Example 4.1. Compute the convolution $x * h$ where

$$x(t) = \begin{cases} -1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of h. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.1(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 4.1(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.2}$$

Second, we consider the case of $-1 \leq t < 0$. From Figure 4.1(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{t} \\ &= -e^{-t}[e^{t}-e^{-1}] \\ &= e^{-t-1}-1. \end{aligned} \tag{4.3}$$

Third, we consider the case of $0 \leq t < 1$. From Figure 4.1(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\ &= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\ &= e^{-t}[e^{-1}-1+e^{t}-1] \\ &= 1+(e^{-1}-2)e^{-t}. \end{aligned} \tag{4.4}$$

Fourth, we consider the case of $t \geq 1$. From Figure 4.1(h), we can see that

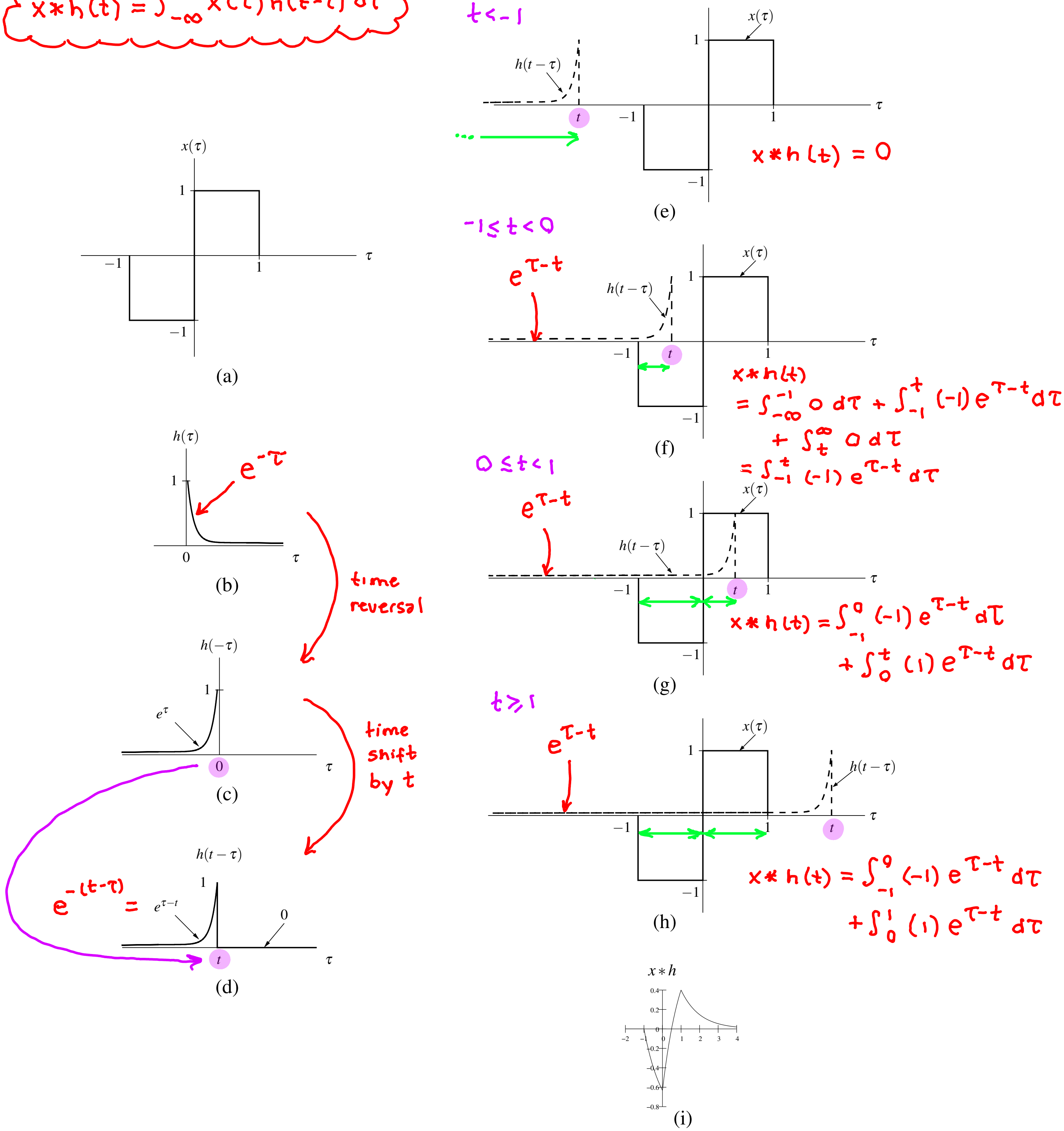
$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\ &= e^{-t}[e^{-1}-1+e-1] \\ &= (e-2+e^{-1})e^{-t}. \end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x * h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1} - 1 & -1 \le t < 0 \\ (e^{-1} - 2)e^{-t} + 1 & 0 \le t < 1 \\ (e - 2 + e^{-1})e^{-t} & 1 \le t. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.1(i).

$$x*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$$



$t<-1$: $x*h(t)=0$

$-1\le t<0$:
$$x*h(t)=\int_{-\infty}^{-1}0\,d\tau+\int_{-1}^{t}(-1)e^{\tau-t}\,d\tau+\int_{t}^{\infty}0\,d\tau=\int_{-1}^{t}(-1)e^{\tau-t}\,d\tau$$

$0\le t<1$:
$$x*h(t)=\int_{-1}^{0}(-1)e^{\tau-t}\,d\tau+\int_{0}^{t}(1)e^{\tau-t}\,d\tau$$

$t\geq 1$:
$$x*h(t)=\int_{-1}^{0}(-1)e^{\tau-t}\,d\tau+\int_{0}^{1}(1)e^{\tau-t}\,d\tau$$

Figure 4.1: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<-1$, (f) $-1\le t<0$, (g) $0\le t<1$, and (h) $t\geq 1$; and (i) the convolution result $x*h$.

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