Example 6.1 (Fourier transform of the unit-impulse function)**.** Find the Fourier transform X of the function

$$x(t) = A\delta(t - t_0),$$

where A and t_0 are real constants. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

substitute given x into Fourier transform analysis equation

$$X(\omega) = \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-j\omega t}\,dt$$

pull constant A out of integral

$$= A\int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\,dt.$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

sifting property

$= Ae^{-j\omega t}\big|_{t=t_0}$

$$X(\omega) = Ae^{-j\omega t_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega, \quad \text{where} \quad X(\omega) = Ae^{-j\omega t_0}. \qquad \blacksquare$$