Example 6.10 (Frequency-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = \cos(\omega_0 t)\cos(20\pi t),$$

where ω_0 is a real constant.

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

$$\begin{aligned}
X(\omega) &= \left(\mathcal{F}\{\cos(\omega_0 t)\overbrace{(\tfrac{1}{2})(e^{j20\pi t} + e^{-j20\pi t})}^{\cos(20\pi t)}\}\right)(\omega) && \text{distribute} \\
&= \left(\mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos(\omega_0 t) + \tfrac{1}{2}e^{-j20\pi t}\cos(\omega_0 t)\}\right)(\omega) && \text{linearity property} \\
&= \tfrac{1}{2}\left(\mathcal{F}\{e^{j20\pi t}\cos(\omega_0 t)\}\right)(\omega) + \tfrac{1}{2}\left(\mathcal{F}\{e^{-j20\pi t}\cos(\omega_0 t)\}\right)(\omega).
\end{aligned}$$

table of FT pairs

From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad ①$$

frequency domain shifting property

From this transform pair and the frequency-domain shifting property of the Fourier transform, we have

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega - 20\pi) + \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega + 20\pi) && \text{FT pair ①} \\
&= \tfrac{1}{2}\left[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]\right]\Big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]\right]\Big|_{v=\omega+20\pi} && \text{substitute} \\
&= \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]\right) \\
&= \tfrac{\pi}{2}\left[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi) + \delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)\right]. \quad \blacksquare
\end{aligned}$$