Example 7.25. Using a Laplace transform table and properties of the Laplace transform, find the Laplace transform X of the function x shown in Figure 7.13.

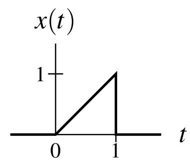

Figure 7.13: Function for the Laplace transform example.

Second solution (which incurs less work by avoiding differentiation). First, we express x using unit-step functions to yield

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

add and subtract $u(t-1)$

To simplify the subsequent Laplace transform calculation, we choose to rewrite x as

$$\begin{aligned} x(t) &= tu(t) - tu(t-1) + u(t-1) - u(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1). \end{aligned}$$

group two middle terms together

taking LT

(This is motivated by a preference to compute the Laplace transform of $(t-1)u(t-1)$ instead of $tu(t-1)$.) Taking the Laplace transform of both sides of the preceding equation, we obtain

$tu(t)$ time shifted by 1

$u(t)$ time shifted by 1 and then multiplied by t (requires differentiation)

$$X(s) = \underbrace{\mathcal{L}\{tu(t)\}(s)}_{①} - \underbrace{\mathcal{L}\{(t-1)u(t-1)\}(s)}_{②} - \underbrace{\mathcal{L}\{u(t-1)\}(s)}_{③}. \quad (*)$$

We have

$$① \quad \mathcal{L}\{tu(t)\}(s) = \frac{1}{s^2}, \quad \leftarrow \text{from LT table}$$

$$\begin{aligned} ② \quad \mathcal{L}\{(t-1)u(t-1)\}(s) &\overset{\text{time shifting}}{=} e^{-s}\mathcal{L}\{tu(t)\}(s) \\ &\overset{\text{LT table}}{=} e^{-s}\left(\frac{1}{s^2}\right) \\ &\overset{\text{multiply}}{=} \frac{e^{-s}}{s^2}, \quad \text{and} \end{aligned}$$

$$\begin{aligned} ③ \quad \mathcal{L}\{u(t-1)\}(s) &\overset{\text{time shifting}}{=} e^{-s}\mathcal{L}\{u(t)\}(s) \\ &\overset{\text{LT table}}{=} e^{-s}\left(\frac{1}{s}\right) \\ &\overset{\text{multiply}}{=} \frac{e^{-s}}{s}. \end{aligned}$$

Combining the above results, we have

substituting ①, ②, and ③ into (*)

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ &= \frac{1 - e^{-s} - se^{-s}}{s^2}. \end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane. ■