Example 4.12. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(t) = A\delta(t - t_0),$$

where A and t_0 are real constants and $A \neq 0$. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\mathsf{inv}} = \delta. \tag{4.34}$$

$\mathcal{H}$ is invertible if and only if a solution for h_{inv} exists

So, let us attempt to solve this equation for h_{inv}. Substituting the given function h into (4.34) and using straightforward algebraic manipulation, we can write

$$\begin{aligned}
& h * h_{\mathsf{inv}}(t) = \delta(t) \\
\Rightarrow\quad & \int_{-\infty}^{\infty} h(\tau)h_{\mathsf{inv}}(t-\tau)d\tau = \delta(t) \\
\Rightarrow\quad & \int_{-\infty}^{\infty} A\delta(\tau - t_0)h_{\mathsf{inv}}(t-\tau)d\tau = \delta(t) \\
\Rightarrow\quad & \int_{-\infty}^{\infty} \delta(\tau - t_0)h_{\mathsf{inv}}(t-\tau)d\tau = \tfrac{1}{A}\delta(t).
\end{aligned}$$

definition of convolution

substitute given function h

divide both sides by $A \neq 0$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$h_{inv}(t-\tau)|_{\tau=t_0} = \frac{1}{A}\delta(t)$ sifting property

$$h_{\mathsf{inv}}(t - t_0) = \tfrac{1}{A}\delta(t). \tag{4.35}$$

Substituting $t + t_0$ for t in the preceding equation yields

$$\begin{aligned}
h_{\mathsf{inv}}([t + t_0] - t_0) &= \tfrac{1}{A}\delta(t + t_0) \quad\Leftrightarrow \\
h_{\mathsf{inv}}(t) &= \tfrac{1}{A}\delta(t + t_0).
\end{aligned}$$

impulse response of inverse system

Since $A \neq 0$, the function h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■