Example 6.40 (Simple RL network)**.** Consider the resistor-inductor (RL) network shown in Figure 6.26 with input v_1 and output v_2. This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response H of the system. (b) Find the response v_2 of the system to the input $v_1(t) = \operatorname{sgn} t$.

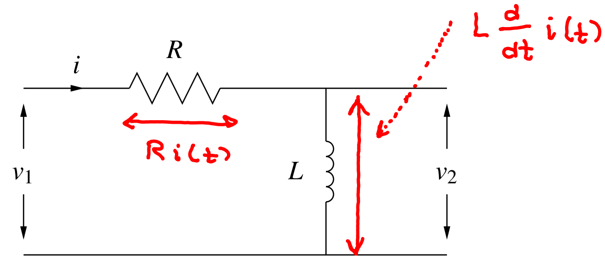

Figure 6.26: Simple RL network.

Solution. (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\tfrac{d}{dt}i(t) \quad \text{and} \tag{6.35}$$
$$v_2(t) = L\tfrac{d}{dt}i(t). \tag{6.36}$$

(Recall that the voltage v across an inductor L is related to the current i through the inductor as $v(t) = L\frac{d}{dt}i(t)$.) Taking the Fourier transform of (6.35) and (6.36) yields

using time-domain differentiation property
$\frac{d}{dt}x(t) \overset{FT}{\longleftrightarrow} j\omega X(\omega)$

$$\begin{aligned} V_1(\omega) &= RI(\omega) + j\omega LI(\omega) \\ &= (R + j\omega L)I(\omega) \quad \text{and} \end{aligned} \tag{6.37}$$

$$V_2(\omega) = j\omega LI(\omega). \tag{6.38}$$

From (6.37) and (6.38), we have

* Since system is LTI, $V_2(\omega) = V_1(\omega)H(\omega) \Rightarrow H(\omega) = \frac{V_2(\omega)}{V_1(\omega)}$

$$\begin{aligned} H(\omega) &= \overset{*}{\frac{V_2(\omega)}{V_1(\omega)}} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \tag{6.39}$$

substitute (6.38) in numerator and (6.37) in denominator

cancel I's

Thus, we have found the frequency response of the system.

(b) Now, suppose that $v_1(t) = \operatorname{sgn} t$ (as given). Taking the Fourier transform of the input v_1 (with the aid of Table 6.2), we have

$= \mathcal{F}\{\operatorname{sgn} t\}(\omega)$ from FT table

$$V_1(\omega) = \frac{2}{j\omega}. \tag{6.40}$$

From the definition of the system, we know

from *

$$V_2(\omega) = H(\omega)V_1(\omega). \tag{6.41}$$

Substituting (6.40) and (6.39) into (6.41), we obtain

$V_1(\omega)$ $H(\omega)$

substitute

$$\begin{aligned} V_2(\omega) &= \left(\frac{j\omega L}{R + j\omega L}\right)\left(\frac{2}{j\omega}\right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$

cancel factors of jω

Taking the inverse Fourier transform of both sides of this equation, we obtain

taking inverse FT

$$\begin{aligned} v_2(t) &= \mathcal{F}^{-1}\left\{\frac{2L}{R+j\omega L}\right\}(t) \\ &= \mathcal{F}^{-1}\left\{\frac{2}{R/L+j\omega}\right\}(t) && \text{divide numerator and denominator by } L \\ &= 2\mathcal{F}^{-1}\left\{\frac{1}{R/L+j\omega}\right\}(t). && \text{linearity} \end{aligned}$$

Using Table 6.2, we can simplify to obtain

from FT table

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

$$v_2(t) = 2e^{-(R/L)t}u(t).$$

Thus, we have found the response v_2 to the input $v_1(t) = \operatorname{sgn} t$. ■