Example 5.1 (Fourier series of a periodic square wave)**.** Find the Fourier series representation of the periodic square wave x shown in Figure 5.1.

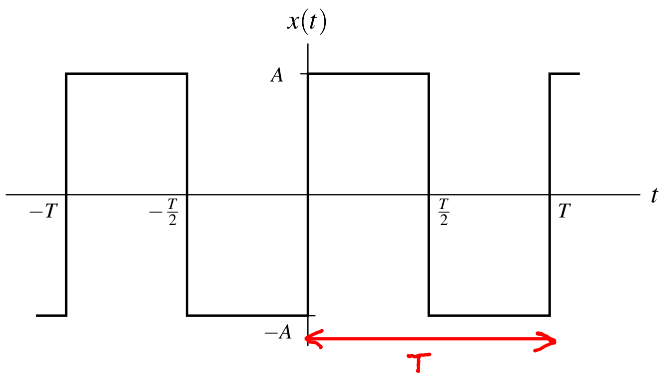

Figure 5.1: Periodic square wave.

Solution. Let us consider the single period of $x(t)$ for $0 \leq t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & 0 \leq t < \frac{T}{2} \\ -A & \frac{T}{2} \leq t < T. \end{cases}$$

Let $\omega_0 = \frac{2\pi}{T}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\ &= \tfrac{1}{T}\left(\int_0^{T/2} Ae^{-jk\omega_0 t}dt + \int_{T/2}^{T}(-A)e^{-jk\omega_0 t}dt\right) \\ &= \begin{cases} \dfrac{1}{T}\left(\left[\dfrac{-A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Bigg|_0^{T/2} + \left[\dfrac{A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Bigg|_{T/2}^{T}\right) & k \neq 0 \\ \dfrac{1}{T}\left([At]|_0^{T/2} + [-At]|_{T/2}^{T}\right) & k = 0. \end{cases} \end{aligned}$$

Fourier series analysis equation

split into 2 integrals and substitute given x

integrate

1

Now, we simplify the expression for c_k for each of the cases $k \neq 0$ and $k = 0$ in turn. First, suppose that $k \neq 0$. We have

$$\begin{aligned} c_k &= \frac{1}{T}\left(\left[\frac{-A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Bigg|_0^{T/2} + \left[\frac{A}{jk\omega_0}e^{-jk\omega_0 t}\right]\Bigg|_{T/2}^{T}\right) \\ &= \frac{-A}{j2\pi k}\left(\left[e^{-jk\omega_0 t}\right]\Big|_0^{T/2} - \left[e^{-jk\omega_0 t}\right]\Big|_{T/2}^{T}\right) \\ &= \frac{jA}{2\pi k}\left(\left[e^{-j\pi k} - 1\right] - \left[e^{-j2\pi k} - e^{-j\pi k}\right]\right) \\ &= \frac{jA}{2\pi k}\left[2e^{-j\pi k} - e^{-j2\pi k} - 1\right] \\ &= \frac{jA}{2\pi k}\left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1\right]. \end{aligned}$$

from 1 above

factor out constant and $T\omega_0 = 2\pi$

simplify

②

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

from ②

$$
\begin{aligned}
c_k &= \frac{jA}{2\pi k}[2(-1)^k - 1^k - 1] \\
&= \frac{jA}{2\pi k}[2(-1)^k - 2] \\
&= \frac{jA}{\pi k}[(-1)^k - 1] \\
&= \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0. \end{cases}
\end{aligned}
$$

simplify

$(-1)^k - 1 = \begin{cases} -2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$

Now, suppose that $k = 0$. We have

from ① above

$$
\begin{aligned}
c_0 &= \tfrac{1}{T}\left([At]|_0^{T/2} + [-At]|_{T/2}^{T} \right) \\
&= \tfrac{1}{T}\left[\tfrac{AT}{2} - \tfrac{AT}{2}\right] \\
&= 0.
\end{aligned}
$$

simplify

Thus, the Fourier series of x is given by

$$
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(2\pi/T)kt},
$$

where

$$
c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}. \end{cases}
$$

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