Example 7.18 (Initial and final value theorems)**.** A bounded causal function x with a (finite) limit at infinity has the Laplace transform

$$X(s) = \frac{2s^2+3s+2}{s^3+2s^2+2s} \quad \text{for } \operatorname{Re}(s) > 0.$$

Determine $x(0^+)$ and $\lim_{t\to\infty} x(t)$.

Solution. Since x is causal (i.e., $x(t) = 0$ for all $t < 0$) and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$\begin{aligned}
x(0^+) &= \lim_{s\to\infty} sX(s) && \text{substitute given } X \\
&= \lim_{s\to\infty} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] && \text{multiply} \\
&= \lim_{s\to\infty} \frac{2s^2+3s+2}{s^2+2s+2} && \text{take limit (highest power terms dominate)} \\
&= 2.
\end{aligned}$$

Since x is bounded and causal and has well-defined limit at infinity, we can apply the final value theorem. From this theorem, we have

$$\begin{aligned}
\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) && \text{substitute given } X \\
= \lim_{s\to 0} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] && \text{multiply} \\
= \left.\frac{2s^2+3s+2}{s^2+2s+2}\right|_{s=0} && \text{evaluate at } s=0 \\
= 1.
\end{aligned}$$

In passing, we note that the inverse Laplace transform x of X can be shown to be

$$x(t) = [1 + e^{-t}\cos t]u(t).$$

As we would expect, the values calculated above for $x(0^+)$ and $\lim_{t\to\infty} x(t)$ are consistent with this formula for x. ■