Theorem 4.12 (Eigenfunctions of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{st}$, where s is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \tag{4.49}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(s)$.

Proof. We have

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{commutative property of convolution} \\
&= h * x(t) && \text{definition of convolution} \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau && \text{substitute given function } x \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau && \text{factor out } e^{st} \\
&= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau}_{\text{call this } H(s)} \\
&= H(s)e^{st}.
\end{aligned}$$

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