University of Victoria

Department of Electrical and Computer Engineering

ECE 515 Information Theory

Fall 2024

Midterm Test

INSTRUCTIONS:
One sheet of 8.5" × 11" paper and a calculator allowed. There are 20 marks total.
State any assumptions that may be necessary. Please write clearly.

Question 1 [12 marks total]

Consider two binary random variables X and Y with joint probability distribution $p(x, y)$ as follows: $p(0,0) = 1/4$, $p(0,1) = 1/4$, $p(1,0) = 1/4$, and $p(1,1) = 1/4$.

a) [2 marks] Find $H(X)$.

b) [2 marks] Find $H(Y)$.

c) [2 marks] Find $H(X|Y)$.

d) [2 marks] Find $H(XY)$.

e) [2 marks] Find $I(X;Y)$.

f) [2 marks] Find $H(Y|X)$.

Question 2 [4 marks]

A discrete source X has an alphabet of size $K = 17$ symbols. Determine the maximum and minimum possible entropy of this source and the corresponding symbol probabilities.

Question 3 [4 marks]

Let X and Y be discrete random variables. Show that $H(X) + H(Y) \geq H(XY)$ (a Venn diagram is not sufficient). What can be said about X and Y when equality holds?

ECE 515 Fall 2024 Midterm Test Solutions

1. $p(0,0) = p(0,1) = p(1,0) = p(1,1) = 1/4$

$p(x=0) = p(x=1) = \frac{1}{2}$

$p(y=0) = p(y=1) = 1/2$

a) $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$ bit

b) $H(Y) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$ bit

c) $$H(X|Y) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(y_j)}$$

$$= -\log \frac{1/4}{1/2} = \log 2 = 1 \text{ bit}$$

d) $H(XY) = H(Y) + H(X|Y) = 2$ bits

e) $I(X;Y) = H(X) - H(X|Y) = 1 - 1 = 0$ bits

f) $$H(Y|X) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i)}$$

$$= -\log \frac{1/4}{1/2} = \log 2 = 1 \text{ bit}$$

2. Source X with $K=17$ symbols

- the minimum entropy occurs when the source is deterministic

$$p(x_i)=1 \text{ for some } i \text{ and } 0 \text{ otherwise}$$

in this case $H(X) = -\log 1 = 0$ bits

- the maximum entropy occurs when the source symbols are equiprobable

$$p(x_i) = \frac{1}{17} \text{ for all } i$$

in this case $H(X) = \log_2 17 = 4.09$ bits

3. $H(XY) = H(X) + H(Y|X)$

$H(X) + H(Y) = H(X) + H(Y|X) + I(X;Y)$

$\therefore\ H(X) + H(Y) = H(XY) + I(X;Y)$

since $I(X;Y) \geq 0$

$H(X) + H(Y) \geq H(XY)$

equality occurs when $I(X;Y) = 0$

which occurs when X and Y are independent