Example 7.43. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t),$$

where the prime symbol denotes derivative. If $x(t) = 5u(t)$, $y(0^-) = 1$, and $y'(0^-) = -1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
& \mathcal{L}_u\left\{y'' + 3y' + 2y\right\}(s) = \mathcal{L}_u x(s) && \text{linearity} \\
\Rightarrow\ & \mathcal{L}_u\left\{y''\right\}(s) + 3\mathcal{L}_u\left\{y'\right\}(s) + 2\mathcal{L}_u y(s) = \mathcal{L}_u x(s) && \text{take ULT} \\
\Rightarrow\ & \left[s^2Y(s) - sy(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = X(s) && \text{multiply} \\
\Rightarrow\ & s^2Y(s) - sy(0^-) - y'(0^-) + 3sY(s) - 3y(0^-) + 2Y(s) = X(s) && \text{move terms not containing Y to right-hand side and factor out Y on left-hand side} \\
\Rightarrow\ & \left[s^2 + 3s + 2\right]Y(s) = X(s) + sy(0^-) + y'(0^-) + 3y(0^-) && \text{divide both sides by } s^2+3s+2 \\
\Rightarrow\ & Y(s) = \frac{X(s) + sy(0^-) + y'(0^-) + 3y(0^-)}{s^2 + 3s + 2}. \quad (1)
\end{aligned}$$

Since $x(t) = 5u(t)$ (3), we have

take ULT of (3)   ULT table

$$X(s) = \mathcal{L}_u\{5u(t)\}(s) = \tfrac{5}{s}. \quad (2)$$

Substituting this expression for X and the given initial conditions into the above equation yields

$$Y(s) = \frac{\left(\frac{5}{s}\right) + s - 1 + 3}{s^2 + 3s + 2} = \frac{s^2 + 2s + 5}{s(s+1)(s+2)}.$$

substituting (2) into (1)

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= sY(s)\big|_{s=0} && \text{from formula for simple pole case} \\
&= \left.\frac{s^2 + 2s + 5}{(s+1)(s+2)}\right|_{s=0} \\
&= \tfrac{5}{2}, \\
A_2 &= (s+1)Y(s)\big|_{s=-1} && \text{from formula for simple pole case} \\
&= \left.\frac{s^2 + 2s + 5}{s(s+2)}\right|_{s=-1} \\
&= -4, \quad \text{and} \\
A_3 &= (s+2)Y(s)\big|_{s=-2} && \text{from formula for simple pole case} \\
&= \left.\frac{s^2 + 2s + 5}{s(s+1)}\right|_{s=-2} \\
&= \tfrac{5}{2}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(s) = \frac{5/2}{s} - \frac{4}{s+1} + \frac{5/2}{s+2}.$$

recall:

$$Y(s) = \frac{5}{2}\left(\frac{1}{s}\right) - 4\left(\frac{1}{s+1}\right) + \frac{5}{2}\left(\frac{1}{s+2}\right)$$

Taking the inverse unilateral Laplace transform of Y yields

taking inverse ULT

$$\begin{aligned} y(t) &= \mathcal{L}_u^{-1}Y(t) \\ &= \tfrac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s}\right\}(t) - 4\mathcal{L}_u^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s+2}\right\}(t) \\ &= \tfrac{5}{2} - 4e^{-t} + \tfrac{5}{2}e^{-2t} \quad \text{for } t \geq 0. \end{aligned}$$

linearity

from ULT table

$$1 \xleftrightarrow{\text{ULT}} \frac{1}{s} \; ; \; e^{-at} \xleftrightarrow{\text{ULT}} \frac{1}{s+a}$$

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