**Example 6.30** (Frequency spectrum of a time-shifted signum function)**.** The function

$$x(t) = \operatorname{sgn}(t-1)$$

has the Fourier transform

$$X(\omega) = \tfrac{2}{j\omega}e^{-j\omega}. \qquad (1)$$

(a) Find and plot the magnitude and phase spectra of $x$. (b) Determine at what frequency (or frequencies) $x$ has the most information.

*Solution.* (a) First, we find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

take magnitude of both sides of (1)

$$\begin{aligned} |X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| && |ab| = |a||b| \\ &= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| && |e^{j\theta}| = 1 \\ &= \left|\tfrac{2}{j\omega}\right| && \left|\tfrac{a}{b}\right| = \tfrac{|a|}{|b|} \\ &= \tfrac{2}{|\omega|}. \end{aligned}$$

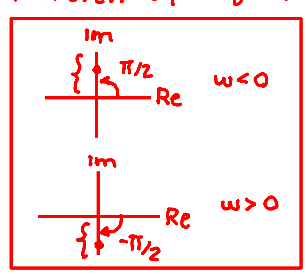
Next, we find the phase spectrum $\arg X(\omega)$. First, we observe that $\arg X(\omega)$ is not well defined if $\omega = 0$. So, we assume that $\omega \neq 0$. From the expression for $X(\omega)$, we can write (for $\omega \neq 0$)

take argument of both sides of (1)

$$\begin{aligned} \arg X(\omega) &= \arg\left\{\tfrac{2}{j\omega}e^{-j\omega}\right\} && \arg(ab) = \arg a + \arg b \\ &= \arg e^{-j\omega} + \arg\tfrac{2}{j\omega} && \arg(e^{j\theta}) = \theta \\ &= -\omega + \arg\tfrac{2}{j\omega} && \tfrac{1}{j} = -j \\ &= -\omega + \arg(-\tfrac{j2}{\omega}) && (*) \\ &= \begin{cases} -\frac{\pi}{2} - \omega & \omega > 0 \\ \frac{\pi}{2} - \omega & \omega < 0 \end{cases} && \text{definition of signum function} \\ &= -\tfrac{\pi}{2}\operatorname{sgn}\omega - \omega. \end{aligned}$$

In the above simplification, we used the fact that

$$(*) \qquad \arg\tfrac{2}{j\omega} = \arg(-\tfrac{j2}{\omega}) = \begin{cases} -\frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} & \omega < 0. \end{cases}$$




Finally, using numerical calculation, we can plot the graphs of $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown in Figures 6.10(a) and (b).

(b) Since $|X(\omega)|$ is largest for $\omega = 0$, $x$ has the most information at the frequency 0.

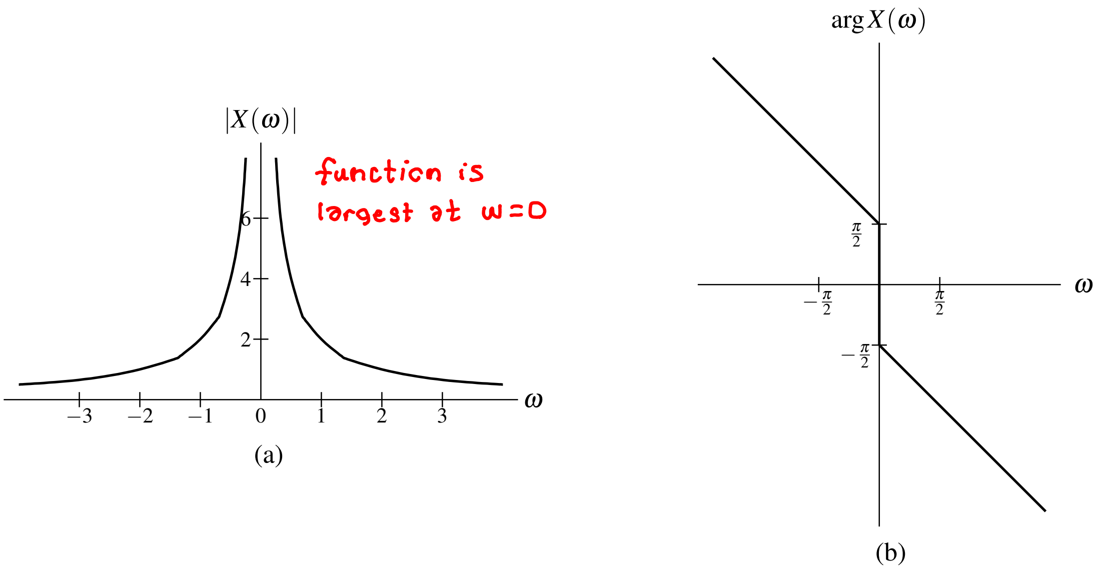



Figure 6.10: Frequency spectrum of the time-shifted signum function. (a) Magnitude spectrum and (b) phase spectrum of $x$.

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