Theorem 4.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. Using the fact that δ is the convolutional identity, we can write

convolutional identity

$$\mathcal{H}x(t) = \mathcal{H}\{x * \delta\}(t).$$

Rewriting the convolution in terms of an integral, we have

rewrite convolution as integral

$$\mathcal{H}x(t) = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(\cdot - \tau)d\tau\right\}(t).$$

Since $\mathcal{H}$ is a linear operator, we can pull the integral and $x(\tau)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

interchange $\mathcal{H}$ with both $x(\tau)$ and integral (linearity)

$$\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(\cdot - \tau)\}(t)d\tau.$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by τ (i.e., $\underbrace{\mathcal{H}\{\delta(\cdot - \tau)\}}_{\text{shift by } \tau \text{ then } \mathcal{H}} = \underbrace{H\delta(\cdot - \tau)}_{\mathcal{H} \text{ then shift by } \tau}$) and then use the fact that $h = \mathcal{H}\delta$ to obtain

interchange $\mathcal{H}$ and time shift (time invariance)

$$\begin{aligned}
\mathcal{H}x(t) &= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\delta(t - \tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\
&= x * h(t).
\end{aligned}$$

$h = \mathcal{H}\delta$ (by definition)

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ■