Example 7.3. Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-at}u(t)\}(s) \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}\,dt \\ &= \left[\left(-\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_0^{\infty}. \end{aligned}$$

definition of LT

combine exponentials and use u to change limits

integrate

At this point, we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges to a finite value. This yields

real exponential $\begin{cases} 0 & \sigma+a>0 \\ \infty & \sigma+a<0 \end{cases}$

complex sinusoid finite but limit does not exist

$$\begin{aligned} X(s) &= \left[\left(-\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_0^{\infty} \\ &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_0^{\infty} \\ &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)\infty}e^{-j\omega\infty} - 1\right]. \end{aligned}$$

factor and split exponentials

take difference

Thus, we can see that the above expression only converges for $\sigma + a > 0$ (i.e., $\mathrm{Re}(s) > -a$). In this case, we have that

if $\mathrm{Re}(s) > -a$

$$\begin{aligned} X(s) &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)[0-1] \\ &= \left(\tfrac{-1}{s+a}\right)(-1) \\ &= \frac{1}{s+a}. \end{aligned}$$

rewrite in terms of s ($s = \sigma + j\omega$)

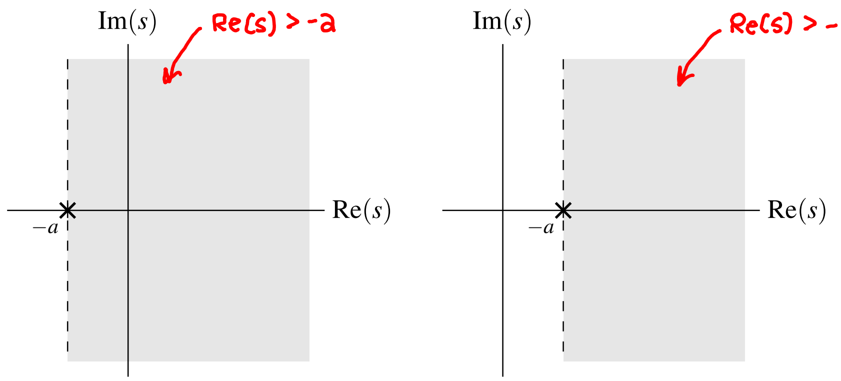
simplify

Thus, we have that

$$e^{-at}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \mathrm{Re}(s) > -a.$$

Note: We must specify this region of convergence since $\frac{1}{s+a}$ is not correct for all $s \in \mathbb{C}$

The region of convergence for X is illustrated in Figures 7.2(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.



(a) (b)

Figure 7.2: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

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