Example 6.18 (Time-domain integration property of the Fourier transform)**.** Use the time-domain integration property of the Fourier transform in order to find the Fourier transform X of the function $x = u$.

Solution. We begin by observing that x can be expressed in terms of an integral as

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau. \qquad 1$$

Now, we consider the Fourier transform of x. We have

from 1

$$X(\omega) = \left(\mathcal{F}\left\{\int_{-\infty}^{t} \delta(\tau)\,d\tau\right\}\right)(\omega).$$

time-domain integration property

From the time-domain integration property, we can write

$$X(\omega) = \tfrac{1}{j\omega}\mathcal{F}\delta(\omega) + \pi\mathcal{F}\delta(0)\delta(\omega).$$

Evaluating the two Fourier transforms on the right-hand side using Table 6.2, we obtain

$F\delta(\omega) = 1$

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \tfrac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

drop 1's

Thus, we have shown that $u(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$. ■