Example 3.35. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = tx(t). \quad ①$$

② *Solution.* Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

equal for als x_1, x_2, a_1, a_2

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = a_1tx_1(t) + a_2tx_2(t) \text{ and}$$

from definition of $\mathcal{H}$ in ①

$$\begin{aligned}\mathcal{H}x'(t) &= tx'(t) \\ &= t\left[a_1x_1(t) + a_2x_2(t)\right] \\ &= a_1tx_1(t) + a_2tx_2(t).\end{aligned}$$

from definition of $\mathcal{H}$ in ①

from definition of x' in ②

Since $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the superposition property holds and the system is linear. ■

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1x_1 + a_2x_2\} = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$