The Z transform (3)

Today

- Analysis of stability and causality of LTI systems in the Z domain
- The inverse Z Transform
  - Section 3.3 (read class notes first)
  - Examples 3.9, 3.11
- Properties of the Z Transform
  - Section 3.4
  - *All examples*

Stability, causality, and the ROC

- **We can evaluate the stability and causality of LTI systems in the Z-domain.**

- Suppose our LTI system is *given* by h[n], by $H(e^{j\omega})$ in the frequency domain, and by H(z) in the z-domain

  - The system is causal if h[n]=0 for n<0 (right-sided)

    The ROC of a causal system is the exterior of a circle (property 5), and it contains $z=\infty$

  - The system is anti-causal if h[n]=0 for n>0 (left-sided)

    The ROC of an anti-causal system is the interior of a circle (property 6) and it contains z=0.

Stability, causality, and the ROC (cont'd)

- The LTI system is stable if and only if h[n] is absolutely summable (which is equivalent to the fact that $H(e^{j\omega})$ exists)
- Using Property 2 of ROC, we conclude that:

The ROC of a stable system includes the unit circle (|z|=1)

See example 3.8

The inverse Z-transform

- by expressing the Z-transform as the Fourier Transform of an exponentially weighted sequence, we obtain

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- The **formal expression** of the inverse Z-transform requires the use of contour integrals in the complex plane.

Computational methods for the inverse Z-Transform

- For rational Z-transforms we can compute the inverse Z-transforms using alternative procedures:
  - Inspection (Z Transform pairs)
  - Partial Fraction Expansion
  - Power Series Expansion

Inspection method

- Makes use of common Z-Transform pairs in Table 3.1 and of the properties of the Z-Transform (Table 3.2).
  - Most useful Z-Transform pairs: 1, 5, 6
  - Most useful property: time shifting
- The inspection method can be used by itself when determining the inverse ZT of simple sequences
- Most often, it represents the final step in the expansion-based methods

Example for the inspection method

- Consider a causal LTI system specified by its system function H(z). Compute its unit impulse response h[n].

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4} z^{-1}}$$

Inverse ZT via partial fraction expansion

- We will study only the case of first-order poles (all poles are distinct) and M<N.
- Equations 3.45, 3.46, 3.47 are not required
- General idea:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \xrightarrow[\text{partial fraction expansion}]{} X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- The partial fraction expansion process computes all coefficients A_k

Example for partial fraction expansion

- Compute the inverse Z-transform for:

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{3}\right)} \quad |z| > \frac{1}{3}$$

Inverse ZT via power expansion

- Main idea: the expression of the Z-transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

can be interpreted as a power-series involving both positive and negative powers of z.

- The coefficients in this power series are x[n]
- For right sided signals : power expansion in negative powers of z
- For left-sided signals: power expansion in positive powers of z

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \xrightarrow[\text{power series expansion}]{} X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

Example 1 for power series expansion

- Determine the sequence x[n] corresponding to the following ZT:

$$X(z) = (1 + 2z)(1 + 3z^{-1})$$

Example 2 for power series expansion

- Determine the sequence x[n] that corresponds to the following Z Transform, knowing that this sequence is right-sided

$$X(z) = \frac{1}{1 + \frac{1}{2} z^{-1}}$$

Properties of the Z Transform

- Similar to the properties of the Fourier transform
- Additional information about how the region of convergence is affected by transforms

Table 3.1 SOME COMMON z-TRANSFORM PAIRS

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Table 3.2 SOME z-TRANSFORM PROPERTIES

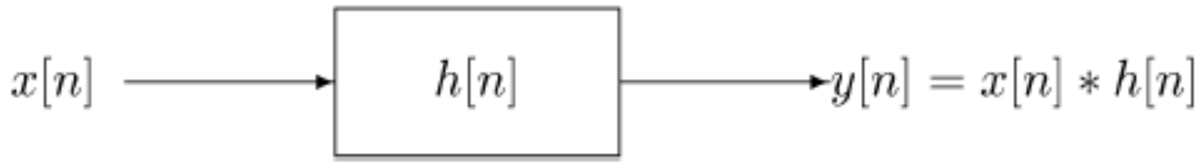
TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Important consequence of the conjugation property

- If x[n] is real and has a rational Z-transform, the complex poles and zeros of the Z-transform occur only in conjugate pairs

Convolution property and system functions



Y(z)=H(z)X(z)

ROC is at least the intersection of the ROCs of H(z) and X(z)

- can be bigger if there is zero-pole cancelation

Example: H(z)=1/(z-a) ROC: |z|>a

X(z)=z-a ROC: z≠∞

Y(z)=1 ROC: all z

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} \text{ is called the system function}$$

H(z) + ROC tell us everything about the system

Example 1.

Consider a signal y[n] which is related to two signals $x_1[n]$ and $x_2[n]$ by

$$y[n] = x_1[n+3] * x_2[-n+1]$$

$$\text{where } x_1[n] = \left(\frac{1}{2}\right)^n u[n] \text{ and } x_2[n] = \left(\frac{1}{3}\right)^n u[n]$$

Use the properties of the Z-transform to determine the Z-transform Y(z) of y[n].

Summary (1)

Evaluation of causality and stability in the Z-domain

causal LTI: H(z) has the ROC represented by the exterior of a circle and including $z=\infty$

anti-causal LTI: H(z) has the ROC represented by the interior of a circle and including $z=0$

stable LTI: the ROC of H(z) includes the unit circle.

Summary (2)

- The inverse ZT is in general a contour integral.
- For rational ZTs, it is not necessary to explicitly compute this integral
- 3 methods:
  - Inspection (ZT pairs and properties of the ZT)
  - Partial fraction expansion
  - Power series expansion
- Which method should we choose?
  - Power series expansion is an excellent tool when the power series is finite
  - For infinite length sequences, we will work mostly with inspection and partial fraction expansion, since long division is computationally expensive

Properties of the Z-Transform

- Tables 3.2, 3.3
  - Will be provided for the midterm and final exams, but you need to be familiar with them
- Very important for a variety of problems
  - Direct ZT and inverse ZT computation
  - LTI systems (conjugation)
  - Pole-zero plot
  - Etc.