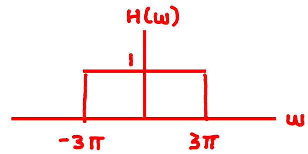
Example 5.10 (Lowpass filtering)**.** Suppose that we have a LTI system with input x, output y, and frequency response H , where

$$H(\omega) = \begin{cases} 1 & |\omega| \le 3\pi \\ 0 & \text{otherwise.} \end{cases}$$



Further, suppose that the input x is the periodic function

$$x(t) = 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t).$$

(a) Find the Fourier series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

don't use Fourier series analysis equation!

Solution. (a) We begin by finding the Fourier series representation of x. Using Euler's formula, we can re-express x as

Euler

simplify and reorder terms

ω0 must be as large as possible

rewrite exponentials as jkω0

$$\begin{aligned}
x(t) &= 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t) \\
&= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\
&= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\
&= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\
&= \tfrac{1}{4}e^{j\underbrace{(-3)}_{k=-3}(2\pi)t} + \tfrac{1}{2}e^{j\underbrace{(-2)}_{k=-2}(2\pi)t} + e^{j\underbrace{(-1)}_{k=-1}(2\pi)t} + e^{j\underbrace{(0)}_{k=0}(2\pi)t} + e^{j\underbrace{(1)}_{k=1}(2\pi)t} + \tfrac{1}{2}e^{j\underbrace{(2)}_{k=2}(2\pi)t} + \tfrac{1}{4}e^{j\underbrace{(3)}_{k=3}(2\pi)t}.
\end{aligned}$$

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & k = 0 \\ 1 & k \in \{-1, 1\} \\ \frac{1}{2} & k \in \{-2, 2\} \\ \frac{1}{4} & k \in \{-3, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

(due to eigenfunction properties of LTI systems)

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(k\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$$H(k\omega_0) = 1 \left\{ \begin{aligned}
b_0 &= a_0 H([0][2\pi]) = 1(1) = 1, \\
b_1 &= a_1 H([1][2\pi]) = 1(1) = 1, \\
b_{-1} &= a_{-1} H([-1][2\pi]) = 1(1) = 1,
\end{aligned} \right.$$

$$H(k\omega_0) = 0 \left\{ \begin{aligned}
b_2 &= a_2 H([2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_{-2} &= a_{-2} H([-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_3 &= a_3 H([3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\
b_{-3} &= a_{-3} H([-3][2\pi]) = \tfrac{1}{4}(0) = 0.
\end{aligned} \right.$$

we were given

$$H(\omega) = \begin{cases} 1 & \omega \in [-3\pi, 3\pi] \\ 0 & \text{otherwise} \end{cases}$$

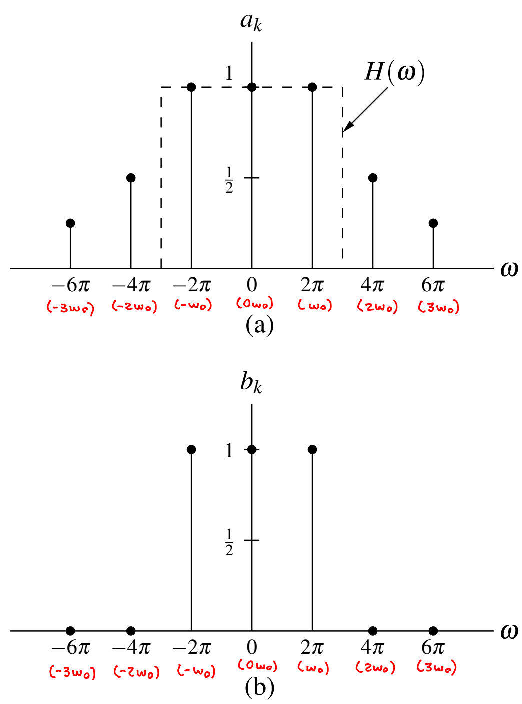
Thus, we have

$$b_k = \begin{cases} 1 & k \in \{-1,0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 5.10(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes.

$\omega_0 = 2\pi$

$b_K = a_K H(k\omega_0)$



$\omega_0 = 2\pi$

Figure 5.10: Frequency spectra of the (a) input function x and (b) output function y.

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NOTE: THE APPROACH USED TO SOLVE THIS PROBLEM DID NOT INVOLVE CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!