Example 7.4. Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) \\ &= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt \\ &= \int_{-\infty}^{0} -e^{-at}e^{-st}\,dt \\ &= \int_{-\infty}^{0} -e^{-(s+a)t}\,dt \\ &= \left[\left(\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}. \end{aligned}$$

definition of LT
use u to change limits
combine exponentials
integrate

In order to more easily determine when the above expression converges to a finite value, we substitute $s = \sigma + j\omega$. This yields

real exponential
$\begin{cases} 0 & \sigma+a<0 \\ \infty & \sigma+a>0 \end{cases}$
complex sinusoid
finite but limit not well defined

$$\begin{aligned} X(s) &= \left[\left(\frac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0} \\ &= \left(\frac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{-\infty}^{0} \\ &= \left(\frac{1}{\sigma+a+j\omega}\right)\left[1 - e^{(\sigma+a)\infty}e^{j\omega\infty}\right]. \end{aligned}$$

split exponential
take difference

Thus, we can see that the above expression only converges for $\sigma + a < 0$ (i.e., $\operatorname{Re}(s) < -a$). In this case, we have

if $\operatorname{Re}(s) < -a$

$$\begin{aligned} X(s) &= \left(\frac{1}{\sigma+a+j\omega}\right)[1-0] \\ &= \frac{1}{s+a}. \end{aligned}$$

rewrite in terms of s ($s = \sigma + j\omega$)

Thus, we have that

$$-e^{-at}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) < -a.$$

Note: We must specify this region of convergence since $\frac{1}{s+a}$ is not correct for all $s \in \mathbb{C}$

The region of convergence for X is illustrated in Figures 7.3(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.

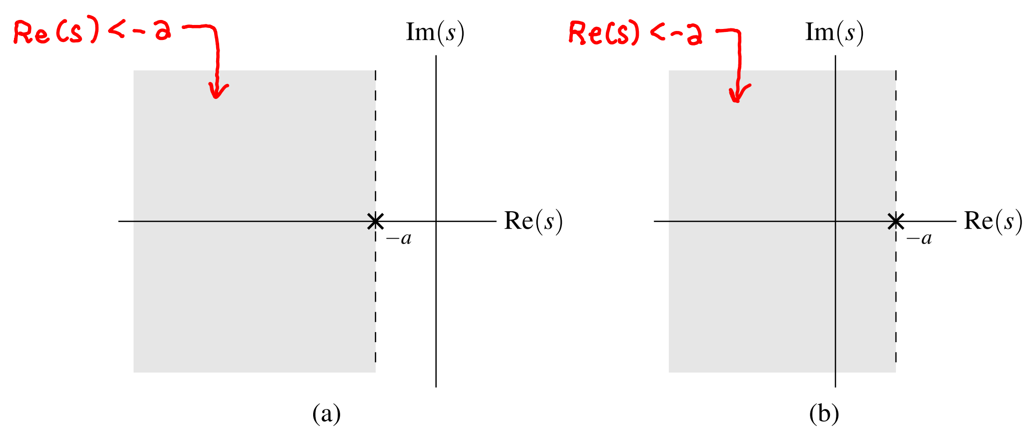

Figure 7.3: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

different | same | different (and this is critical for invertibility of LT)

NOTE:
Example 7.3: $e^{-at}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a}$ for $\operatorname{Re}(s) > -a$
Example 7.4: $-e^{-at}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a}$ for $\operatorname{Re}(s) < -a$