Example 6.7 (Linearity property of the Fourier transform)**.** Using properties of the Fourier transform and the transform pair

$$e^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0), \quad 1$$

find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t),$$

where A and ω_0 are real constants.

Solution. We recall that 2 $\cos\alpha = \frac{1}{2}[e^{j\alpha}+e^{-j\alpha}]$ for any real α. Thus, we can write

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{A\cos(\omega_0 t)\})(\omega) \quad \text{from Euler } 2 \\ &= \left(\mathcal{F}\{\tfrac{A}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})\}\right)(\omega). \end{aligned}$$

Then, we use the linearity property of the Fourier transform to obtain

linearity

$$X(\omega) = \tfrac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\}(\omega) + \tfrac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}(\omega).$$

from given FT pair 1

Using the given Fourier transform pair, we can further simplify the above expression for $X(\omega)$ as follows:

$$\begin{aligned} X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega+\omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega-\omega_0)] \\ &= A\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} A\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \qquad \blacksquare$$