Example 3.41. Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2 x(t), \quad 1$$

where $\mathcal{D}$ denotes the derivative operator. For each function x given below, determine if x is an eigenfunction of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(t) = \cos 2t$; and
(b) $x(t) = t^3$.

Solution. (a) We have

from definition of $\mathcal{H}$ in 1

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{\cos 2t\}(t) \\ &= \mathcal{D}\{-2\sin 2t\}(t) \\ &= -4\cos 2t \\ &= -4x(t).\end{aligned}$$

$\frac{d}{dt} \cos t = -\sin t$

$\frac{d}{dt} \sin t = \cos t$

from definition of x

So, we have $\mathcal{H}x = -4x$.

Therefore, x is an eigenfunction of $\mathcal{H}$ with the eigenvalue -4.

(b) We have

from definition of $\mathcal{H}$ in 1

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t) \\ &= \mathcal{D}\{3t^2\}(t) \\ &= 6t \\ &= \tfrac{6}{t^2}x(t).\end{aligned}$$

$\frac{d}{dt} t^3 = 3t^2$

$\frac{d}{dt} 3t^2 = 6t$

from definition of x

$\left(\frac{6t\, x(t)}{x(t)} = \frac{6t\, x(t)}{t^3}\right)$

not a constant

Therefore, x is not an eigenfunction of $\mathcal{H}$. ■

A function x is said to be an eigenfunction of the system $\mathcal{H}$ with eigenvalue λ if

$$\mathcal{H}x = \lambda x.$$