Example 6.15 (Frequency-domain convolution property)**.** Let x and y be functions related as

$$y(t) = x(t)\cos(\omega_c t),$$

where ω_c is a nonzero real constant. Let $Y = \mathcal{F}y$ and $X = \mathcal{F}x$. Find an expression for Y in terms of X.

Solution. To allow for simpler notation in what follows, we define

$$v(t) = \cos(\omega_c t) \quad 1$$

table of FT pairs

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$2 \quad V(\omega) = \pi[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)]. \leftarrow \text{from table of FT pairs}$$

From the definition of v, we have

$$3 \quad y(t) = x(t)v(t). \leftarrow \text{since } y(t) = x(t)\underbrace{\cos(\omega_c t)}_{v(t)}$$

Taking the Fourier transform of both sides of this equation, we have

$$Y(\omega) = \mathcal{F}\{x(t)v(t)\}(\omega).$$

taking FT of both sides of 3

Using the frequency-domain convolution property of the Fourier transform, we obtain

frequency-domain convolution property

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi} X * V(\omega) \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda. \end{aligned}$$

definition of convolution

Substituting the above expression for V, we obtain

substitute V from 2

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)])d\lambda \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)]d\lambda \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\underline{\omega-\lambda}-\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\underline{\omega-\lambda}+\omega_c)d\lambda\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-\omega_c)d\lambda\right] \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-\omega_c)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+\omega_c)]d\lambda\right] \\ &= \tfrac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)] \\ &= \tfrac{1}{2}X(\omega-\omega_c)+\tfrac{1}{2}X(\omega+\omega_c). \end{aligned}$$

cancel π's

split into two integrals

δ is even

regroup

sifting property

expand

■