Example 3.36. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = |x(t)|. \quad ①$$

②

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$\begin{aligned} a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) &= a_1|x_1(t)| + a_2|x_2(t)| \quad \text{and} && \text{from definition of } \mathcal{H} \text{ in } ① \\ \mathcal{H}x'(t) &= |x'(t)| && \text{from definition of } \mathcal{H} \text{ in } ① \\ &= |a_1x_1(t) + a_2x_2(t)|. && \text{from definition of } x' \text{ in } ② \end{aligned}$$

At this point, we recall the triangle inequality (i.e., for $a, b \in \mathbb{C}$, $|a+b| \leq |a| + |b|$). Thus, $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = -1 \quad \text{and} \quad \mathcal{H}x'(t) = 1.$$

Counterexample

Therefore, the superposition property does not hold and the system is not linear. ■

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1x_1 + a_2x_2\} = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2.$$

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