Example B.2 (Repeated pole)**.** Find the partial fraction expansion of the function

$$f(z) = \frac{4z+8}{(z+1)^2(z+3)}.$$

strictly proper with 2nd order pole at −1 and 1st order pole at −3

Solution. Since f has a repeated pole, we know that f has a partial fraction expansion of the form

terms contributed by pole at −1

$$f(z) = \frac{A_{1,1}}{z+1} + \frac{A_{1,2}}{(z+1)^2} + \frac{A_{2,1}}{z+3}. \quad (1)$$

term contributed by pole at −3

where $A_{1,1}$, $A_{1,2}$, and $A_{2,1}$ are constants to be determined. To calculate these constants, we proceed as follows:

coefficient number

pole order

$$\begin{aligned}
A_{1,1} &= \frac{1}{(2-1)!}\left[\left(\tfrac{d}{dz}\right)^{2-1}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \frac{1}{1!}\left[\tfrac{d}{dz}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \left[\tfrac{d}{dz}\left(\frac{4z+8}{z+3}\right)\right]\Big|_{z=-1} \\
&= \left[4(z+3)^{-1} + (-1)(z+3)^{-2}(4z+8)\right]\Big|_{z=-1} \\
&= \left[\frac{4}{(z+3)^2}\right]\Big|_{z=-1} \\
&= \tfrac{4}{4} \\
&= 1,
\end{aligned}$$

formula for case of repeated pole

substitute for f

differentiate

$$\begin{aligned}
A_{1,2} &= \frac{1}{(2-2)!}\left[\left(\tfrac{d}{dz}\right)^{2-2}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \frac{1}{0!}\left[(z+1)^2 f(z)\right]\Big|_{z=-1} \\
&= \left[\frac{4z+8}{z+3}\right]\Big|_{z=-1} \\
&= \tfrac{4}{2} \\
&= 2, \quad \text{and}
\end{aligned}$$

formula for case of repeated pole

$$\begin{aligned}
A_{2,1} &= (z+3)f(z)\big|_{z=-3} \\
&= \frac{4z+8}{(z+1)^2}\Big|_{z=-3} \\
&= \tfrac{-4}{4} \\
&= -1.
\end{aligned}$$

formula for case of simple pole

substitute for f

Thus, the partial fraction expansion of f is given by

$$f(z) = \frac{1}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{z+3}. \quad \blacksquare$$

substitute computed coefficients into (1)