3.3 Suppose that we have two functions x and y related as

$$y(t) = x(at-b),$$

where a and b are real constants and $a \neq 0$.
(a) Show that y can be formed by first time shifting x by b and then time scaling the result by a.
(b) Show that y can also be formed by first time scaling x by a and then time shifting the result by $\frac{b}{a}$.

Answer (a). (shift then scale)

When working with time transformed functions, always give each transformed function a name

Let f denote the result of time shifting x by b. So, by definition, we have

$$f(t) = x(t-b). \quad (1)$$

Let g denote the result of time scaling f by a. So, by definition, we have

$$g(t) = f(at).$$

Substituting the above formula for f into the equation for g, we obtain

$$\begin{aligned} g(t) &= f(at) \\ &= x(at-b) \\ &= y(t). \end{aligned}$$

substituting (1)

Therefore, y can be formed in the manner specified in the problem statement.

Answer (b). (scale then shift)

Let f denote the result of time scaling x by a. So, by definition, we have

$$f(t) = x(at).$$

Let g denote the result of time shifting f by $\frac{b}{a}$. So, by definition, we have

$$g(t) = f\left(t - \tfrac{b}{a}\right).$$

Substituting the above formula for f into the equation for g, we obtain

$$\begin{aligned} g(t) &= f\left(t - \tfrac{b}{a}\right) \\ &= x\left(a\left[t - \tfrac{b}{a}\right]\right) \\ &= x(at-b) \\ &= y(t). \end{aligned}$$

substituting (1)

Therefore, y can be formed in the manner specified in the problem statement.