Example 7.38 (Simple RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.24 with input v_1 and output v_2. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function H of this system. (b) Determine whether the system is BIBO stable. (c) Determine the step response of the system.

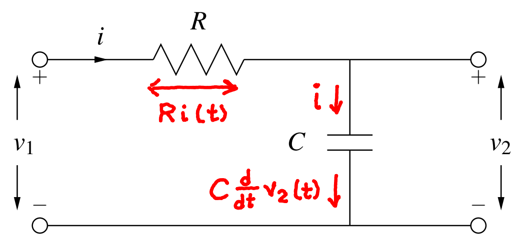


Figure 7.24: Simple RC network.

Solution. (a) From basic circuit analysis, we have

$$\left.\begin{aligned} v_1(t) &= Ri(t) + v_2(t) \quad \text{and} && (7.14a)\\ i(t) &= C\tfrac{d}{dt}v_2(t). && (7.14b) \end{aligned}\right\}$$

taking LT

Taking the Laplace transform of (7.14) yields

$$\left.\begin{aligned} V_1(s) &= RI(s) + V_2(s) \quad \text{and} && (7.15a)\\ I(s) &= CsV_2(s). && (7.15b) \end{aligned}\right\}$$

substitute (7.15b) into (7.15a) to eliminate I

Substituting (7.15b) into (7.15a) and rearranging, we obtain

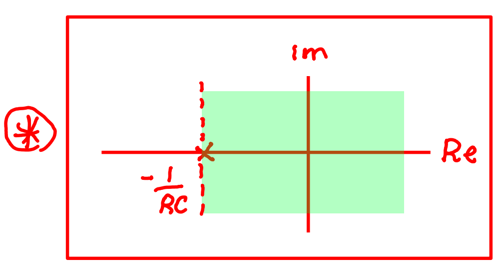
$$\begin{aligned} & V_1(s) = R[CsV_2(s)] + V_2(s) \\ \Rightarrow\quad & V_1(s) = RCsV_2(s) + V_2(s) \\ \Rightarrow\quad & V_1(s) = [1+RCs]V_2(s) \\ \Rightarrow\quad & \frac{V_2(s)}{V_1(s)} = \frac{1}{1+RCs}. \end{aligned}$$

factor

rearrange

Thus, we have that the system function H is given by

$V_2(s) = V_1(s)H(s) \Longrightarrow$
$H(s) = \dfrac{V_2(s)}{V_1(s)}$



$$\begin{aligned} H(s) &= \frac{1}{1+RCs} \\ &= \frac{\frac{1}{RC}}{s+\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{s-(-\frac{1}{RC})}. \end{aligned}$$

divide numerator and denominator by RC

explicitly show poles

Since the system can be physically realized, it must be causal. Therefore, the ROC of H must be a right-half plane. Thus, we may infer that the ROC of H is $\operatorname{Re}(s) > -\frac{1}{RC}$. So, we have

see (*)

$$H(s) = \frac{1}{1+RCs} \quad \text{for } \operatorname{Re}(s) > -\tfrac{1}{RC}.$$

(b) Since resistance and capacitance are (strictly) positive quantities, $R > 0$ and $C > 0$. Thus, $-\frac{1}{RC} < 0$. Consequently, the ROC contains the imaginary axis and the system is stable.

(c) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

Since system is LTI

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left(\frac{1}{1+RCs}\right)V_1(s). \end{aligned}$$

substitute for H

To compute the step response, we need to consider an input equal to the unit-step function. So, $v_1 = u$, implying that $V_1(s) = \frac{1}{s}$. Substituting this expression for V_1 into the above expression for V_2, we have

$v_1(t) = u(t) \overset{LT}{\longleftrightarrow} V_1(s) = \frac{1}{s}$

$$\begin{aligned} V_2(s) &= \left(\frac{1}{1+RCs}\right)\left(\frac{1}{s}\right) \\ &= \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}. \end{aligned}$$

divide numerator and denominator by RC

Now, we need to compute the inverse Laplace transform of V_2 in order to determine v_2. To simplify this task, we find the partial fraction expansion for V_2. We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= sV_2(s)\big|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s+\tfrac{1}{RC})V_2(s)\big|_{s=-\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{-\frac{1}{RC}} \\ &= -1. \end{aligned}$$

Thus, we have that V_2 has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

$$v_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}(t).$$

LT table

RHP

Using Table 7.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

inverse LT

$$\begin{aligned} v_2(t) &= u(t) - e^{-t/(RC)}u(t) \\ &= \left(1 - e^{-t/(RC)}\right)u(t). \end{aligned}$$

■

$u(t) \overset{LT}{\longleftrightarrow} \frac{1}{s}$ for $Re(s) > 0$

$e^{-at}u(t) \overset{LT}{\longleftrightarrow} \frac{1}{s+a}$ for $Re(s) > -a$