ECE 515 Fall 2025 Assignment 4 Solutions

1.(a) - the last step in the algorithm will combine 2 symbols while the other steps will combine 3 symbols

- for a ternary Huffman code, the number of source symbols should be

$$K' = 3 + 2c$$

where $c = \left\lceil \frac{K-3}{2} \right\rceil$

- to have a binary symbol at the last step there should be 2 symbols for the last step

therefore, the ternary Huffman algorithm should be modified so that

$$K' = 3 + 2c + 1$$

where $c = \left\lceil \frac{K-4}{2} \right\rceil$

(b) X has $K=7$ symbols

$$c = \left\lceil \frac{7-4}{2} \right\rceil = 2$$

$$K' = 3+2(2)+1 = 8$$

$\therefore$ add a symbol x_8 with $p(x_8)=0$

x_1 .23

x_2 .22

x_3 .20

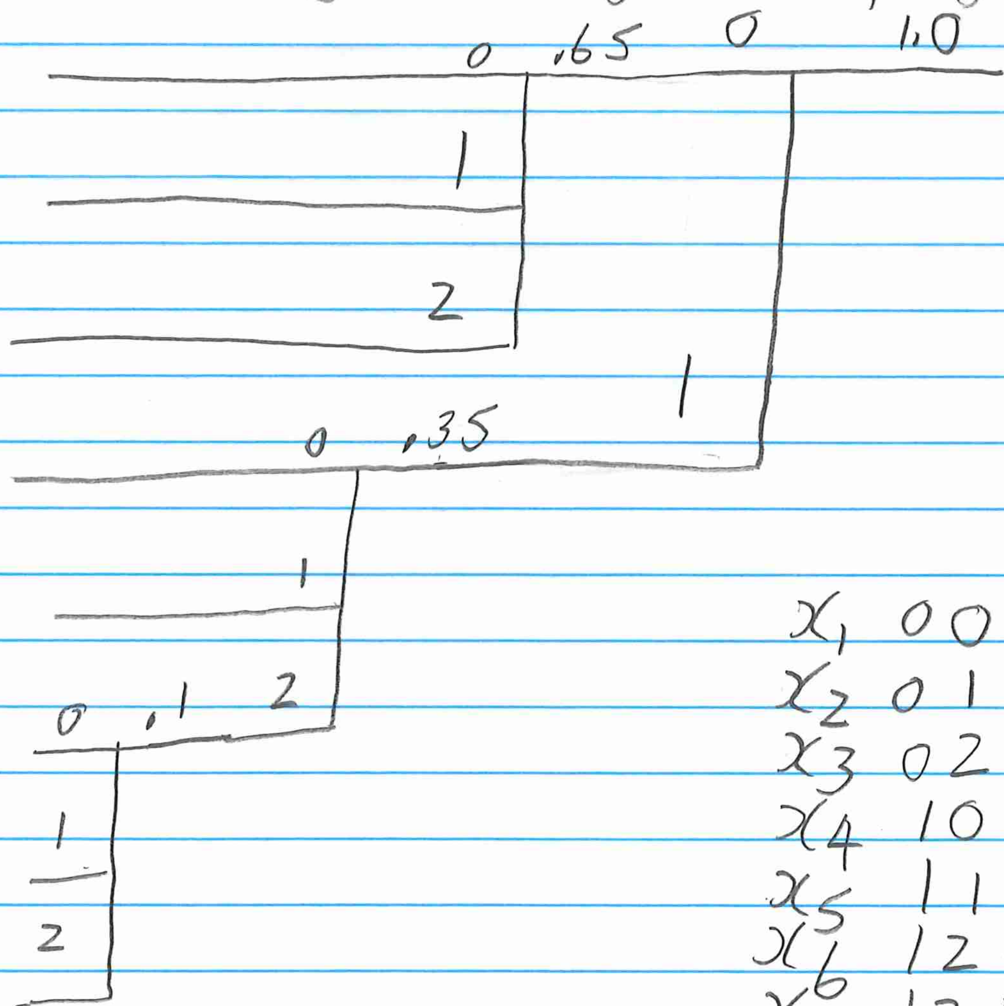
x_4 .15

x_5 .10

x_6 .05

x_7 .05

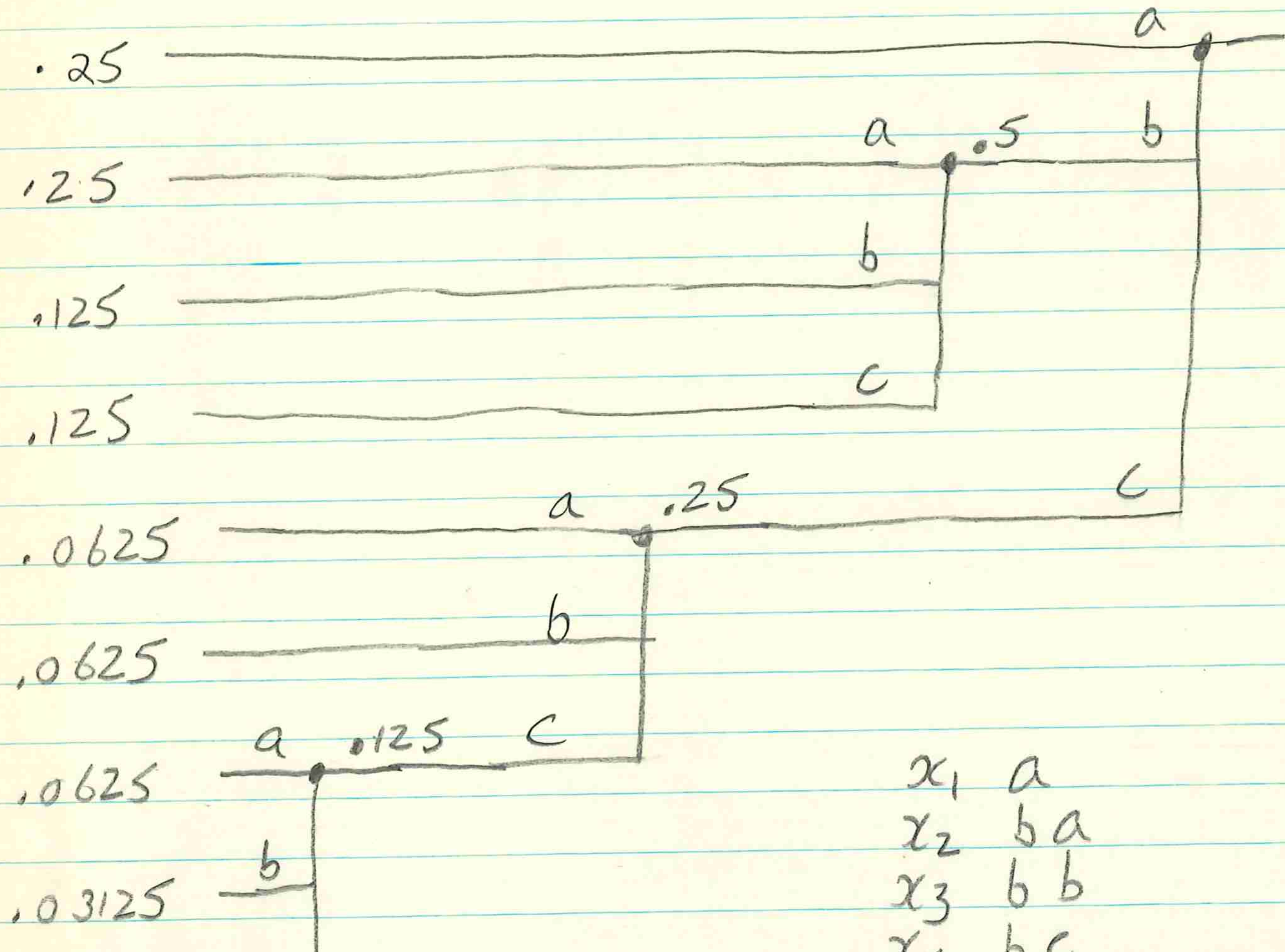
x_8 0



x_1	00
x_2	01
x_3	02
x_4	10
x_5	11
x_6	120
x_7	121
x_8	121

2. probabilities $1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/32, 1/32$

(a) Huffman code $J=3$ $K=9$ $K = J + 3(J-1)$ so no extra symbols are required



x_1 a
x_2 b a
x_3 b b
x_4 b c
x_5 c a
x_6 c b
x_7 c c a
x_8 c c b
x_9 c c c

$H(X) = .631 + .473 + .473 + .197 = 1.774$ trits

$L(C) = .25(1) + .625(2) + .125(3) = 1.875$ trits

$\zeta = 94.6\ \%$

Fano code

.25	a		
.25	b	a	
.125	b	b	
.125	c	a	
.0625	c	b	a
.0625	c	b	b
.0625	c	c	a
.03125	c	c	b
.03125	c	c	c

x_1 a
x_2 ba
x_3 bb
x_4 ca
x_5 cba
x_6 cbb
x_7 cca
x_8 ccb
x_9 ccc

$L(C) = .25 + .5(2) + .25(3) = 2.0$ trits

$\zeta = \frac{H(X)}{L(C)} = \frac{1.774}{2.0} = 88.7\ \%$

the unused symbol has a significant impact on the efficiency

To improve the Fano code, ensure there are not two symbols in a group except at the bottom (end)

.25	a		
.25	b	a	
.125	b	b	
.125	b	c	
.0625	c	a	
.0625	c	b	
.0625	c	c	a
.03125	c	c	b
.03125	c	c	c

x_1 a
x_2 ba
x_3 bb
x_4 bc
x_5 ca
x_6 cb
x_7 cca
x_8 ccb
x_9 ccc

This is identical to the Huffman code

$\eta = 94.6\ \%$

Shannon code

symbol	$p(x_k)$	P_k	l_k	codeword
x_1	$1/4$	0	$\lceil 1.26 \rceil = 2$	00
x_2	$1/4$	$1/4$	2	02
x_3	$1/8$	$1/2$	$\lceil 1.89 \rceil = 2$	11
x_4	$1/8$	$5/8$	2	12
x_5	$1/16$	$6/8$	$\lceil 2.52 \rceil = 3$	202
x_6	$1/16$	$13/16$	3	210
x_7	$1/16$	$14/16$	3	212
x_8	$1/32$	$15/16$	$\lceil 3.15 \rceil = 4$	2210
x_9	$1/32$	$31/32$	4	2220

$$L(C) = 3/4(2) + \frac{3}{16}(3) + \frac{1}{16}(4) = 2.3125 \text{ trits}$$

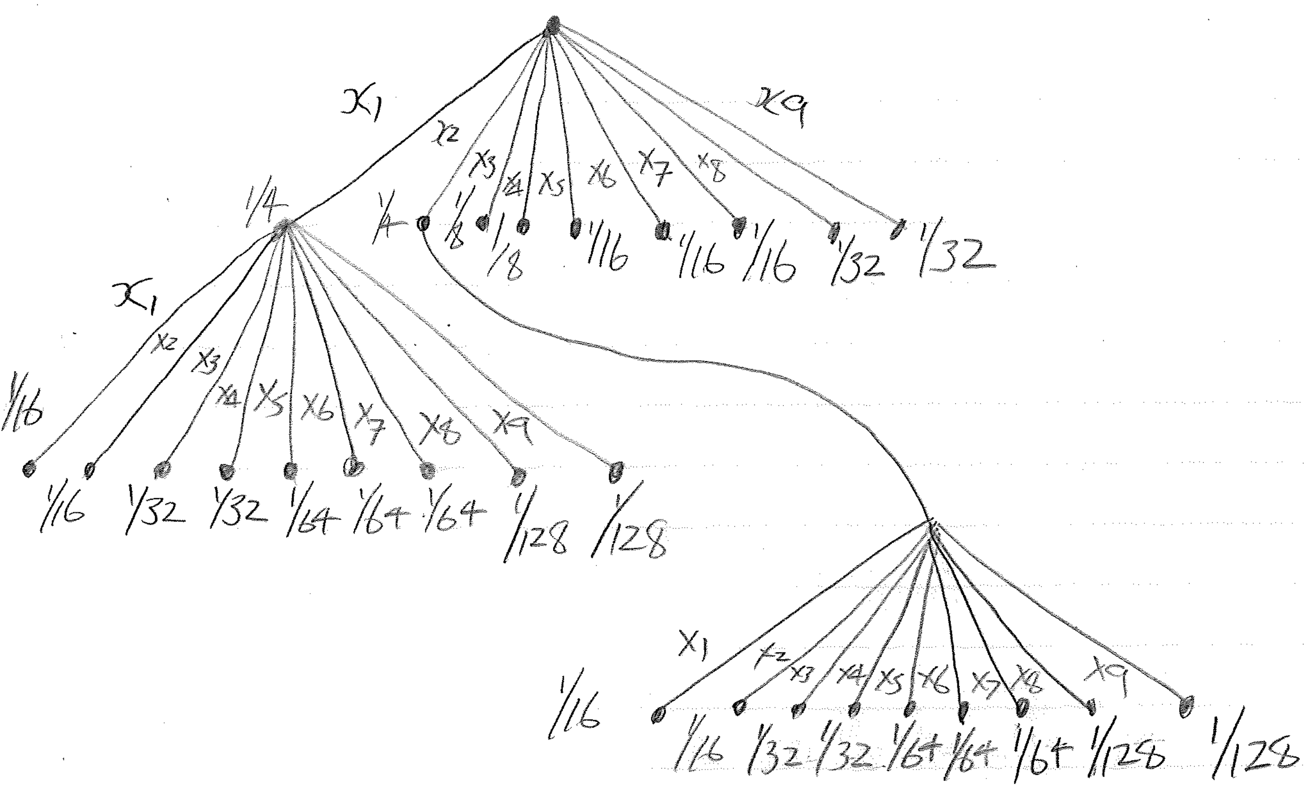
$$\zeta = \frac{1.774}{2.3125} = 76.7\%$$

much worse than the Fano code

Tunstall code

we require $3^L > K = 9$

choose $L = 3$



aaa	$x_1 x_1$	baa	$x_2 x_1$	caa	x_3
aab	$x_1 x_2$	bab	$x_2 x_2$	cab	x_4
aac	$x_1 x_3$	bac	$x_2 x_3$	cac	x_5
aba	$x_1 x_4$	bba	$x_2 x_4$	cba	x_6
abb	$x_1 x_5$	bbb	$x_2 x_5$	cbb	x_7
abc	$x_1 x_6$	bbc	$x_2 x_6$	cbc	x_8
aca	$x_1 x_7$	bca	$x_2 x_7$	cca	x_9
acb	$x_1 x_8$	bcb	$x_2 x_8$		
acc	$x_1 x_9$	bcc	$x_2 x_9$	ccb	for
				ccc	termination

$$ABR = \frac{3}{2(0.5)+0.5} = \frac{3}{1.5} = 2$$

$$\eta = \frac{H(X)}{ASR} = \frac{1.774}{2.0} = 88.7\%$$

(b) uniquely decodable code with no aa, bb or cc

- since any symbol can follow another let a be the first symbol for all codewords

then symbols can end with b or c

this gives b, c only possible 1 bit words

bc, cb only possible 2 bit words

bcb
cbc only possible 3 bit words without a

bcbc
cbcb only possible 4 bit words without a

.25	ab
.25	ac
.125	abc
.125	acb
.0625	abcb
.0625	acbc
.0625	abcbc
.03125	acbcb
.03125	acbcbc

Assign words from smallest to largest

$$L(C) = .5(2) + .25(3) + .125(4) + .09375(5) + .03125(6)$$

$$= 2.90625$$

$$\eta = \frac{H(x)}{L(C)} = 1.774 = 61.0\%$$

- significant penalty to satisfy the codeword restriction

3. Arithmetic code

$p(x_1) = 0.7$ $p(x_2) = 0.2$ $p(x_3) = 0.1$

$0 \le x_1 < .7$ $.7 \le x_2 < .9$ $.9 \le x_3 < 1$

symbol	range	low	high
x_1	1	0	.7
x_1	.7	0	.49
x_2	.49	.343	.441
x_3	.098	.4312	.441
x_2	.0098	.43806	.44002
x_1	.00196	.43806	.439432
x_3	.001372	.4392948	.439432
x_1	.0001372	.4392948	.43939084

low in binary

.01110000011101011 00

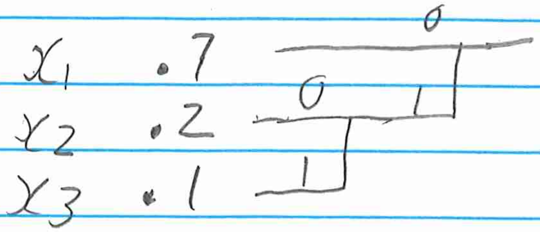
high in binary

.011100000111110111 1

they differ in the 13th bit so the codeword is

011100000111 11

b) the corresponding Huffman code is



x_1 0
x_2 10
x_3 11

$x_1 x_1 x_2 x_3 x_2 x_1 x_3 x_1$ is encoded to

001011100110 (12 bits)

the arithmetic and Huffman codewords differ by 1 bit

for $u = x_1 x_1 x_2 x_3 x_2 x_1 x_3 x_1$ $p(u) = (.7)^4 (.2)^2 (.1)^2$

$= 9.604 \times 10^{-5}$

$l_u = \lceil -\log_2 p(u) \rceil = \lceil 13.35 \rceil = 14$ bits

- the theoretical codeword length differs from the arithmetic codeword length by only 1 bit.
- the Huffman and arithmetic code solutions differ by only one bit

4. source $x_1\ x_2\ x_3$ with $p(x_1) = .8$ $p(x_2) = .02$
$p(x_3) = .18$

decode $.110001$ $P_0 = 0$ $P_1 = .8$ $P_2 = .82$ $P_3 = 1$

in decimal the codeword is $.765625$

number	symbol	low	high	range
.765625	x_1	0	.8	.8
.95703125	x_3	.82	1	.18
.761284722	x_1	0	.8	.8
.951605903	x_3	.82	1	.18
.731143904	x_1	0	.8	.8
.91392988	x_3	.82	1	.18
.521832667	x_1			

the output sequence is $x_1 x_3 x_1 x_3 x_1 x_3 x_1$

$H(X) = .8157$

thus 7 symbols can be expected to compress to

$$7H(X) = 5.7 \text{ bits} \approx 6 \text{ bits}$$

which is similar to the length of the arithmetic codeword the theoretical codeword length is

$p(u) = (.8)^4(.18)^3 = .002389$

$-\log_2 p(u) = 8.710$

$l_u = \lceil 8.710 \rceil = 9$ bits ~~which is 3 bits~~

which is 3 bits longer than the actual codeword length

5. Suffix code

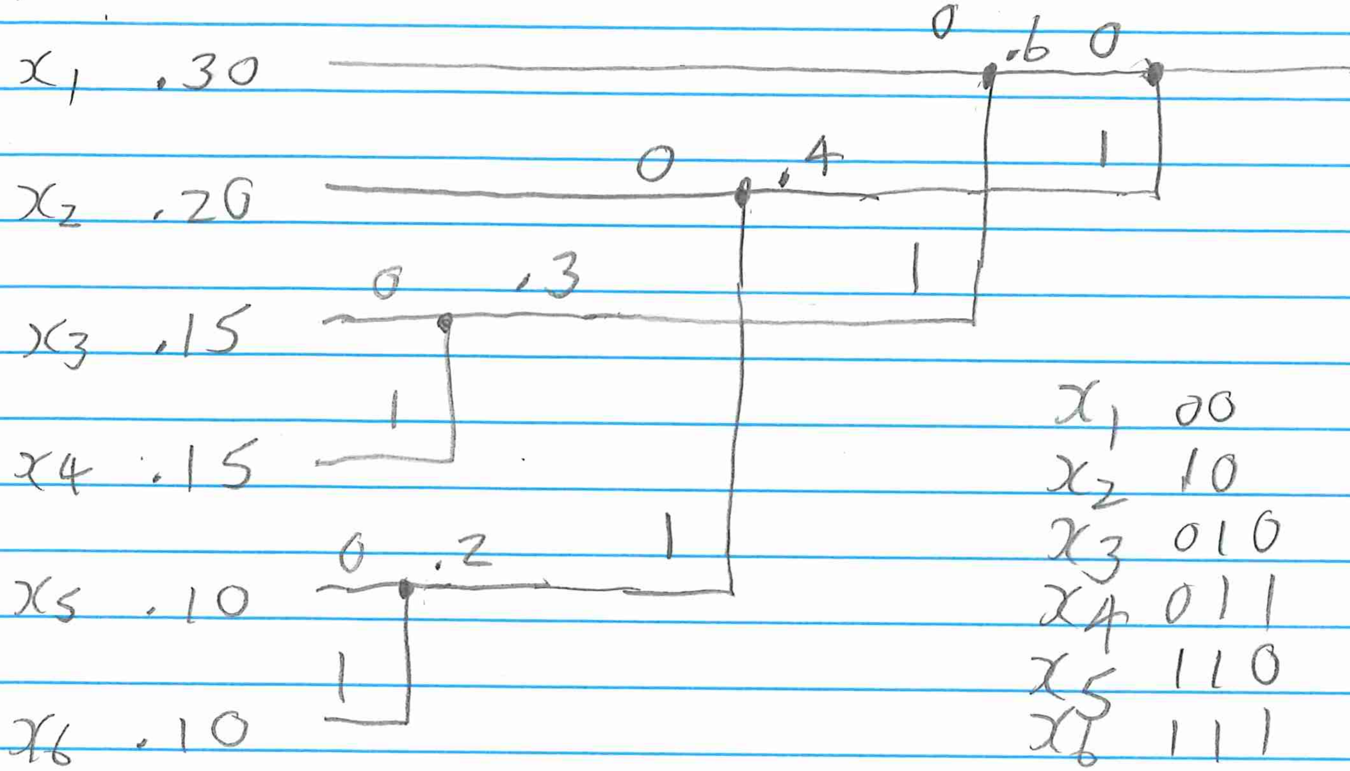
(a) Note that if each codeword in a prefix code is reversed, the resulting code satisfies the suffix condition

This code must be uniquely decodable

If it were not, two sequences of codewords could be reversed, resulting in a non-unique decoding for the corresponding prefix code

(b) Since reversing codewords does not change their lengths, a minimum average codeword length suffix code can be obtained by first finding the Huffman code and then reversing the codewords

(c) prefix code



x_1	00
x_2	10
x_3	010
x_4	011
x_5	110
x_6	111

suffix code

x_1	00
x_2	01
x_3	010
x_4	110
x_5	011
x_6	111

6. $L=6$ $p(x_1)=.9$ $p(x_2)=.1$

$H(X)=.4690$

code rate $R=\frac{L}{N}>H(X)$

possible values of N are

7, 8, 9, 10, 11, 12

require the lowest possible actual code rate

choose $2^L-1=63$ typical sequences

encode each to a codeword

the remaining codeword is used for the

atypical sequences

begin with $N=12$ to ensure a low code rate

$p(\text{typical}) = .9^{12} + 12(.9)^{11}(.1) + 50(.9)^{10}(.1)^2 = .8333$

there are $2^{12}-63 = 4033$ atypical sequences

representing these require $2 \times 6 = 12$ bits

the actual code rate is

$$R = \frac{6(.8333) + 18(.1667)}{12} = .6667$$

for $N=11$ $p(\text{typical}) = .9^{11} + 11(.9)^{10}(.1) + 51(.9)^{9}(.1)^2 = .8949$

the actual code rate is

$$R = \frac{6(.8949) + 18(.1051)}{11} = .6601$$

$N=10$ $R = .6802$

$N=9$ $R = .7253$

$N=8$ $R = .7841$

$N=7$ $R = .8629$

7. Show that

$$\frac{H(p(X)) + D(p(X)\|q(X))}{\log_b J} \leq L(\hat{C}) < \frac{H(p(X)) + D(p(X)\|q(X))}{\log_b J} + 1$$

consider the Shannon code

$$\hat{l}_k = \lceil -\log_J q(x_k) \rceil$$

$$L(\hat{C}) = \sum_{k=1}^{K} p(x_k) \left\lceil \frac{-\log_b q(x_k)}{\log_b J} \right\rceil$$

$$< \sum_{k=1}^{K} p(x_k) \left(\frac{-\log_b q(x_k)}{\log_b J} + 1 \right)$$

$$= \sum_{k=1}^{K} p(x_k) \left[\frac{\log_b\left(\frac{p(x_k)}{q(x_k)} \cdot \frac{1}{p(x_k)} \right)}{\log_b J} + 1 \right]$$

$$= -\sum_{k=1}^{K} p(x_k) \frac{\log_b p(x_k)}{\log_b J}$$

$$+ \sum_{k=1}^{K} p(x_k) \frac{\log_b \left(\frac{p(x_k)}{q(x_k)} \right)}{\log_b J} \qquad + \sum_{k=1}^{K} p(x_k)$$

$$= \frac{H(p(X)) + D(p(X) \| q(X))}{\log_b J} \qquad + 1$$

the lower bound is obtained by considering that

$$-\sum_{k=1}^{K} p(x_k) \log_J q(x_k) \leq \sum_{k=1}^{K} p(x_k) \lceil -\log_J q(x_k) \rceil$$