Example 6.11 (Time scaling property of the Fourier transform)**.** Using the Fourier transform pair

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right), \quad (1)$$

find the Fourier transform X of the function

$$x(t) = \operatorname{rect}(at),$$

where a is a nonzero real constant.

Solution. Let $v(t) = \operatorname{rect} t$ (2) so that $x(t) = v(at)$ (3). Also, let $V = \mathcal{F}v$. From the given transform pair, we know that

$$V(\omega) = (\mathcal{F}\{\operatorname{rect} t\})(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right). \quad \text{(6.9)}$$

← from FT of (2) using FT pair (1)

From the definition of v and the time-scaling property of the Fourier transform, we have

(4) → $$X(\omega) = \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

← from FT of (3) using time scaling property

Substituting the expression for V in (6.9) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

substituting (6.9) into (4)

Thus, we have shown that

$$\operatorname{rect}(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right). \qquad \blacksquare$$