Example 5.9. Consider a LTI system with the frequency response

$$H(\omega) = e^{-j\omega/4}.$$

Find the response y of the system to the input x, where

$$x(t) = \tfrac{1}{2}\cos(2\pi t).$$

Euler $\left[\cos\theta = \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\right]$

Solution. To begin, we rewrite x as

$$x(t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$c_k = \begin{cases} \frac{1}{4} & k \in \{-1, 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Fourier series with only two nonzero terms

Thus, we can write

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t} && \text{from eigenfunction properties of LTI systems} \\
&= c_{-1} H(-\omega_0) e^{-j\omega_0 t} + c_1 H(\omega_0) e^{j\omega_0 t} && \text{expand summation} \\
&= \tfrac{1}{4} H(-2\pi) e^{-j2\pi t} + \tfrac{1}{4} H(2\pi) e^{j2\pi t} && \text{substitute for } c_{-1}, c_1, \omega_0 \\
&= \tfrac{1}{4} e^{j\pi/2} e^{-j2\pi t} + \tfrac{1}{4} e^{-j\pi/2} e^{j2\pi t} && \text{evaluate } H(\ldots) \\
&= \tfrac{1}{4}[e^{-j(2\pi t - \pi/2)} + e^{j(2\pi t - \pi/2)}] && \text{combine exponentials} \\
&= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\
&= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\
&= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right). && \text{express in terms of cos (Euler)}
\end{aligned}$$

Observe that $y(t) = x\left(t - \frac{1}{4}\right)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\omega) = e^{-\omega/4}$ is an ideal delay of $\frac{1}{4}$ (i.e., a system that performs a time shift of $\frac{1}{4}$). ■

NOTE: THE APPROACH USED IN THE SOLUTION TO THIS PROBLEM DID NOT REQUIRE CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!