Example 4.14. Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely integrable. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} e^{at}\,dt \\
&= \int_{0}^{\infty} e^{at}\,dt \\
&= \begin{cases} \int_0^\infty e^{at}\,dt & a \neq 0 \\ \int_0^\infty 1\,dt & a = 0 \end{cases} \\
&= \begin{cases} \left[\frac{1}{a}e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases}
\end{aligned}$$

split integration interval and use fact that $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

drop zero integral

identify two cases for integration

integrate

Now, we simplify the preceding equation for each of the cases $a \neq 0$ and $a = 0$. Suppose that $a \neq 0$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \left[\tfrac{1}{a}e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right).
\end{aligned}$$

what is $e^{a\infty}$?

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} \\
&= -\tfrac{1}{a}.
\end{aligned}$$

assuming $a<0$

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= [t]\big|_0^\infty \\
&= \infty.
\end{aligned}$$

Thus, we have shown that

combining above results

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

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