_0v)\}\gamma_0/n,$
- (c)  $\text{var}(CV_{nn} - s_n) \approx \gamma_0/n + \gamma_1/n^2 + 2\gamma_2/n^2,$
- (d)  $\text{var}(RCV_{nv}^* - s_n) \approx \gamma_0/n + t(t-1)^{-1}(v-1)^{-1}\gamma_1/n^2 + 2\gamma_2/n^2.$

As a consequence of Theorem 6.2 we get the following result.

**COROLLARY 6.3.** *It follows that*

- (a)  $\text{var}(CV_{nv}^* - s_n) \approx \text{var}(CV_{nn} - s_n) + (v-1)^{-1}\gamma_1/n^2,$
- (b)  $\text{var}(LT_{nv}^* - s_n) \approx \text{var}(CV_{nn} - s_n) + \{n_0/(m_0v)\}\gamma_0/n,$
- (c)  $\text{var}(LT_{nv}^* - s_n) \approx \text{var}(CV_{nv} - s_n) + \{n_0/(m_0v)\}\gamma_0/n.$

Theorem 6.2 and Corollary 6.3 show that  $LT_{nv}^*$  is a poorer estimate of  $s_n$  than  $CV_{nv}^*$ . Also  $\text{var}(CV_{nv}^* - s_n) > \text{var}(CV_{nn} - s_n)$ , but the difference becomes small as  $v$  increases.

The expressions for the variances of  $CV_{nv} - s_n$  and  $LT_{nv} - s_n$  are quite complicated. Here we give only the expression for  $\text{var}(LT_{nv} - s_n)$ . Let

$$\bar{T}_1(z_1) = \int T(z_1, z) / dF(z),$$

$$\gamma_3 = nE \left\{ \int \bar{T}_1(z_1) dD_n(z_1) \right\}^2, \quad \gamma_4 = nE \left\{ \int T(z) dD_n(z) \right\} \left\{ \int \bar{T}_1(z_1) dD_n(z_1) \right\}.$$

**THEOREM 6.4.** *We have*

- (a)  $\text{var}(CV_{nv} - s_n) \approx \text{var}(CV_{nv}^* - s_n) + O\{(v-1)^{-1}n^{-2}\},$
- (b)  $\text{var}(LT_{nv} - s_n) \approx \{1 + n_0/(m_0v)\}\gamma_0/n + \{m_0/(n_0v)\}\gamma_3/n - 2\gamma_4/(nv).$

Corollary 6.3 shows that  $CV_{nv}^*$ , the corrected  $v$ -fold cross-validated estimate, has a larger variance than  $CV_{nn}$ , the ordinary cross-validated estimate of  $s_n$ . Even though we have been unable to show in our general setting that  $CV_{nv}^*$  has a smaller variance than

$CV_{nv}$ , we believe this to be so. In the regression case when the regression function  $\mu$  is linear,  $\gamma_0 = 2\sigma^4$ ,  $\gamma_1 = 16\sigma^4$  and  $\gamma_2 = \gamma_3 = \gamma_4 = 0$ . Then the following can be shown:

$$\begin{aligned}\text{var}(CV_{nn} - s_n) &\simeq 2\sigma^4\{n^{-1} + 8n^{-2}\}, \\ \text{var}(CV_{nv}^* - s_n) &\simeq 2\sigma^4\{n^{-1} + 8n^{-2} + 8(v-1)^{-1}n^{-2}\}, \\ \text{var}(CV_{nv} - s_n) &\simeq 2\sigma^4\{n^{-1} + 8n^{-2} + 8(v-1)^{-1}n^{-2} + 4(v-1)^{-2}n^{-2} + 2(v-1)^{-3}n^{-2}\}.\end{aligned}$$

As we have discussed earlier, if the regression function  $\mu$  is not linear and we are fitting a linear model, then the expressions for  $\gamma_0, \dots, \gamma_4$  involve the model bias.

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#### APPENDIX

##### Outline of proofs of theorems

For Theorem 6.1, note for (a) that  $CV_{nv} - s_n$  can be written as

$$\begin{aligned}\int T(z) dD_n(z) + \sum p_\alpha \int \{T(z, F_{n_\alpha}) - T(z, F)\} d\bar{D}_{n_\alpha}(z) \\ + \sum p_\alpha \int \{T(z, F_{n_\alpha}) - T(z, F_n)\} dF(z) = I_1 + I_2 + I_3, \quad (\text{A}\cdot 1)\end{aligned}$$

say. Now  $E(I_1) = E(I_2) = 0$ , whereas

$$\begin{aligned}I_3 = \sum p_\alpha \int \{\bar{T}_1(z_1) dD_{n_\alpha}(z_1) - \bar{T}_1(z_1) dD_n(z_1)\} \\ + \sum p_\alpha \int \{\bar{T}_2(z_2, z_1) dD_{n_\alpha}(z_2) dD_{n_\alpha}(z_1) - \bar{T}_2(z_2, z_1) dD_n(z_2) dD_n(z_1)\}.\end{aligned}$$

The first term in the last expression is zero since  $\sum p_\alpha D_{n_\alpha} = D_n$ . Since  $p_\alpha \simeq v^{-1}$  and  $n_\alpha \simeq (v-1)v^{-1}n$  for all  $\alpha$ , the expectation of the second term approximately is

$$\sum p_\alpha (1/n_\alpha - 1/n) c_0 \simeq (v-1)^{-1} c_0/n.$$

The proof of (b) is the same as in (a).

For (c) note that

$$\begin{aligned}CV_{nv}^* - s_n = \int T(z) dD_n(z) + \sum p_\alpha \int \{T(z, F_{n_\alpha}) - T(z, F)\} d\bar{D}_{n_\alpha}(z) \\ + \sum p_\alpha \int \{T(z, F_n) - T(z, F_{n_\alpha})\} dD_n(z) \\ = I_4 + I_5 + I_6, \quad (\text{A}\cdot 2)\end{aligned}$$

say. Now  $E(I_4) = E(I_5) = 0$ . The result holds since it can be shown that

$$E(I_6) \simeq (v-1)^{-1} n^{-2} c_1 + O(n^{-3}).$$

The proofs of (d) and (e) follow as in (c).

For (a) of Theorem 6.2, as in (A.2),  $cv_{nv}^* - s_n = I_4 + I_5 + I_6$ . Since  $E(cv_{nv}^* - s_n) = O(n^{-2})$  as given in Theorem 6.1 and since we are keeping terms of order  $n^{-2}$  and higher,

$$\text{var}(cv_{nv}^* - s_n) \doteq E\{(cv_{nv}^* - s_n)^2\}.$$

Now,

$$E\{(cv_{nv}^* - s_n)^2\} = E(I_4^2) + E(I_5^2) + E(I_6^2) + 2E(I_4I_5) + 2E(I_4I_6) + 2E(I_5I_6).$$

The result is proved via

$$\begin{aligned} E(I_4^2) &= \gamma_0/n, & E(I_5^2) &\doteq v(v-1)^{-1}\gamma_1/n^2, \\ E(I_6^2) &\doteq O(n^{-3}), & E(I_4I_5) &\doteq v(v-1)^{-1}\gamma_2/n^2, \\ E(I_4I_6) &\doteq -(v-1)^{-1}\gamma_2/n^2, & E(I_5I_6) &\doteq O(n^{-3}). \end{aligned}$$

For (b), the proof follows from part (a) of Theorem 6.4 by putting  $v = n$ .

For (c) note that  $LT_{nv}^* - s_n$  can be written as

$$\begin{aligned} v^{-1} \sum \int T(z) d\bar{D}_{n\alpha}(z) + v^{-1} \sum \int \{T(z, F_{n\alpha}) - T(z, F)\} d\bar{D}_{n\alpha}(z) \\ + v^{-1} \sum \int \{T(z, F_n) - T(z, F_{n\alpha})\} dD_n(z) = I_7 + I_8 + I_9, \end{aligned}$$

say. The result holds since it can be shown that

$$E\{(LT_{nv} - s_n)^2\} = E(I_7^2) + O(n^{-2}) = \{1 + n_0/(m_0v)\}\gamma_0n^{-1} + O(n^{-2}).$$

The proof of (d) is the same as in parts (a) and (c).

For Theorem 6.4 the proof of (a) follows using (A.1) and calculations similar to the ones in Theorem 6.2.

For (b) note that

$$\begin{aligned} LT_{nv} - s_n &= v^{-1} \sum \int T(z) d\bar{D}_{n\alpha}(z) + v^{-1} \sum \int \{T(z, F_{n\alpha}) - T(z, F)\} d\bar{D}_{n\alpha}(z) \\ &\quad + v^{-1} \sum \int \{T(z, F_{n\alpha}) - T(z, F_n)\} dF(z) \\ &= I_{10} + I_{11} + I_{12}, \end{aligned}$$

say. Since we are interested only in the terms of order  $n^{-1}$ ,

$$E(LT_{nv} - s_n) \doteq E(I_{10}^2) + E(I_{12}^2) + 2E(I_{10}I_{12}).$$

The proof follows from

$$E(I_{10}^2) = \{1 + n_0/(m_0v)\}\gamma_0n^{-1}, \quad E(I_{10}I_{12}) \doteq -(nv)^{-1}\gamma_4, \quad E(I_{12}^2) \doteq \{m_0/(n_0v)\}\gamma_3/n.$$

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