Example 5.3. Consider the periodic function x with fundamental period $T = 3$ as shown in Figure 5.3. Find the Fourier series representation of x.

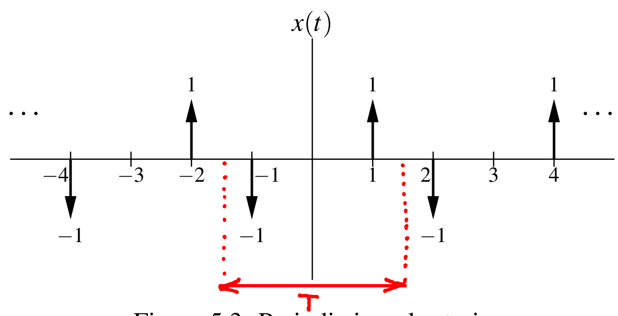

Figure 5.3: Periodic impulse train.

Solution. The function x has the fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \le t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \le t < \frac{3}{2}$). From the Fourier series analysis equation, we have

$$\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt \\
&= \tfrac{1}{3}\int_{-3/2}^{3/2} x(t)e^{-j(2\pi/3)kt}\,dt \\
&= \tfrac{1}{3}\int_{-3/2}^{3/2} [-\delta(t+1)+\delta(t-1)]\,e^{-j(2\pi/3)kt}\,dt \\
&= \tfrac{1}{3}\left[\int_{-3/2}^{3/2} -\delta(t+1)e^{-j(2\pi/3)kt}\,dt + \int_{-3/2}^{3/2} \delta(t-1)e^{-j(2\pi/3)kt}\,dt\right] \\
&= \tfrac{1}{3}\left[-e^{-jk(2\pi/3)(-1)} + e^{-jk(2\pi/3)(1)}\right] \\
&= \tfrac{1}{3}\left[e^{-j(2\pi/3)k} - e^{j(2\pi/3)k}\right] \\
&= \tfrac{1}{3}\left[2j\sin\left(-\tfrac{2\pi}{3}k\right)\right] \\
&= \tfrac{2j}{3}\sin\left(-\tfrac{2\pi}{3}k\right) \\
&= -\tfrac{2j}{3}\sin\left(\tfrac{2\pi}{3}k\right).
\end{aligned}$$

Fourier series analysis equation

consider interval $[-T/2, T/2)$

substitute given x

split into 2 integrals

extend limits and apply sifting property

simplify

Euler $[\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})]$

simplify

sin is odd

Thus, x has the Fourier series representation

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3}\sin\left(\tfrac{2\pi}{3}k\right)e^{j(2\pi/3)kt}.
\end{aligned}$$

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