Example 2.2. For two functions x_1 and x_2, the expression x_1+x_2 denotes the function that is the sum of the functions x_1 and x_2. The expression $(x_1+x_2)(t)$ denotes the function x_1+x_2 evaluated at t. Since the addition of functions can be defined pointwise (i.e., we can add two functions by adding their values at corresponding pairs of points), the following relationship always holds:

adding functions   adding numbers

$$(x_1+x_2)(t) = x_1(t)+x_2(t) \quad \text{for all } t.$$

Similarly, since subtraction, multiplication, and division can also defined pointwise, the following relationships also hold:

subtracting functions   subtracting numbers

$$(x_1-x_2)(t) = x_1(t)-x_2(t) \quad \text{for all } t,$$

multiplying functions   multiplying numbers

$$(x_1x_2)(t) = x_1(t)x_2(t) \quad \text{for all } t, \quad \text{and}$$

dividing functions   dividing numbers

$$(x_1/x_2)(t) = x_1(t)/x_2(t) \quad \text{for all } t.$$

It is important to note, however, that not all mathematical operations involving functions can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be functions. The convolution operation (for functions), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to functions, then the expression $(x_1 \diamond x_2)(t)$ is mathematically valid, but the expression $x_1(t) \diamond x_2(t)$ is not. The latter expression is not valid since the $\diamond$ operator requires two functions as operands, but the provided operands $x_1(t)$ and $x_2(t)$ are numbers (namely, the values of the functions x_1 and x_2 each evaluated at t). Due to issues like this, one must be careful in the use of mathematical notation related to functions. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■