**Example A.12** (Poles and zeros of a rational function)**.** Find and plot the poles and (finite) zeros of the function

$$f(z) = \frac{z^2(z^2+1)(z-1)}{(z+1)(z^2+3z+2)(z^2+2z+2)}.$$

*Solution.* We observe that $f$ is a rational function, so we can easily determine the poles and zeros of $f$ from its factored form. We now proceed to factor $f$. First, we factor $z^2+3z+2$. To do this, we solve for the roots of $z^2+3z+2=0$ to obtain

or factor by hand

$$z = \frac{-3 \pm \sqrt{3^2-4(1)(2)}}{2(1)} = -\tfrac{3}{2} \pm \tfrac{1}{2} = \{-1,-2\}.$$ $z^2+3z+2=(z+2)(z+1)$

(For additional information on how to find the roots of a quadratic equation, see Section A.16.) So, we have

$$z^2+3z+2 = (z+1)(z+2).$$ 1

Second, we factor $z^2+2z+2$. To do this, we solve for the roots of $z^2+2z+2=0$ to obtain

$$z = \frac{-2 \pm \sqrt{2^2-4(1)(2)}}{2(1)} = -1 \pm j = \{-1+j,-1-j\}.$$

So, we have

$$z^2+2z+2 = (z+1-j)(z+1+j).$$ 2

Lastly, we factor $z^2+1$. Using the well-known factorization for a sum of squares, we obtain

$$z^2+1 = (z+j)(z-j).$$ 3 $a^2+b^2=(a+jb)(a-jb)$

Combining the above results, we can rewrite $f$ as

1, 2, 3

$$f(z) = \frac{z^2(z+j)(z-j)(z-1)}{(z+1)^2(z+2)(z+1-j)(z+1+j)}. = \frac{z^2(z+j)^1(z-j)^1(z-1)^1}{(z+1)^2(z+2)^1(z+1-j)^1(z+1+j)^1}$$

From this expression, we can trivially deduce that $f$ has:
- first order zeros at 1, $j$, and $-j$,
- a second order zero at 0, } from numerator
- first order poles at $-1+j$, $-1-j$, $-2$, and
- a second order pole at $-1$. } from denominator

The zeros and poles of this function are plotted in Figure A.9. In such plots, the poles and zeros are typically denoted by the symbols “x” and “o”, respectively.

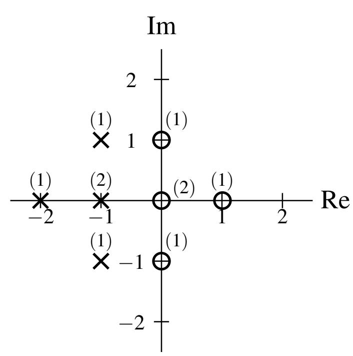


Figure A.9: Plot of the poles and zeros of $f$ (with their orders indicated in parentheses).

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