Example 4.16. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(t) = \delta(t-1).$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(t) = e^t \cos(\pi t).$$

Solution. (a) We find the system function H using (4.49). Substituting the given function h into (4.49), we obtain

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t) e^{-st} dt && \text{(4.49)} \\ &= \int_{-\infty}^{\infty} \delta(t-1) e^{-st} dt && \text{substitute given } h \\ &= \left[e^{-st}\right]\Big|_{t=1} && \text{sifting property} \\ &= e^{-s}. \end{aligned}$$

(b) We can rewrite x to obtain

$$\begin{aligned} x(t) &= e^t \cos(\pi t) && \text{Euler} \\ &= e^t \left[\tfrac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right] && \text{exponent rules} \\ &= \tfrac{1}{2} e^{(1+j\pi)t} + \tfrac{1}{2} e^{(1-j\pi)t}. \end{aligned}$$

So, the input x is now expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t},$$

where

$$a_k = \tfrac{1}{2} \quad \text{and} \quad s_k = \begin{cases} 1 + j\pi & k = 0 \\ 1 - j\pi & k = 1. \end{cases}$$

Now, we use H and the eigenfunction properties of LTI systems to find y. Calculating y, we have

$$\mathcal{H}\{a_k e^{s_k t}\}(t) = a_k H(s_k) e^{s_k t}$$

$$\begin{aligned} y(t) &= \sum_{k=0}^{1} a_k H(s_k) e^{s_k t} && \text{expand summation} \\ &= a_0 H(s_0) e^{s_0 t} + a_1 H(s_1) e^{s_1 t} && \text{substitute for } a_k, s_k \\ &= \tfrac{1}{2} H(1+j\pi) e^{(1+j\pi)t} + \tfrac{1}{2} H(1-j\pi) e^{(1-j\pi)t} && \text{evaluate } H(\ldots) \\ &= \tfrac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)t} + \tfrac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)t} \\ &= \tfrac{1}{2} e^{t-1+j\pi t-j\pi} + \tfrac{1}{2} e^{t-1-j\pi t+j\pi} \\ &= \tfrac{1}{2} e^{t-1} e^{j\pi(t-1)} + \tfrac{1}{2} e^{t-1} e^{-j\pi(t-1)} && \text{rearrange} \\ &= e^{t-1}\left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] && \text{Euler} \\ &= e^{t-1} \cos[\pi(t-1)]. \end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(s) = e^{-s}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

NOTE: THIS SOLUTION DID NOT REQUIRE THE COMPUTATION OF A CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!

