Relationship Between the Laplace and Fourier Transforms (General Case)

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

Now, consider the general case of an arbitrary complex value for s in (7.2). Let us express s in Cartesian form as $s = \sigma + j\omega$ where σ and ω are real. Substituting $s = \sigma + j\omega$ into (7.2), we obtain

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt \\ &= \mathcal{F}\{e^{-\sigma t} x(t)\}(\omega). \end{aligned}$$

substituting $\sigma + j\omega$ for s in LT definition

split exponential in two

definition of FT

Thus, we have shown

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t} x(t)\}(\omega). \tag{7.5}$$

Thus, the Laplace transform of x can be viewed as the (CT) Fourier transform of $x'(t) = e^{-\sigma t} x(t)$ (i.e., x weighted by a real exponential function).