Review Article

Symmetric-Hyperbolic System of Conservative Equations for a Viscous Heat Conducting Fluid

By

T. Ruggeri, Bologna, Italy

Summary

Some critical considerations on the models of "extended irreversible thermodynamics" are given. By developing a methodology ("invariance of the generators") based both on the ideas of the "extended irreversible thermodynamics" and the "entropy principle" in its general formulation of "rational thermodynamics", a theory for a newtonian thermo-viscous fluid is proposed. The theory has the following properties, new when compared with previous ones: a) the system of equations is hyperbolic for any value of the field variables, provided that the usual thermodynamic stability condition (maximum entropy at equilibrium) holds; all wave-propagation speeds are then real and finite; b) the system is conservative and it is possible to seek for weak solutions and, in particular, for shock-waves; moreover, the system is symmetric-hyperbolic in the sense of Friedrichs; special properties hold therefore for weak solutions and shocks; c) the only thermodynamic variables at non-equilibrium, modified with respect to the corresponding ones at equilibrium, are the entropy density and chemical potential; consequentely, there exists only a single absolute temperature, playing an important role in relaxations; d) the entropy principle is automatically satisfied.

1. Classical Approach: Fourier-Navier-Stokes Equations

The equations for the motion of a thermoviscous fluid, in conservative form, are:

$$\left.\begin{array}{c} \dfrac{\partial \varrho}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\varrho u_i\right) = 0, \\[2mm] \dfrac{\partial}{\partial t}\left(\varrho u_k\right) + \dfrac{\partial}{\partial x_i}\left(\varrho u_i u_k + t_{ik}\right) = 0, \\[2mm] \dfrac{\partial \varepsilon}{\partial t} + \dfrac{\partial}{\partial x_i}\left(\varepsilon u_i + q_i + t_{ij}u_j\right) = 0; \end{array}\right\} \tag{1.1}$$

$t_{ik} = p\delta_{ik} - \sigma_{ik}$ being the pressure tensor; p the pressure, $\sigma \equiv (\sigma_{ik})$ the viscous tensor; $\varepsilon = \varrho\left(\dfrac{1}{2}u^2 + e\right)$ the total energy, e the internal energy and ϱ, $\boldsymbol{u}$, $\boldsymbol{q}$ respectively the mass density, the velocity of the fluid particle and the heat flux.

To the Eq. (1.1), the constitutive equations must be added; in particular one has to determine the constitutive dependence of the heat flux and of the viscous tensor. A classical way to determine the constitutive equations of Fourier and Navier-Stokes (FNS) is the following:

The Gibbs relation is supposed to hold:

$$\theta \, dS = de + p \, dV \tag{1.2}$$

(θ = absolute temperature, S = entropy per unit mass, $V = 1/\varrho$ = specific volume). From (1.1-2) the following entropy balance is obtained:

$$\frac{\partial}{\partial t}(\varrho S) + \frac{\partial}{\partial x_i}\left(\varrho S u_i + \frac{q_i}{\theta}\right) = s, \tag{1.3}$$

where the "entropy source" s is:

$$s = -\frac{1}{\theta^2} \boldsymbol{q} \cdot \nabla \theta + \frac{1}{\theta} \sigma^D : \left\{\frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}) - \frac{1}{3} \boldsymbol{I} \operatorname{div} \boldsymbol{u}\right\} + \frac{1}{3\theta} \operatorname{tr} \sigma \cdot \operatorname{div} \boldsymbol{u} \tag{1.4}$$

with the notation: $\sigma^D = \sigma - \frac{1}{3} \boldsymbol{I} \operatorname{tr} \sigma$, denoting the deviator stress corresponding to σ; $\boldsymbol{I} = 3 \times 3$ identity matrix and $\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}(\boldsymbol{AB})$.

Since the entropy inequality requires s to be positive for any thermodynamic process, requiring linear constitutive relations between (i) heat flux and temperature gradient and (ii) stress and velocity gradient, one obtains the *FNS* constitutive equations:

$$\left.\begin{aligned} &\nabla \theta = -k \boldsymbol{q}, \\[1ex] &\frac{1}{2}(\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}) - \frac{1}{3} \boldsymbol{I} \operatorname{div} \boldsymbol{u} = \alpha \sigma^D, \\[1ex] &\operatorname{div} \boldsymbol{u} = \beta \operatorname{tr} \sigma, \end{aligned}\right\} \tag{1.5}$$

satisfying the entropy inequality

$$s = \frac{kq^2}{\theta^2} + \frac{\alpha}{\theta} \sigma^D : \sigma^D + \frac{\beta}{3\theta}(\operatorname{tr} \sigma)^2 > 0, \tag{1.6}$$

provided that the scalar functions k, α, β are positive ($1/k = \chi$ is the thermal conductivity, $1/(2\alpha) = \eta$, $1/(3\beta) = \zeta$ the usual viscous coefficients). The system (1.1), (1.5) (denoted in the following as *FNS* system), with the associated constitutive relations for all thermodynamic variables is a first order system of 14 partial differential equations for 14 unknowns (for example two thermodynamic variables, the three components of $\boldsymbol{u}$, the 3-components of $\boldsymbol{q}$, the five components of σ^D and $\operatorname{tr} \sigma$).

As is known, the system is not hyperbolic because of the form of constitutive relations (1.5); the consequence is an infinite wave propagation speed. Several authors proposed alternative equations to replace the *FNS* system in order to eliminate the paradox and hyperbolize the system. Starting from Maxwell's idea and from the well known paper by C. Cattaneo [1] (in the case of a rigid heat conductor), a large body of literature exists to which one can hardly objectively refer to. In any case, the most interesting contribution, for its simplicity

both physically and mathematically, is the paper by I. Müller [2], which has provided also the basis for further developments (in the non relativistic context see e.g., the papers by Carrassi-Morro [3], Lebon and coworkers [4]—[6]; for relativistic theories see e.g. Müller [7], Kranyš [8], Israel [9], Israel-Stewart [10] and, in a different context, Boillat [11], Massa-Morro [12] and Dixon [13]). Since in the present paper we are concerned with non relativistic fluids, we shall relate to the work by Müller [2] who gave the foundation to the "extended irreversible thermodynamics".

Different approaches to similar problems, known as "rational thermodynamics" are by Noll [14], Coleman-Noll [15], Coleman [16], Gurtin [17], Truesdell [18], Müller [19], Green-Lindsay [20] and, recently, by Grioli [21].

Other formulations, which are similar to "extended irreversible thermodynamics", employ "internal variables" (see e.g. Kluitenberg [22], Morro [23]).

A review of modern thermodynamics is given in the exhaustive article by Hutter [24].

Now we shall examine briefly the methodology developed in [2], useful in the following.

2. Müller's Theory [2]

The basic idea of the theory is as follows: If one tries to introduce into (1.5) the time derivatives of q, σ^D and tr σ by preserving the entropy principle, one must necessarily consider the non-equilibrium entropy density (different from equilibrium one) not only to depend upon the two thermodynamic variables but also on q and σ. Müller supposes that the non-equilibrium quantities are not far from those at equilibrium. To introduce the least possible modification of the classical theory he supposes that the new entropy density:

$$S_N \equiv S_N(\varrho, e, \boldsymbol{q}, \sigma^D, \text{tr } \sigma) \tag{2.1}$$

is equal to S plus the quadratic terms of a development in the now variables ("linearity" equilibrium — non equilibrium). It follows that the Gibbs relation is modified into

$$dS_N = de + p \, dV - \alpha_1 \boldsymbol{q} \cdot d\boldsymbol{q} - \alpha_2 \sigma^D : d\sigma^D - \alpha_3(\text{tr } \sigma) \, d(\text{tr } \sigma), \tag{2.2}$$

where α_i are functions of e and V only.

Forming the time derivative of S_N and taking into account (1.1), one reaches an entropy balance of the type (1.3) with S replaced by S_N on the l.h.s. and with a r.h.s. which involves s and the time derivatives of q, σ^D and tr σ. In a similar way as in the classical case, Müller obtains equations which replace to FNS systems and read

$$\left.\begin{aligned}
a\varrho\theta \, \frac{dq_i}{dt} + h\theta^2 \, \frac{\partial \sigma_{ij}^D}{\partial x_j} + L\theta^2 \, \frac{\partial \text{tr } \sigma}{\partial x_i} + \frac{\partial \theta}{\partial x_i} &= -kq_i, \\
b\varrho \, \frac{d\sigma_{ij}^D}{dt} - h\theta \left(\frac{\partial q_i}{\partial x_j} \right) - \left(\frac{\partial u_i}{\partial x_j} \right) &= -\alpha \sigma_{ij}^D, \\
c\varrho \, \frac{d \, \text{tr } \sigma}{dt} - 3L\theta \, \text{div } \boldsymbol{q} - \text{div } \boldsymbol{u} &= -\beta \, \text{tr } \sigma,
\end{aligned}\right\} \tag{2.3}$$

where:

$$\langle A_{ij}\rangle = A^{D}_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}\operatorname{tr}\boldsymbol{A}\,\delta_{ij}$$

and $d/dt = \partial/\partial t + u_i\,\partial/\partial x_i$ denotes the substantial derivative. Eqs. (2.3) go over into (1.5) when the phenomenological coefficients a, b, c, h and L vanish (a, b, and c are relaxation terms).

In particular for an inviscid fluid the equation that replaces the Fourier equation becomes a Cattaneo-like equation:

$$\tau\,d\boldsymbol{q}/dt + \chi\operatorname{grad}\theta = -\boldsymbol{q}, \qquad (\tau = a\varrho\theta\chi). \tag{2.4}$$

3. Some Critical Observations Regarding the Theories of "Extended Irreversible Thermodynamics"

Müller's theory has several good physical behavioural properties: It satisfies the entropy principle, is in agreement with statistical mechanics (in the frame of the thirteen momenta of Grad's theory) and provides linear high frequency waves propagating with finite speed through a constant state [3]. Moreover, because of the supposed linear relationship between the thermodynamic variable at equilibrium, it differs least from the classical theory.

This non-linear theory, and many other similar ones, however, are subject to criticisms both, from a physical, and, even more so, from a mathematical point of view. We show these objections and formulate a theory which conceptually does not differ from that of Müller, but, in our opinion, is free of the following shortcomings inherent to "irreversible thermodynamics".

i) The hydrodynamic system (1.1), (2.3) is generally non-hyperbolic. In fact, as shown in [25], even in the pure viscous case (no heat flux), the characteristic speeds are real if, and only if, some of the field variables are suitably bounded. This circumstance is not acceptable since, even if the limitations are satisfied by the initial data, they may be found not to be fulfilled by the solutions at a generic time.

In well behaving theories hyperbolicity conditions involve only the constitutive functions: for instance, for an inviscid non-heat conducting fluid, it is known that hyperbolicity implies $(\partial p/\partial\varrho)_s > 0$. Therefore, in such theories, even in case new bounded real velocities appear, other speeds, when calculated for suitable values of the fields may become complex. As a consequence instability effects arise, and the Cauchy problem is ill-posed [26], [27]. The theory may thus fail in problems of discontinuity (acceleration) waves and high frequency linear waves propagating across non-constant states because hyperbolicity is not guaranteed for all times [28]—[30].

ii) While the Eq. (1.1) exhibit a "conservative form", Eq. (2.3) generally are not conservation laws[1]. This lack prevents us from establishing a weak

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[1] In view of the mass balance law the conservation law for any function $g(x_i, t)$ has the form:

$$\varrho\,dg/dt = \partial(\varrho g)/\partial t + \partial(\varrho u_i g)/\partial x_i;$$

it is easily shown that Müller's system is conservative if a, b, c are constants and $h = L \equiv 0$.

formulation to the solutions [28], [31]. It is also known that for such nonlinear systems the regularity of solutions is not preserved, in general, even if the initial data are highly regular, not even in a neighbourhood of the initial time (see e.g. [32]). In particular, the lack of conservative form prevents us from studying shock waves which are a special class of C^1 piece-wise weak solutions.

We recall that even if there is no shock initially, it will be created after some time, for instance, by degenerating acceleration [28]—[30], [34] or asymptotic waves [33], [35]. It is true that it is possible (at least in principle) to study "shock structure" solutions also for non-conservative systems, but recent results, obtained by Anile and Majorana [36], have shown (in the case of plane wave) that for Mach numbers greater than some limit value, Müller's theory does not allow continuous "shock structure" solutions. The *FNS* theory always possesses such solutions, even though they are in poor agreement with the experimental data (see [37]). This, fact in our opinion, is not surprising, since, although Müller's theory is a refinement of *FNS*, it cannot be put in a conservative form which is possible for the non-hyperbolic *FNS*-theory. On the other hand, "shock structure" solutions approach true shock waves when some parameter vanishes (usually viscosity and thermal conductivity coefficients) or when the jumps are very large (strong shocks) [38], [39]; but this limit is not possible if the system is non-conservative.

Finally, we note that all thermodynamic theories should agree with statistical mechanics which is based on conservation laws (conservation of momenta). We think that in Müller's theory the conservative form of the equations has been lost because of the assumption of equilibrium non-equilibrium linearity.

iii) In the extended thermodynamic theories, the Gibbs relation is modified into (2.2), with the consequence that, for reasons of integrability, a new absolute temperature must be introduced at non-equilibrium, different from that at equilibrium, even if it does not appear in the field equations. However, while one accomplishes the idea of an entropy density at non-equilibrium different from that at equilibrium (especially when q and σ are considered as true field variables), it is difficult to understand the physical consequences of the two distinct absolute temperatures.

The aim of the present paper is to remove the shortcomings listed in i)—iii). In particular we try to set up a model that, preserving the conceptual bases of Müller's ideas, provides a systems of the type (1.1), (2.3): conservative, hyperbolic (without limitations on the field) and with a unique absolute thermodynamic temperature (regardless of equilibrium and non-equilibrium).

The problem is now to establish a methodology which differs from that developed in [2]. We shall abandon the a priori hypothesis of the development in power series, truncated at the quadratic terms but wish to preserve the physical condition required by the entropy principle. It is straightforward to see that the entropy principle and the idea that some thermodynamic variables must be modified at non-equilibrium are assumptions too general obtain a simple and meaningful result. The key will be to make use of the "symmetry" properties of the physical theories (see e.g. hydrodynamics) possessing a supplementary conservation law (e.g.: entropy balance). In the next section we recall some recent results on these subjects, which will be useful in ensuing developments.

4. Entropy Principle and Systems Possessing
a Supplementary Conservation Law: "Generators"

One of the main results of modern thermomechanics is to regard the entropy inequality as a constraint to the constitutive equations and not as an identification of a privileged time orientation. First supporters of this point of view were Coleman and Noll [15]; later, a more general principle was proposed by Müller in [19]. He suggests to regard entropy density and entropy flux which need not necessarily equal q/θ as constitutive functions. The resulting system possesses one equation more than there are unknowns: therefore, a structure compatibility arises with consequences on the constitutive relations.

Another constraint on the latter arises from the entropy inequality.

The approach can be generalized to any partial differential first-order system as follows: One looks for the functional dependence $F^\alpha \equiv F^\alpha(u)$, in order that the quasi-linear first order conservative system[2]

$$\partial_\alpha F^\alpha(u) = f(u),$$

$$\alpha = 0, i; \quad i = 1, 2, 3; \quad \partial_0 = \partial/\partial t; \quad \partial_i = \partial/\partial x_i \tag{4.1}$$

becomes compatible with the supplementary conservation law

$$\partial_\alpha h^\alpha(u) = g(u), \tag{4.2}$$

where F^α and f are column vectors belonging to $\mathbb{R}^N$ and $u(x^\alpha)$ is the unknown $\mathbb{R}^N$-vector of the system.

If (4.2) were not a consequence of (4.1) we would have $N + 1$ equations for N unknowns, and the problem would generally be unsolvable. Recent investigations [40]—[45] have shown, that compatibility is ensured by the existence of a vector $u' \in \mathbb{R}^N$, the components of which are functions of u, such that

$$u' \cdot dF^\alpha = dh^\alpha, \tag{4.3}$$

$$u' \cdot f \equiv g. \tag{4.4}$$

Then, by supposing local invertibility of the mapping relating u and u', it is easy to see that there exist four scalar functions h'^α ($\alpha = 0, 1, 2, 3$) (in the sense of the metric of $\mathbb{R}^N$ vectors) defined as

$$h'^\alpha = u' \cdot F^\alpha - h^\alpha \tag{4.5}$$

with the property that

$$F^\alpha = \frac{\partial h'^\alpha}{\partial u'} \Leftrightarrow F^\alpha \cdot du' \equiv dh'^\alpha. \tag{4.6}$$

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[2] (4.1) is the usual definition of conservative from to which we refer in the text. Sometimes in the literature it is called "generalized conservative form", because of the source term on the r.h.s. which is responsible for dissipation (see Appendix).

In other words, the system (4.1), when compatible with a supplementary law has the structure such that $\boldsymbol{F}^\alpha$ is the gradient of h'^α with respect to the field $\boldsymbol{u}'$. It follows from (4.6) and (4.1) that in terms of the field $\boldsymbol{u}'$ the system assumes the form

$$\frac{\partial^2 h'^\alpha}{\partial \boldsymbol{u}' \, \partial \boldsymbol{u}'} \, \partial_\alpha \boldsymbol{u}' = \boldsymbol{f}, \tag{4.7}$$

in which the coefficient matrices as Hessian matrices, are symmetric. The field $\boldsymbol{u}'$ is unique (except for constant factors) and has the property that it transforms the original system (4.1) to a new form (4.7) in which the essential property of a conservation law is preserved. This is why we call the field $\boldsymbol{u}'$ the "main field" of the system, [42]. Its components are the multipliers through which Eq. (4.2) is obtained as a linear combination of the field equations (see (4.3), (4.4)); in the context of thermomechanics the components of $\boldsymbol{u}'$ can be identified with the Lagrange multipliers introduced by I. Shih Liu in [46]. The differential operator in (4.7) is known as soon as the h'^α are assigned as functions of $\boldsymbol{u}'$. These quantities (which behave much like potentials) are sufficiently important that we call then "generating functions" of the system.

We emphasize that the systems (4.1), together with their supplementary conservation law (4.2) are completely identified when $\boldsymbol{u}'$, h'^α and the source $\boldsymbol{f}$ are known. In fact, if these quantities are assigned, one can reach the system through (4.6), and from (4.5) and (4.4) the functions h^α and g, i.e. the supplementary conservation law (4.2) can be determined. Therefore, we shall call "generators" the set of $2N + 4$ quantities[3]:

$$\{\boldsymbol{u}', h'^\alpha, \boldsymbol{f}\}.$$

The previous statements hold for any system (4.1) that is compatible with a supplementary conservation law. Moreover, if the matrix $(\partial \boldsymbol{F}^0/\partial \boldsymbol{u})$ is not singular (so that it is possible to choose $\boldsymbol{F}^0$ as field $\boldsymbol{u}$) and h^0 is a *convex function* of $\boldsymbol{F}^0$, then it is possible to show that the matrix $\partial^2 h'^0/\partial \boldsymbol{u}' \, \partial \boldsymbol{u}'$ (coefficient matrix of the time derivative of $\boldsymbol{u}'$ in (4.7) is positive definite[4], and consequentely *the system (4.7) is hyperbolic.* In particular, this is a special case of a *symmetric system in the sense of Friedrichs,* (see Appendix).

Such systems exhibit peculiar properties concerning weak solutions and shock waves. It is remarkable that the physically significant systems examined so far possess the convexity property mentioned above and are therefore symmetric in the field $\boldsymbol{u}'$. The system of fluid dynamics is analysed in [47], [48], relativistic hydrodynamics and magnetohydrodynamics are treated in [42], [43]; Born-Infeld non-linear electrodynamics in [49] and hyperelastic continuum mechanics in [50].

Manifestly, FNS fluids possess the properties mentioned for a generic system (condition (4.6)), but they have a non-convex function $h^0(\boldsymbol{F}^0)$, since the FNS system is not hyperbolic (in this case $\boldsymbol{F}^0$ does not identify a basis in R^{14}).

It is easy to see that the FNS systems (1.1), (1.5) ($N = 14$) assumes the form

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[3] We point out that a generic conservative system is identified with the $6N$ quantities: $\boldsymbol{u}$, $\boldsymbol{F}^\alpha$, $\boldsymbol{f}$ are known.

[4] In fact, in this case from (4.3) $\boldsymbol{u}' = \partial h^0/\partial \boldsymbol{F}^0$ and from (4.5) $h'^0 = \boldsymbol{u}' \cdot \boldsymbol{F}^0 - h^0 = (\partial h^0/\partial \boldsymbol{F}^0) \cdot \boldsymbol{F}^0 - h^0$ and therefore h'^0 is the Legendre conjugate function of h^0 and then a convex function of $\boldsymbol{u}'$; therefore the matrix $\partial^2 h'^0/\partial \boldsymbol{u}' \, \partial \boldsymbol{u}'$ is positive definite.

(4.1) and the entropy law the form (4.2) when we choose:

$$\boldsymbol{F}^0 \equiv (\varrho, \varrho u_k, \varepsilon, 0_3, 0_5, 0_1)^T \tag{4.8}$$

$$\boldsymbol{F}^i \equiv \Big(\varrho u_i, \varrho u_i u_k + t_{ik}, \varepsilon u_i + q_i$$
$$+ t_{ij}u_j, \theta\delta_{ik}, \frac{1}{2}\left(u_j\delta_{ki} + u_k\delta_{ji}\right) - \frac{1}{3}\delta_{jk}u_i, u_i\Big)^T \tag{4.9}$$

$$\boldsymbol{f} \equiv (0_1, 0_3, 0_1, -kq_k, \alpha\sigma_{jk}^D, \beta \operatorname{tr} \sigma)^T \tag{4.10}$$

$$h^0 = -\varrho S; \qquad h^i = -(\varrho S u_i + q_i/\theta) \tag{4.11}$$

$$g = -\left(k\,\frac{q^2}{\theta^2} + \alpha\,\frac{\sigma^D : \sigma^D}{\theta} + \beta\,\frac{(\operatorname{tr}\sigma)^2}{3\theta}\right) \tag{4.12}$$

where $0_1, 0_3, 0_5$ denote respectively the scalar, vector and traceless symmetric matrix null elements.

In [45] the main field and the generating functions for the FNS system has been evaluated as:

$$\boldsymbol{u}' \equiv \frac{1}{\theta}\left(G - \frac{u^2}{2},\, u_k,\, -1,\, \frac{q_k}{\theta},\, -\sigma_{jk}^D,\, -\frac{\operatorname{tr}\sigma}{3}\right)^T \tag{4.13}$$

$$h'^0 = p/\theta; \qquad h'^i = \frac{1}{\theta}\left(t_{ik}u_k + q_i\right); \tag{4.14}$$

$$G = e + pV - \theta S \tag{4.15}$$

in which G is the chemical potential (free enthalpy).

5. The New Model, Exploited Via "Generators"

As we pointed out in the previous section the mathematical quantities that determine the FNS system are the main field $\boldsymbol{u}'$ supplied by (4.13), the generating functions h'^α given by (4.14) and the source $\boldsymbol{f}$ given by (4.10). Following Müller's approach, we may introduce relaxation terms by modifying some thermodynamic variables, e.g. $S_N(\varrho, e, \boldsymbol{q}, \sigma^D, \operatorname{tr}\sigma)$ different from the corresponding equilibrium quantities. The simplest idea is then to modify h'^α, $\boldsymbol{u}'$ and $\boldsymbol{f}'$ of the classical theory by substituting into them the new non-equilibrium variables. Since in this formulation only the absolute temperature and the chemical potential appear as thermodynamic variables and since it does not seem to be reasonable to redefine the former, the chemical potential alone (and with it the entropy density) will be changed.

In other words, we are looking for a system of type (4.1), if it exists, such that the new generating functions and the new main field are:

$$h_N'^\alpha = h'^\alpha \tag{5.1}$$

$$\boldsymbol{u}_N' \equiv \frac{1}{\theta}\left(G_N - \frac{u^2}{2},\, u_k,\, -1,\, \frac{q_k}{\theta},\, -\sigma_{jk}^D,\, -\frac{\operatorname{tr}\sigma}{3}\right)^T. \tag{5.2}$$

$G_N(\varrho, e, \boldsymbol{q}, \sigma^D, \operatorname{tr} \sigma)$ is a function yet to be determined, which we shall interpret as the non-equilibrium chemical potential.

Essentially, we assume that the non-equilibrium generating functions, the main field and the source, have the same form as the corresponding FNS quantities, with the only difference that the equilibrium chemical potential be replaced by G_N: criterion of invariance of the "generators".

As a consequence of the considerations on symmetric systems, the advantages of this simple criterion are the following: i) the new system is conservative; ii) a supplementary conservation equation (for the entropy) is garanted; iii) the entropy inequality is also satisfied; in fact the "entropy source" $s = -g$ does not change from the FNS theory since the last 9 components of $\boldsymbol{u}_N'$ and the source f are also unchanged (see (4.4), (4.10), (5.2)); iv) the new Gibbs relation emerges automatically, without new assumptions.

By differentiating h'^α with respect to $\boldsymbol{u}_N'$ we obtain $\boldsymbol{F}_N{}^\alpha$, i.e. the new conservative system, then through (4.5) we reach $h_N{}^\alpha$ and from (4.4) g_N, i.e. the new supplementary law and the new entropy density.

Even though the method is conceptually simple, calculations are unwieldy. We shall follow the most synthetical way. The index N will denote quantities in the new model, to be determined (e.g. $\boldsymbol{u}_N'$, $h_N'^\alpha$, etc. ...), while quantities without an index are those of the FNS theory.

Besides, $\varDelta \boldsymbol{u}' = \boldsymbol{u}_N' - \boldsymbol{u}'$ and so on.

From (4.6) and (5.1) it follows:

$$\boldsymbol{F}_N{}^\alpha \cdot d\boldsymbol{u}_N' = \boldsymbol{F}^\alpha \cdot d\boldsymbol{u}', \tag{5.3}$$

or

$$\boldsymbol{F}^\alpha \cdot d(\varDelta \boldsymbol{u}') + \varDelta \boldsymbol{F}^\alpha \cdot d\boldsymbol{u}_N' = 0 \quad \rightleftarrows \tag{5.4}$$

$$\boldsymbol{F}^0 \cdot d(\varDelta \boldsymbol{u}') + \varDelta \boldsymbol{F}^0 \cdot d\boldsymbol{u}_N' = 0, \tag{5.5}$$

$$\boldsymbol{F}^i \cdot d(\varDelta \boldsymbol{u}') + \varDelta \boldsymbol{F}^i \cdot d\boldsymbol{u}_N' = 0. \tag{5.6}$$

By contracting (5.5) with the components u_i of the velocity and subtracting the result from (5.6) we obtain

$$(\boldsymbol{F}^i - u_i \boldsymbol{F}^0) \cdot d(\varDelta \boldsymbol{u}') + (\varDelta \boldsymbol{F}^i - u_i \varDelta \boldsymbol{F}^0) \, d\boldsymbol{u}_N' = 0. \tag{5.7}$$

In our case the only non-vanishing component of $\varDelta \boldsymbol{u}'$ is the first (see (4.13), (5.2)), while the first component of $\boldsymbol{F}^i - u_i \boldsymbol{F}^0$ is zero (see (4.8), (4.9)); therefore, the first term in (5.7) vanishes. We point out that the components of $\boldsymbol{u}_N'$ are a basis of R^{14} (i.e. the components of $d\boldsymbol{u}_N'$ are linearly independent) and thus (5.7) gives

$$\boldsymbol{F}_N{}^i = \boldsymbol{F}^i + u_i(\boldsymbol{F}_N{}^0 - \boldsymbol{F}^0). \tag{5.8}$$

Condition (5.8) together with (5.5) resolves the problem of determining the differential system. In fact, through integration (5.5) supplies $\boldsymbol{F}_N{}^0$ and (5.8) provides $\boldsymbol{F}_N{}^i$.

Before entering the details of calculations let us investigate the supplementary law $\partial_\alpha h_N{}^\alpha = g_N$. From (4.5) and (5.1) we have

$$\boldsymbol{u}' \cdot \boldsymbol{F}^\alpha - h^\alpha = \boldsymbol{u}_N' \cdot \boldsymbol{F}_N{}^\alpha - h_N{}^\alpha, \tag{5.9}$$

which may also be written as

$$\Delta h^0 = \boldsymbol{u}' \cdot \Delta \boldsymbol{F}^0 + \Delta \boldsymbol{u}' \cdot \boldsymbol{F}_N{}^0, \tag{5.10}$$

$$\Delta h^i = \boldsymbol{u}' \cdot \Delta \boldsymbol{F}^i + \Delta \boldsymbol{u}' \cdot \boldsymbol{F}_N{}^i. \tag{5.11}$$

As before we may deduce

$$\Delta h^i - u_i \, \Delta h^0 = \boldsymbol{u}' \cdot (\Delta \boldsymbol{F}^i - u_i \, \Delta \boldsymbol{F}^0) + \Delta \boldsymbol{u}' \cdot (\boldsymbol{F}_N{}^i - u_i \boldsymbol{F}_N{}^0). \tag{5.12}$$

By taking into account (5.8) the r.h.s. of (5.12) is seen to be zero, and it follows that

$$h_N{}^i = h^i + u_i(h_N{}^0 - h^0), \tag{5.13}$$

which together with (5.10) determines the supplementary law.

Summarizing our results we have the following scheme:

$$\left.\begin{array}{l} \boldsymbol{F}^0 \cdot d(\Delta \boldsymbol{u}') + \Delta \boldsymbol{F}^0 \cdot d\boldsymbol{u}_N{}' = 0, \\[4pt] \boldsymbol{F}_N{}^i = \boldsymbol{F}^i + u_i(\boldsymbol{F}_N{}^0 - \boldsymbol{F}^0), \\[4pt] h_N{}^0 = h^0 + \boldsymbol{u}' \cdot \Delta \boldsymbol{F}^0 + \Delta \boldsymbol{u}' \cdot \boldsymbol{F}_N{}^0, \\[4pt] h_N{}^i = h^i + u_i(h_N{}^0 - h^0), \end{array}\right\} \tag{5.14}$$

which is solved as follows: the first differential equation gives $\boldsymbol{F}_N{}^0$, the second $\boldsymbol{F}_N{}^i$, the third $h_N{}^0$ and the fourth $h_N{}^i$. Since $\boldsymbol{f}$ is not modified, the source term of the supplementary law, from (4.4), is

$$g_N = \boldsymbol{u}_N{}' \cdot \boldsymbol{f}. \tag{5.15}$$

We start the detailed calculations by observing that $\boldsymbol{F}_N{}^0$ will be of the form

$$\boldsymbol{F}_N{}^0 \equiv (\varrho, \varrho u_k, \varepsilon, \varrho w_k, -\varrho T_{jk}^D, -3\varrho\mu)^T, \tag{5.16}$$

where $\boldsymbol{w}$ is a vector, $\boldsymbol{T}^D$ a symmetric traceless tensor and μ a scalar; they are the unknowns that must be determined.

By developing the first of (5.14), we find at once

$$d\mathscr{S} = \boldsymbol{w} \cdot d(\boldsymbol{q}/\theta^2) + \boldsymbol{T}^D : d(\sigma^D/\theta) + \mu d(\operatorname{tr}\sigma/\theta), \tag{5.17}$$

where we have introduced

$$\mathscr{S} = (G - G_N)/\theta. \tag{5.18}$$

Eq. (5.17) informs us that $\mathscr{S}$ can depend only on the variables

$$\boldsymbol{q}_* = \boldsymbol{q}/\theta^2; \qquad \sigma_*{}^D = \sigma^D/\theta; \qquad \sigma_* = (\operatorname{tr}\sigma)/\theta \tag{5.19}$$

and

$$\boldsymbol{w} = \frac{\partial \mathscr{S}}{\partial \boldsymbol{q}_*}; \qquad \boldsymbol{T}^D = \frac{\partial \mathscr{S}}{\partial \sigma_*{}^D}; \qquad \mu = \frac{\partial \mathscr{S}}{\partial \sigma_*}. \tag{5.20}$$

Therefore, knowledge of the unique constitutive function

$$\mathscr{S} \equiv \mathscr{S}(\boldsymbol{q}_*, \sigma_*{}^D, \sigma_*)$$

determines $\boldsymbol{F}_N{}^0$ and, through (5.18) the new chemical potential G_N

$$G_N(\varrho, e, \boldsymbol{q}, \sigma^D, \operatorname{tr}\sigma) = G(\varrho, e) - \theta\mathscr{S}\left(\frac{\boldsymbol{q}}{\theta^2}, \frac{\sigma^D}{\theta}, \frac{\operatorname{tr}\sigma}{\theta}\right). \tag{5.21}$$

The second of (5.14) yields immediately

$$\Delta F^i = \boldsymbol{F}_N{}^i - \boldsymbol{F}^i \equiv (0_3, 0_6, 0_3, \varrho u_i w_k, -\varrho u_i T_{jk}^D, -3\varrho\mu u_i)^T \tag{5.22}$$

while from the third (5.14) we have

$$h_N{}^0 = -\varrho S + \varrho(\boldsymbol{q}\cdot\boldsymbol{w})/\theta^2 + \varrho(\sigma^D : T^D)/\theta + \varrho(\mu\operatorname{tr}\sigma)/\theta - \varrho\mathscr{S}, \tag{5.23}$$

and since $h_N{}^0$ has to be interpreted as $-\varrho S_N$, the new entropy density becomes

$$S_N = S + \mathscr{S} - \boldsymbol{q}_* \cdot (\partial\mathscr{S}/\partial\boldsymbol{q}_*) - \sigma_*^D : (\partial\mathscr{S}/\partial\sigma_*{}^D) - \sigma_*(\partial\mathscr{S}/\partial\sigma_*). \tag{5.24}$$

Finally the last equation in (5.14) provides $h_N{}^i$:

$$h_N{}^i = -\varrho S_N u_i - q_i/\theta. \tag{5.25}$$

Summarizing the final results, the new system is formed by the same Eqs. (1.1) and the following new equations replacing the FNS constitutive relations[5]:

$$\left.\begin{aligned}
&\varrho\,\frac{d}{dt}\left(\frac{\partial\mathscr{S}}{\partial\boldsymbol{q}_*}\right) + \operatorname{grad}\theta = -k\boldsymbol{q}, \\[2mm]
&\varrho\,\frac{d}{dt}\left(\frac{\partial\mathscr{S}}{\partial\sigma_*{}^D}\right) - \frac{1}{2}\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}) + \frac{1}{3}\,\boldsymbol{I}\operatorname{div}\boldsymbol{u} = -\alpha\sigma^D, \\[2mm]
&\varrho\,\frac{d}{dt}\left(\frac{\partial\mathscr{S}}{\partial\sigma_*}\right) - \operatorname{div}\boldsymbol{u} = -\beta\operatorname{tr}\sigma
\end{aligned}\right\} \tag{5.26}$$

with the supplementary law (see also (5.15))

$$\varrho\,\frac{dS_N}{dt} + \operatorname{div}(\boldsymbol{q}/\theta) = k(q^2/\theta^2) + \alpha(\sigma^D : \sigma^D)/\theta + \beta(\operatorname{tr}\sigma)^2/(3\theta), \tag{5.27}$$

which automatically fulfills the entropy principle.

Furthermore we have the new entropy density given by (5.24) and the new chemical potential listed in (5.21).

From (5.24) and (5.21) we observe that:

$$G_N = e + pV - \theta S_N - \theta\{\boldsymbol{q}_* \cdot (\partial\mathscr{S}/\partial\boldsymbol{q}_*) - \sigma^D{}_* : (\partial\mathscr{S}/\partial\sigma_*{}^D) - \sigma_*(\partial\mathscr{S}/\partial\sigma_*)\}$$

and, therefore, (comparing with (4.15)) discover that apart from the new entropy density also the terms related to relaxation contribute to the new chemical potential.

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[5] We emphasize that the Lagrangian derivatives arise naturally in our system (5.26).

6. Convexity and Hyperbolicity of the System

Until now we have shown that the new system (Eqs. (1.1), (5.26)) is conservative, that the entropy principle is satisfied and that S_N and G_N may be determined from (5.24), (5.21), starting only from one constitutive function $\mathscr{S}$. The crucial point, i.e. the hyperbolicity of the system, must now be attacked. To prove hyperbolicity using the usual definition would be a considerable task; however, as a consequence of the construction of the system, it is enough to show that h_N^0 is a convex function of the field $\boldsymbol{F}_N^0$, since this condition ensure hyperbolicity (through the properties of symmetric systems). Thus, we investigate whether the quadratic form

$$Q = \delta^2 h_N^0 = \frac{\partial^2 h_N^0}{\partial \boldsymbol{F}_N^0 \, \partial \boldsymbol{F}_N^0} \, \delta \boldsymbol{F}_N^0 \cdot \delta \boldsymbol{F}_N^0$$

is positive for any non vanishing variation $\delta \boldsymbol{F}_N^0$. Because (4.3) implies for $\alpha = 0$ $\boldsymbol{u}_N' = \partial h_N^0 / \partial \boldsymbol{F}_N^0$, we may equivalently prove that

$$Q = \delta \boldsymbol{u}_N' \cdot \delta \boldsymbol{F}_N^0 > 0 .$$

Straightforward calculations yield for our system

$$Q = Q_f - \delta \mathscr{S} \delta \varrho + \delta \boldsymbol{q}_* \cdot \delta \left(\varrho \, \frac{\partial \mathscr{S}}{\partial \boldsymbol{q}_*} \right) + \delta \boldsymbol{\sigma}_*^D : \delta \left(\varrho \, \frac{\partial \mathscr{S}}{\partial \boldsymbol{\sigma}_*^D} \right) \tag{6.1}$$
$$+ \delta \sigma^* \delta \left(\varrho \, \frac{\partial \mathscr{S}}{\partial \sigma_*} \right) ,$$

where Q_f denotes the analogous quantity when it is evaluated for an inviscid fluid at thermal equilibrium

$$Q_f = \delta \left(\frac{G - u^2/2}{\theta} \right) \delta \varrho + \delta \left(\frac{\boldsymbol{u}}{\theta} \right) \cdot \delta(\varrho \boldsymbol{u}) - \delta \varepsilon \delta \left(\frac{1}{\theta} \right) . \tag{6.2}$$

It is known that Q_f is positive provided that the specific heat at constant pressure c_p is positive and $(\partial p / \partial \varrho)_s > 0$; equivalent is to say that the negative chemical potential at equilibrium is a convex function of p and θ (such conditions always hold at equilibrium and ensure hyperbolicity of the equations governing a perfect fluid). The proof is given in [48].

By developing (6.1) we obtain

$$Q = Q_f + \varrho \{ \delta \boldsymbol{q}_* \cdot \delta(\partial \mathscr{S} / \partial \boldsymbol{q}_*) + \delta \boldsymbol{\sigma}_*^D : \delta(\partial \mathscr{S} / \partial \boldsymbol{\sigma}_*^D) + \delta \sigma_* \delta(\partial \mathscr{S} / \partial \sigma_*) \} . \tag{6.3}$$

We conclude that Q is positive, if and only if $\mathscr{S}$ is a convex function of its arguments.

Therefore by chosing $\mathscr{S}$ as a convex function of $\boldsymbol{q}_*$, $\boldsymbol{\sigma}_*^D$, and σ_* one finds that h_N^0 is also a convex function of $\boldsymbol{F}_N^0$ and then obtains a symmetric-hyperbolic system.

A question arises what physical meaning the convexity condition for the function $\mathscr{S}$ might correspond to. We recall that the convexity of a generic function Ψ with respect to a vector field $\boldsymbol{X} \in \mathbb{R}^n$ is expressed also by the condition

$$\Psi(\boldsymbol{X}_0) - \Psi(\boldsymbol{X}) + (\boldsymbol{X} - \boldsymbol{X}_0) \cdot \frac{\partial \Psi}{\partial \boldsymbol{X}} > 0$$

for any $\boldsymbol{X}$, $\boldsymbol{X}_0$ ($\boldsymbol{X} \neq \boldsymbol{X}_0$) belonging to a convex subset of $\mathbb{R}^n$. Let us identify Ψ' with $\mathscr{S}$, $\boldsymbol{X}$ with $(\boldsymbol{q}_*, \sigma_*^D, \sigma_*) \in \mathbb{R}^9$ and $\boldsymbol{X}_0$ with the equilibrium state $[\boldsymbol{X}_0 \equiv (0_3, 0_5, 0_1)]$ in which $\mathscr{S}$ vanishes. Then the convexity condition becomes

$$\mathscr{S} - \boldsymbol{q}_* \cdot (\partial\mathscr{S}/\partial\boldsymbol{q}_*) - \sigma_*^D : (\partial\mathscr{S}/\partial\sigma_*^D) - \sigma_*(\partial\mathscr{S}/\partial\sigma_*) < 0.$$

From (5.24) the above condition is equivalent to the physically relevant result:

$$\boxed{S_N < S} \qquad \forall \; (\boldsymbol{q}_*, \sigma_*^D, \sigma_*) \neq (0_3, 0_5, 0_1). \tag{6.4}$$

Condition (6.4), which expresses the physical condition of thermodynamic stability (maximum entropy) at equilibrium (required also by Müller in [2]), is guaranteed by the hyperbolicity of our system.

This result seems to confirm that our method, which a priori may appear as rather formal, is based on a sound physical ground.

Even though the system we have reached is quite different from that proposed by Müller, it preserves essentially the main features required by that author: Existence of an entropy principle and the idea of modifying some thermodynamic variables in non-equilibrium.

Now we ask whether or not it is possible to reset also Müller's idea concerning nearness between non-equilibrium and equilibrium variables. In other words we look for a new criterion of nearness between G_N and G, S_N and S.

7. An Interesting Special Case

Our model is specified by the constitutive function $\mathscr{S}$, which depends on the variables $\boldsymbol{q}_*, \sigma_*^D, \sigma_*$. Such a function is a measure of the "distance" between our model and the classical FNS theory. In fact when $\mathscr{S}$ vanishes, the model becomes the FNS theory.

It is reasonable to assume that this function is not very large. If it is required to be convex, it seems interesting to examine as the special case where $\mathscr{S}$ is a quadratic function of the variables

$$\mathscr{S} = \frac{1}{2}\left(\tau_0 q_*{}^2 + \tau_1 \sigma_*^D : \sigma_*^D + \tau_2 \sigma_*{}^2\right), \tag{7.1}$$

in which τ_i are positive constants. The system (5.26) then becomes

$$\chi\left\{\tau_0\varrho\,\frac{d}{dt}\left(\boldsymbol{q}/\theta^2\right) + \operatorname{grad}\theta\right\} = -\boldsymbol{q},$$

$$2\eta\left\{\tau_1\varrho\,\frac{d}{dt}\left(\sigma^D/\theta\right) - \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right) + \frac{1}{3}\boldsymbol{I}\operatorname{div}\boldsymbol{u}\right\} = -\sigma^D, \tag{7.2}$$

$$3\zeta\left\{\tau_2\varrho\,\frac{d}{dt}\left(\operatorname{tr}\sigma/\theta\right) - \operatorname{div}\boldsymbol{u}\right\} = -\operatorname{tr}\sigma,$$

and (5.21), (5.24) becomes:

$$G_N = G - \theta\mathscr{S}; \qquad S_N - S = \mathscr{S}. \tag{7.3}$$

Gibbs' relation is modified into:

$$dS_N = dS - \tau_0 \frac{\boldsymbol{q}}{\theta^2} d\left(\frac{\boldsymbol{q}}{\theta^2}\right) - \tau_1 \frac{\boldsymbol{\sigma}^D}{\theta} : d\left(\frac{\boldsymbol{\sigma}^D}{\theta}\right)$$
$$- \tau_2 \frac{\operatorname{tr}\boldsymbol{\sigma}}{\theta} d\left(\frac{\operatorname{tr}\boldsymbol{\sigma}}{\theta}\right). \tag{7.4}$$

We point out that in (7.4) the measure of "nearness" is quantitatively·defined by the squares of $\boldsymbol{q}/\theta^2$, $\boldsymbol{\sigma}^D/\theta$, and $\operatorname{tr}\boldsymbol{\sigma}/\theta$, instead of the squares of $\boldsymbol{q}$, $\boldsymbol{\sigma}^D$, $\operatorname{tr}\boldsymbol{\sigma}$ as it happens in Müller's theory. Our measure involves also the "coldness" $1/\theta$.

8. Conclusion

Our model, based on the idea of preserving the symmetry structure of the classical FNS theory possesses also the main properties of "extended irreversible thermodynamics": 1) modified thermodynamic variables at non-equilibrium; 2) entropy inequality; 3) "nearness" of equilibrium and non-equilibrium (in the particular case of sect. 7). Moreover, it has the advantage offered by a conservative symmetric hyperbolic system:

i) existence and uniqueness of the solution in a neighbourhood of the initial time, with the same regularity of the initial data, when they belong to a Sobolev space H^s with $s \geqq 4$ [51].

ii) reality and finiteness of the speeds of propagation of the discontinuity waves;

iii) compatibility with shock-wave propagation, boundedness of the shock speed and entropy growth across the shock [41], [42], [52].

Furthermore, the model requires modification of the non-equilibrium entropy and chemical potential only, and mantains a unique thermodynamic absolute temperature, which, through the coldness, plays a fundamental role for relaxation.

Appendix

Some Definitions Relevant to Quasi-Linear Systems

As a matter of convenience, we recall here some definitions for quasi-linear systems.

We consider a quasi-linear system of first order:

$$\boldsymbol{A}^\alpha(\boldsymbol{u})\, \partial_\alpha \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}) \tag{A1}$$

for the unknown N-vector $\boldsymbol{u} \equiv \boldsymbol{u}(x^\alpha)$; $\boldsymbol{A}^\alpha$ are real matrices $N \times N$ ($\alpha = 0, 1, 2, 3$; $i = 1, 2, 3$).

Definition I — The system (A1) is said to be conservative if $\boldsymbol{A}^\alpha = \partial\boldsymbol{F}^\alpha/\partial\boldsymbol{u}$, i.e. exhibits the form (4.1) $\partial_\alpha \boldsymbol{F}^\alpha(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u})$.

Definition II — The system (A1) is said to be hyperbolic in the direction-t if:

i) $\det \boldsymbol{A}^0 \neq 0$,

ii) $\forall \; n \equiv (n_i) : \|n\| = 1$, the following eigenvalue problem

$$(A^i n_i - \lambda A^0) \, r = 0 \tag{A2}$$

has only real proper values λ and N linearly independent eigenvectors r.

Remark I — n and $\lambda \equiv \lambda(u, n)$ in the theory of wave propagation have the physical meaning of unit normal and normal velocity to the wave front respectively.

Definition III — A system of the type (A1) is said to be symmetric (see i.e. [32]) if:

i) $A^\alpha = (A^\alpha)^T$,

ii) A^0 is positive definite.

Remark II — As well known from linear algebra, any symmetric system is also hyperbolic (see Def. II).

Acknowledgment

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Prof. Dr. T. Ruggeri
Istituto di Matematica Applicata
Università di Bologna
Via Vallescura, 2
I - 40136 Bologna
Italy