Example 6.3 (Fourier transform of the rectangular function)**.** Find the Fourier transform X of the function

$$x(t) = \operatorname{rect} t.$$

$\left[\operatorname{rect} t = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}\right]$

Solution. From the definition of the Fourier transform, we can write

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j\omega t} dt.$$

substitute given function x into Fourier transform analysis equation

From the definition of the rectangular function, we can simplify this equation to obtain

change limits since rect t = 0 for $|t| > \frac{1}{2}$

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} \operatorname{rect}(t) e^{-j\omega t} dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t} dt. \end{aligned}$$

rect t = 1 for t in integration interval

Evaluating the integral and simplifying, we have

integrate

$$\begin{aligned} X(\omega) &= \left[-\tfrac{1}{j\omega} e^{-j\omega t} \right] \Big|_{-1/2}^{1/2} \\ &= \tfrac{1}{j\omega} \left(e^{j\omega/2} - e^{-j\omega/2} \right) \\ &= \tfrac{1}{j\omega} \left[2j \sin\left(\tfrac{\omega}{2}\right) \right] \\ &= \tfrac{2}{\omega} \sin\left(\tfrac{\omega}{2}\right) \\ &= \left[\sin\left(\tfrac{\omega}{2}\right) \right] / \left(\tfrac{\omega}{2}\right) \\ &= \operatorname{sinc}\left(\tfrac{\omega}{2}\right). \end{aligned}$$

$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$

rewrite in form of sinc

definition of sinc function

Thus, we have shown that

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\tfrac{\omega}{2}\right). \qquad \blacksquare$$

Note: This is why the sinc function is of great importance.