Suppose that we have a LTI system $\mathcal{H}$ with input x, output y, impulse response h, and system function H. Suppose now that we can express some arbitrary input signal x as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}. \qquad 1$$

(As it turns out, many functions can be expressed in this way.) From the eigenfunction properties of LTI systems, the response of the system to the input $a_k e^{s_k t}$ is $a_k H(s_k) e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathcal{H}x(t) && \text{substitute 1 for x} \\ &= \mathcal{H}\left\{\sum_k a_k e^{s_k t}\right\}(t) && \text{linearity of } \mathcal{H} \\ &= \sum_k a_k \mathcal{H}\{e^{s_k t}\}(t) && \text{complex exponentials are} \\ &= \sum_k a_k H(s_k) e^{s_k t}. && \text{eigenfunctions of LTI systems} \end{aligned}$$

Thus, we have that

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{4.48}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. Furthermore, observe that the relationship between the input $x(t) = \sum_k a_k e^{s_k t}$ and output y in (4.48) does not involve convolution (such as in the equation $y = x * h$). In fact, the formula for y is identical to that for x except for the insertion of a constant multiplicative factor $H(s_k)$. In effect, we have used eigenfunctions to replace convolution with the much simpler operation of multiplication by a constant.