New Algorithms for the Derivation of the Transfer Function Matrix of Two-Dimensional Digital Filters

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Abstract—New algorithms for the derivation of the transfer function matrix of two-dimensional digital filters from their state-space representations are proposed. Two key steps in developing the new algorithms are as follows. First, the transfer function is reformulated in terms of the characteristic polynomials of the matrices involved. Second, an efficient algorithm for the identification of 1-D polynomial coefficients is developed and is, in turn, used to determine the coefficient matrices of the 2-D transfer function. As all the matrices required in the first step can be obtained by inverting one complex matrix in conjunction with a few matrix multiplications, and the second step leads to an unitary Vandermonde matrix, the proposed algorithms are computationally efficient and reliable. The efficiency of the algorithms is illustrated by an example.

I. INTRODUCTION

State-space two-dimensional (2-D) digital filters have been studied quite extensively during the past decade, and several useful methods for their analysis and design have been established [1]. These include methods for stability analysis [2]-[6], analysis of finite wordlength effects [7]-[8], design [9]-[10], model reduction [11]-[12], and relevant computation issues [13]. Since many of the available analysis and design methods are applicable only to 2-D transfer functions but not directly to their state-space descriptions, it is often desirable to derive the transfer function from its state-space models. Several algorithms for such derivation have been proposed [14]-[16] which are basically extensions of the well-known Faddeva algorithm [17] to the 2-D case.

In this paper we present new algorithms for the derivation of the 2-D transfer function from the state-space model. Two key steps in developing the new algorithms can be described as follows. First, the transfer function is reformulated in terms of the characteristic polynomials of several matrices involved that depend on one complex variable. Second, algorithms are proposed that identify the coefficients of a 1-D polynomial of order n when it’s values at (n + 1) points on the unit circle are known. The algorithms essentially entail solving a system of linear equations whose coefficient matrix is an unitary Vandermonde matrix. An example is given to illustrate the efficiency of the algorithms proposed.

II. PRELIMINARIES

Consider the Roesser state-space model of a multi-input multi-output (MIMO) 2-D digital filter

\[
\begin{bmatrix}
    x^h(i + 1, j) \\
    x^p(i, j + 1)
\end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\
    x^p(i, j) \end{bmatrix} + \begin{bmatrix} B_1 \\
    B_2 \end{bmatrix} u(i, j)
\]

\[= Ax + Bu \quad (1a)\]

\[y(i, j) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\
    x^p(i, j) \end{bmatrix} = Cx \quad (1b)\]

where \(A_1 \in R^{n_1 \times n_1}, A_4 \in R^{n_2 \times n_2}, B_1 \in R^{n_1 \times m}, C_1 \in R^p \times n_1\). If we define

\[I(z_1, z_2) = z_1 I_{n_1} \oplus z_2 I_{n_2}\]

where \(\oplus\) denotes the direct sum, then the \(p \times m\) transfer function matrix of the digital filter is given by

\[H(z_1, z_2) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} z_1 I - A_1 & -A_2 \\
    -A_3 & z_2 I - A_4 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\
    B_2 \end{bmatrix} = C \left[I(z_1, z_2) - A \right]^{-1} B \quad (2)\]

Obviously, the \((l, k)\) entry of \(H(z_1, z_2)\) is a scalar 2-D rational function of order \((n_1, n_2)\) given by

\[H_{lk}(z_1, z_2) = C_l \left[I(z_1, z_2) - A \right]^{-1} B_k = \frac{C_l \text{adj} \left[I(z_1, z_2) - A \right]^{-1} B_k}{\text{det} \left[I(z_1, z_2) - A \right]} \quad (3)\]

where \(C_l\) and \(B_k\) are the \(l\)th row of \(C\) and the \(k\)th column of \(B\), respectively, and \(H_{lk}(z_1, z_2)\) can be considered as the transfer function of the single-input, single-output (SISO) state-space digital filter model given by

\[
\begin{bmatrix} x^h(i + 1, j) \\
    x^p(i, j + 1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\
    x^p(i, j) \end{bmatrix} + \begin{bmatrix} b_1 \\
    b_2 \end{bmatrix} u(i, j)\]
\[ y(i,j) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(i,j) \\ x_2(i,j) \end{bmatrix} = c_1 \begin{bmatrix} x_1(i,j) \\ x_2(i,j) \end{bmatrix} \] (4a)

with \( b = B_k \) and \( c = C_i \). In the rest of this section and Section III, we will focus on the problem of obtaining the transfer function of the SISO state-space filter (4).

If \( H(z_1, z_2) \) is the transfer function of the filter represented by (4), we can write

\[
H(z_1, z_2) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} z_1I_{n_1} - A_1 & -A_2 \\ -A_3 & z_2I_{n_2} - A_4 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\] (4b)

where \( \frac{p_k(z_2)}{g_k(z_2)} \) are polynomials in \( z_2 \) with order not greater than \( n_2 \), and

\[
\sum_{i=0}^{n_1} p_k(z_2)z_2^i = \det \begin{bmatrix} z_1I_{n_1} - A_1 & -A_2 \\ -A_3 & z_2I_{n_2} - A_4 \end{bmatrix}
\] (5)

From (6) it follows that

\[
p_k(z_2) = \det(z_2I_{n_2} - A_4) = P(z_2, A_4)
\] (7)

where \( P(z_2, A_4) \) denotes the characteristic polynomial of \( A_4 \) in variable \( z_2 \). Further, by (5) and the formula of matrix inversion [18] the transfer function \( H(z_1, z_2) \) can be expressed as

\[
H(z_1, z_2) = l(z_2) + g(z_2)[z_1I_{n_1} - E(z_2)]^{-1}f(z_2)
\] (8)

where

\[
E(z_2) = A_1 + A_2[z_2I_{n_2} - A_4]^{-1}A_3
\]

\[
f(z_2) = b_1 + A_2[z_2I_{n_2} - A_4]^{-1}b_2
\]

\[
g(z_2) = c_1 + c_2[z_2I_{n_2} - A_4]^{-1}A_3
\]

\[
l(z_2) = c_2[z_2I_{n_2} - A_4]^{-1}b_2
\] (9)

By using a well-known formula for computing the transfer function of a 1-D SISO state-space model, we can rewrite (8) as

\[
H(z_1, z_2) = l(z_2) + \frac{\det(z_1I_{n_1} - E(z_2) + f(z_2)g(z_2))}{\det(z_1I_{n_1} - E(z_2))} - 1
\]

\[
= \frac{P[z_1, E(z_2)] - f(z_2)g(z_2)}{P[z_1, E(z_2)]} + l(z_2) - 1
\] (10)

where \( P[z_1, E(z_2)] \) and \( P[z_1, E(z_2)] - f(z_2)g(z_2) \) are the characteristic polynomials of \( E(z_2) \) and \( E(z_2) - f(z_2)g(z_2) \), respectively. Note that the denominator in (10) is a monic polynomial in \( z_1 \) but the denominator in (5) is a polynomial in \( z_1 \) with \( p_{n_1}(z_2) \) as the coefficient of \( z_1^{n_1} \). This observation in conjunction with (7) leads to

\[
\sum_{i=0}^{n_1} q_i(z_2)z_1^i = P(z_2, A_4)[P[z_1, E(z_2)] - f(z_2)g(z_2)]
\]

\[
+ [f(z_2) - 1]P[z_1, E(z_2)]
\]

\[
\sum_{i=0}^{n_1} p_i(z_2)z_1^i = P(z_2, A_4)P[z_1, E(z_2)]
\] (11)

III. THE ALGORITHMS

The algorithms are based on (11) and an efficient method for the identification of a 1-D polynomial as described below.

A. Identification of the Coefficients of a 1-D Polynomial

Let \( p(z_2) \) be a polynomial of order \( n_2 \), i.e.,

\[
p(z_2) = \alpha_0 + \alpha_1z_2 + \cdots + \alpha_{n_2}z_2^{n_2}
\] (12)

and let \( \{z_2(k), 0 \leq k \leq n_2\} \) be \((n_2 + 1)\) points that are uniformly distributed on the unit circle of the complex \( z_2 \) plane, i.e.,

\[
z_2(k) = e^{j2\pi k/(n_2+1)}, \quad 0 \leq k \leq n_2
\] (13)

If the values \( p_k = p[z_2(k)], 0 \leq k \leq n_2 \) are known, then coefficients \( \{\alpha_k, 0 \leq k \leq n_2\} \) can be identified as

\[
\alpha = \mathbf{V}^{-1}(z_2)q
\] (14)

where \( \alpha = [\alpha_0 \cdots \alpha_{n_2}]^T, q = [p_0 \cdots p_n]^T \), and \( \mathbf{V}(z_2) \) is the \((n_2 + 1) \times (n_2 + 1)\) Vandermonde matrix whose second to last column is

\[
z_2 = \begin{bmatrix} z_2(0) & z_2(1) & \cdots & z_2(n_2) \end{bmatrix}^T
\] (15)

i.e.,

\[
\mathbf{V}(z_2) = \begin{bmatrix} z_2(0)^{n_2} & \cdots & z_2(0) & 1 \\ \vdots & \vdots & \vdots & \vdots \\ z_2(n_2)^{n_2} & \cdots & z_2(n_2) & 1 \end{bmatrix}
\] (16)

Since \( z_2(k), 0 \leq k \leq n_2 \) are distinct, \( \mathbf{V}(z_2) \) is always nonsingular. More important, it follows from (13) that

\[
\mathbf{V}^H(z_2)\mathbf{V}(z_2) = (n_2 + 1)I_{n_2+1}
\] (17)

where \( \mathbf{V}^H(z_2) \) denotes the complex-conjugate transpose of \( \mathbf{V}(z_2) \). Therefore, \( \mathbf{V}(z_2)/\sqrt{n_2 + 1} \) is an unitary matrix and (14) can be written as

\[
\alpha = \frac{1}{n_2 + 1}\mathbf{V}^H(z_2)q
\] (18)

Equation (18) provides an efficient formula for the identification of the 1-D polynomial \( p(z_2) \).

B. Identification of the Coefficients of \( p_k(z_2) \) and \( q_k(z_2) \)

Throughout this subsection we assume that matrix \( A_4 \) has no eigenvalues on the unit circle. Note that this is
the case if the state-space digital filter at hand is stable [4]. As will be seen in Section III.D, the algorithm to be presented here can be readily modified to deal with the case where \( A_4 \) does have eigenvalues on the unit circle.

Given a point \( z_2 \) on the unit circle, it follows from (9) that \( E(z_2), f(z_2), g(z_2), \) and \( l(z_2) \) can be evaluated as

\[
\begin{bmatrix}
E(z_2) \\
\mathbf{f}(z_2) \\
\mathbf{g}(z_2) \\
l(z_2)
\end{bmatrix} = \begin{bmatrix}
A_1 & b_1 \\
C_1 & 0
\end{bmatrix} + \begin{bmatrix}
A_2 \\
C_2
\end{bmatrix} (z_2 I_{n_2} - A_4)^{-1} \begin{bmatrix}
A_3 \\
b_2
\end{bmatrix}
\]  

and they can, in turn, be used in (11) to obtain the values of \( p_i(z_2) \) and \( q_i(z_2) \) for \( 0 \leq i \leq n_1 \) at the given point \( z_2 \). If this procedure is applied to each of the \( n_2 + 1 \) points defined by (13), then the values of every \( p_i(z_2) \) and \( q_i(z_2) \) on the set \( \{ z_2(k) \}, 0 \leq k \leq n_2 \) can be obtained.

From these observations in conjunction with the analysis in Section III.A., we conclude that all polynomials \( p_i(z_2) \) and \( q_i(z_2) \) can be identified using the following algorithm.

**Algorithm**

**Step 1:** Use (19) to evaluate \( E(z_2), f(z_2), g(z_2), \) and \( l(z_2) \) over the set of points defined in (13).

**Step 2:** Compute the determinant of \( z_2 I_{n_2} - A_4 \) and the characteristic equations of \( E(z_2), f(z_2) - f(z_2) g(z_2) \) for \( z_2 = z_2(k) \), \( 0 \leq k \leq n_2 \).

**Step 3:** Use (11) to obtain \( p_i[z_2(k)] \) and \( q_i[z_2(k)] \) for \( 0 \leq k \leq n_2, 0 \leq i \leq n_1 \).

**Step 4:** For each \( i \) \( (0 \leq i \leq n_1) \), form vector \( q = [p_0 \cdots p_{n_2}]^T \) and identify polynomial \( p_i(z_2) \) by using (18). Similarly, form vector \( q = [q_0 \cdots q_{n_2}]^T \) and identify \( q_i(z_2) \) by using (18).

**C. The MIMO case**

As was mentioned earlier, the transfer function matrix \( H(z_1, z_2) \) given by (2) can be evaluated entry by entry and we can treat each entry as a transfer function of a SISO 2-D digital filter. In this regard, the algorithm can be implemented efficiently by extending (19) to deal with the MIMO case. As a matter of fact, we can write

\[
\begin{bmatrix}
E(z_2) \\
\mathbf{f}(z_2) \\
\mathbf{g}(z_2) \\
l(z_2)
\end{bmatrix} = \begin{bmatrix}
A_1 & B_1 \\
C_1 & 0
\end{bmatrix} + \begin{bmatrix}
A_2 \\
C_2
\end{bmatrix} (z_2 I_{n_2} - A_4)^{-1} \begin{bmatrix}
A_3 \\
b_2
\end{bmatrix}
\]  

where \( f(z_2), g(z_2) \) and \( l(z_2) \) in the \((l, k)\) entry of \( H(z_1, z_2) \) are the \( k\)th column of \( F(z_2) \), the \( l\)th row of \( G(z_2) \), and the \((l, k)\) entry of \( L(z_2) \) in (20), respectively.

**D. The Unstable Case**

When the filter is unstable, eigenvalues of \( A_4 \) with unity modulus may exist. In this case, the \( n_2 + 1 \) points defined by (13) need to be modified to

\[
z_2(k) = r e^{2\pi i k/(n_2+1)}, \quad 0 \leq k \leq n_2
\]  

where \( r > 0 \) denotes the radius of a circle in the \( z_2 \) plane where \( A_4 \) has no eigenvalues. With \( q = [p_0 \cdots p_{n_2}]^T \), equation (14) becomes

\[
\alpha = V_r^{-1}(z_2) q
\]

\[
V_r(z_2) = \begin{bmatrix}
r_1 z_2(0) & \cdots & r_2 z_2(0) & 1 \\
\vdots & \ddots & \vdots & \vdots \\
r_{n_2} z_2(n_2) & \cdots & r_2 z_2(n_2) & 1
\end{bmatrix}
\]

\[
= V(z_2) \text{diag} \{r_1, \cdots, r_{n_2}, r, 1\}
\]

where \( V(z_2) \) is the Vandermonde matrix defined by (16), and \( \text{diag} \{r_1, \cdots, r_{n_2}, r, 1\} \) is the diagonal matrix with \( r_1, \cdots, r_{n_2}, r, 1 \) as the entries along its main diagonal. By (17),

\[
V_r(z_2)^H V_r(z_2) = (n_2 + 1) \text{diag} \{r_1^2, \cdots, r_{n_2}^2, 1\}
\]

which implies that

\[
V_r^{-1}(z_2) = \frac{1}{n_2 + 1} \text{diag} \{r^{-1}, \cdots, r_{n_2}^{-1}, 1\} V_r^H(z_2)
\]

Therefore, equation (18) is modified to

\[
\alpha = \frac{1}{n_2 + 1} \text{diag} \{r^{-1}, \cdots, r_{n_2}^{-1}, 1\} V_r^H(z_2) q
\]

\[
= \frac{1}{n_2 + 1} \text{diag} \{r^{-1}, \cdots, r_{n_2}^{-1}, 1\} V_r^H(z_2) q
\]

Obviously, (18) is a special case of (23) with \( r = 1 \).

**IV. EXAMPLE**

As an example we consider the state-space model of order \((2, 6)\) that was used in [6] to demonstrate the 2-D Lyapunov theory proposed. The model is given by (4) with

\[
A = \begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}
\]

\[
A_1 = \begin{bmatrix}
0.500 & 0.007 \\
-0.007 & 0.500
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}^T
\]

\[
A_2 = \begin{bmatrix}
0.012 & -0.008 & 0.028 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.012 & 0.008 & 0.012
\end{bmatrix}
\]

\[
A_4 = \begin{bmatrix}
0.845 & -2.657 & 2.810 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -0.845 & -2.657 & -2.810
\end{bmatrix}
\]

\[
b = \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

\[
c = \begin{bmatrix}
c_1 & c_2
\end{bmatrix} = \begin{bmatrix}
0.134 & -0.657 & 0.036 & 0.269 & 0.805 & 1 & 2 \\
0.983 & 0.500 & -1 & 0 & 1 & 2 & 3 & 1
\end{bmatrix}^T
\]

\[
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\]
The algorithm presented here and that proposed in [16] lead to the transfer function

\[ H(z_1, z_2) = \begin{bmatrix} z_2^2 & \cdots & z_2 & 1 \\ z_2 & \cdots & z_1 & 1 \end{bmatrix} \begin{bmatrix} z_1^2 & z_1 & 1 \end{bmatrix}^T \]

where

\[ D_1 = \begin{bmatrix} 1.0000 & -1.0000 & 0.2500 \\ 0.0000 & 0.0000 & 0.0001 \\ -2.5821 & 2.5821 & -0.6453 \\ 0.0000 & 0.0000 & 0.0002 \\ 2.3107 & -2.3107 & 0.5778 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.7140 & 0.7140 & -0.1785 \end{bmatrix} \]

\[ N_1 = \begin{bmatrix} 0.0000 & 0.6317 & -0.3094 \\ 7.5360 & -6.3818 & 1.3419 \\ -0.8882 & 2.0949 & -0.7442 \\ -23.9776 & 25.1974 & -6.4878 \\ 8.8500 & -9.2265 & 2.5091 \\ 16.5056 & -18.8216 & 5.3540 \\ -7.9463 & 6.2887 & -1.1409 \end{bmatrix} \]

The amounts of computation required by our algorithm and that in [16] were 36,412 and 282,458 Kflops, respectively. Evidently, our algorithm leads to a significant reduction in the amount of computation relative to that in [16].

V. CONCLUSIONS

New algorithms based on a 1-D polynomial identification technique for the derivation of the transfer function matrix of a 2-D digital filter from its state-space model have been proposed. The computation efficiency of the algorithms has been examined and found to be superior relative to the algorithm described in [16]. Although only the Roesser state-space model has been considered, the method can be readily extended to the Fornasini-Marchesini state-space model.

REFERENCES


