Example 6.19 (Energy of the sinc function)**.** Consider the function $x(t) = \operatorname{sinc}\left(\frac{1}{2}t\right)$, which has the Fourier transform X given by $X(\omega) = 2\pi \operatorname{rect} \omega$. Compute the energy of x.

①

Solution. We could directly compute the energy of x as

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt. = \int_{-\infty}^{\infty} \left|\frac{\sin t/2}{t/2}\right|^2 dt \to ☹
\end{aligned}
$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$
\begin{aligned}
E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega && \text{from given } X \text{ in ①} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \operatorname{rect} \omega|^2 \, d\omega && \text{rect } t = 1 \text{ for } t \in [-\tfrac{1}{2}, \tfrac{1}{2}] \text{ and zero otherwise} \\
&= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 d\omega && \text{cancel one } 2\pi \text{ factor} \\
&= 2\pi \int_{-1/2}^{1/2} d\omega && \text{integrate} \\
&= 2\pi [\omega]|_{-1/2}^{1/2} \\
&= 2\pi [\tfrac{1}{2} + \tfrac{1}{2}] \\
&= 2\pi.
\end{aligned}
$$

Thus, we have

$$
E = \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt = 2\pi. \quad \blacksquare
$$