Example 6.9 (Time-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A, ω_0, and θ are real constants.

$= v(t - [-\frac{\theta}{\omega_0}])$ table of FT pairs

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Solution. Let $v(t) = A\cos(\omega_0 t)$ so that $x(t) = v(t + \frac{\theta}{\omega_0})$. Also, let $V = \mathcal{F}v$. From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad 3$$

Using this transform pair and the linearity property of the Fourier transform, we have that

linearity

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A\cos(\omega_0 t)\}(\omega) && \text{from FT of 1} \\ 4 \qquad &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. && \text{from FT pair 3 (and linearity)} \end{aligned}$$

From the definition of v and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0} V(\omega) && \text{from FT of 2 using time-domain shifting property } [e^{-jw(-\theta/w_0)}] \\ &= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. && \text{substituting expression for } V(w) \text{ from 4} \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \overset{\text{CTFT}}{\longleftrightarrow} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \quad \blacksquare$$