Example 4.5. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t). \tag{4.23}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{4.24}$$

(i.e., $\mathcal{H}$ corresponds to an ideal integrator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t). \qquad (1)$$

Substituting (4.23) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{from (1)} \\
&= x * u(t) && \text{substitute given function } h \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau && \text{definition of convolution} \\
&= \int_{-\infty}^{t} x(\tau)\underbrace{u(t-\tau)}_{1}d\tau + \int_{t^+}^{\infty} x(\tau)\underbrace{u(t-\tau)}_{0}d\tau && \text{split into two integrals} \\
&= \int_{-\infty}^{t} x(\tau)d\tau. && \text{second integral is 0}
\end{aligned}$$

Therefore, the system with the impulse response h given by (4.23) is, in fact, the ideal integrator given by (4.24). ■