The Z transform (3)

Today

- The inverse Z Transform
- Section 3.3 (read class notes first)
- Examples 3.9, 3.11

The inverse Z-transform

- by expressing the Z-transform as the Fourier Transform of an exponentially weighted sequence, we obtain

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- The **formal expression** of the inverse Z-transform requires the use of contour integrals in the complex plane.

Computational methods for the inverse Z-Transform

- For rational Z-transforms we can compute the inverse Z-transforms using alternative procedures:
  - Inspection (Z Transform pairs)
  - Partial Fraction Expansion
  - Power Series Expansion

Inspection method

- Makes use of common Z-Transform pairs in Table 3.1 and of the properties of the Z-Transform (Table 3.2), which we will discuss in the next lecture.
  - Most useful Z-Transform pairs: 1, 5, 6
  - Most useful property: time shifting
- The inspection method can be used by itself when determining the inverse ZT of simple sequences
- Most often, it represents the final step in the expansion-based methods

Example for the inspection method

- Consider a causal LTI system specified by its system function H(z). Compute its unit impulse response h[n].

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4} z^{-1}}$$

Table 3.1 SOME COMMON z-TRANSFORM PAIRS

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Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Table 3.2 SOME z-TRANSFORM PROPERTIES

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Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Inverse ZT via partial fraction expansion

- We will study only the case of first-order poles (all poles are distinct) and M<N.
- Equations 3.45, 3.46, 3.47 are not required
- General idea:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \xrightarrow[\text{partial fraction expansion}]{} X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- The partial fraction expansion process computes all coefficients A_k

Example for partial fraction expansion

- Compute the inverse Z-transform for:

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{3}\right)} \quad |z| > \frac{1}{3}$$

Inverse ZT via power series expansion

- We start from the definition of X(z) $X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$
- We notice that x[n] is the coefficient of n-th power of z^{-1}
- If we have the Z transform expressed as a series of powers of z^{-1}, then we can retrieve x[n] by direct identification
- Main idea

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \xrightarrow[\text{power series expansion}]{} X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

- For rational ZT: long division

Example 1 for power series expansion

- Determine the sequence x[n] corresponding to the following ZT:

$$X(z) = (1 + 2z)(1 + 3z^{-1})$$

Example 2 for power series expansion

- Determine the sequence x[n] that corresponds to the following Z Transform, knowing that this sequence is right-sided

$$X(z) = \frac{1}{1 + \frac{1}{2} z^{-1}}$$

Summary

- The inverse ZT is in general a contour integral.
- For rational ZTs, it is not necessary to explicitly compute this integral
- 3 methods:
  - Inspection (ZT pairs and properties of the ZT)
  - Partial fraction expansion
  - Power series expansion
- Which method should we choose?
  - Power series expansion is an excellent tool when the power series is finite
  - For infinite length sequences, we will work mostly with inspection and partial fraction expansion, since long division is computationally expensive