Example 7.37 (System function to differential equation)**.** A LTI system with input x and output y has the system function

$$H(s) = \frac{s}{s+R/L},$$

where L and R are positive real constants. Find the differential equation that characterizes this system.

Solution. Let X and Y denote the Laplace transforms of x and y, respectively. To begin, we have

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{s}{s+R/L}\right)X(s). \end{aligned}$$

system is LTI

substitute given H

multiply both sides by (s+R/L)

Rearranging this equation, we obtain

$$\begin{aligned} &(s+\tfrac{R}{L})Y(s) = sX(s) \\ \Rightarrow\quad &sY(s)+\tfrac{R}{L}Y(s) = sX(s). \end{aligned}$$

simplify

take inverse LT (and use linearity)

Taking the inverse Laplace transform of both sides of this equation (by using the linearity and time-differentiation properties of the Laplace transform), we have

$$\begin{aligned} &\mathcal{L}^{-1}\{sY(s)\}(t)+\tfrac{R}{L}\mathcal{L}^{-1}Y(t) = \mathcal{L}^{-1}\{sX(s)\}(t) \\ \Rightarrow\quad &\tfrac{d}{dt}y(t)+\tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t). \end{aligned}$$

time-domain differentiation property

Therefore, the system is characterized by the differential equation

$$\tfrac{d}{dt}y(t)+\tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t).$$

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