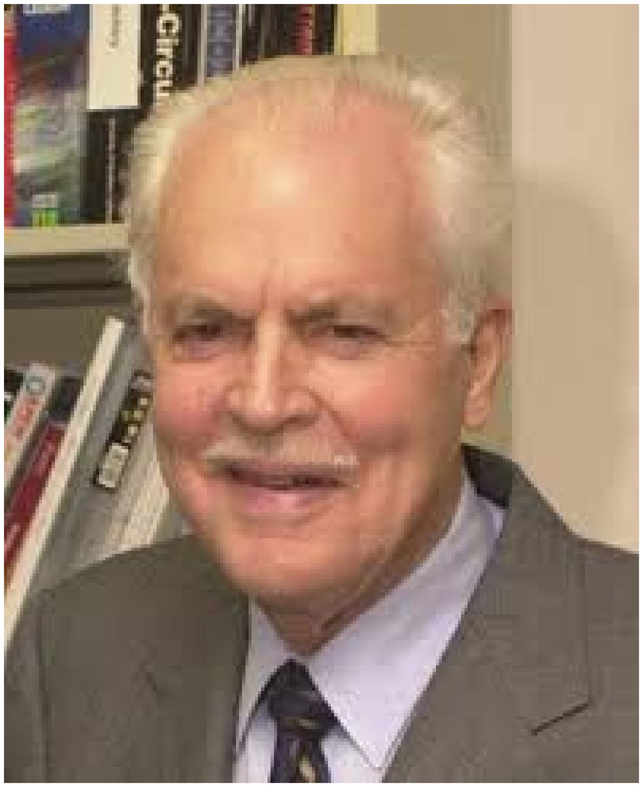
ELEC 405/511
Error Control Coding

Reed-Solomon Codes

Irving Reed (1923-2012) Gus Solomon (1930-1996)



Polynomial Codes Over Certain Finite Fields, 1960

Reed-Solomon Codes

- Nonbinary BCH codes
- Consider GF(q) ($q=p^r$, p prime)
- To construct a t error correcting nonbinary BCH code with symbols from GF(q), we use the same technique as for binary BCH codes.
- Roots of $g(x)$ are in GF(q^m), $n|q^m-1$

  $n-k \leq 2mt$ product of at most $2t$ minimal polynomials of degree m

  $d \geq 2t+1$

- Choose $2t$ consecutive powers of α, an element of order n in GF(q^m), as roots of $g(x)$.
- For RS codes, m=1 and α is a primitive element in GF(q), then

$$n = q-1$$

$$n-k \leq 2t \longrightarrow n-k = 2t$$

$$d \geq 2t+1 \longrightarrow d \geq n-k+1$$

Singleton Bound

- Theorem 4-10 Singleton bound

  The minimum distance for an (n,k) linear code is bounded by

  $$d \leq n-k+1$$

- For an RS code $d \geq n-k+1$, so $d = n-k+1$ and all RS codes meet the Singleton bound
  - they are optimal $(n,k,n-k+1)$ codes, $n = q-1$
- Codes that meet the Singleton bound are called Maximum Distance Separable (MDS)

Reed-Solomon Codes – Minimal Polynomials

- Coefficients of $g(x)$ are in GF(q), roots of $g(x)$ are also in GF(q).
- Minimal polynomial of α is x-α. There are no conjugates since $\alpha^q = \alpha^{q-1}\alpha = \alpha$.
- BCH: $M_1(x) = (x-\alpha)(x-\alpha^q)(x-\alpha^{q^2})\cdots$

  RS: $M_1(x) = (x-\alpha)$
- RS codes are a subclass of BCH codes with m = 1.

Example 8-4 t=2 GF(8)

- n = 8-1 = 7 Form GF(8) from x^3+x+1

α^0	1
α^1	α
α^2	α^2
α^3	$\alpha+1$
α^4	$\alpha^2+\alpha$
α^5	$\alpha^2+\alpha+1$
α^6	α^2+1

$$g(x)=(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$
$$=x^4+\alpha^3x^3+x^2+\alpha x+\alpha^3$$

$$\mathbf{H}=\begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \alpha & \alpha^3 & \alpha^5 \\ 1 & \alpha^3 & \alpha^6 & \alpha^2 & \alpha^5 & \alpha & \alpha^4 \\ 1 & \alpha^4 & \alpha^8 & \alpha^5 & \alpha^2 & \alpha^6 & \alpha^3 \end{bmatrix}$$

- (7,3,5) RS code

Comparison: RS vs Binary BCH

- RS: $n = q^m$-1 $q = 8, m = 1$ (7,3,5)

$$g(x) = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

- Binary BCH: $n = q^m$-1 $q = 2, m = 3$ (7,1,7)

$$g(x) = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)\boxed{(x-\alpha^6)(x-\alpha^5)}$$

- RS code: $q^k = 8^3 = 512$ codewords
- Binary BCH code: $q^k = 2^1 = 2$ codewords

Comparison: RS vs Binary BCH

- (7,3,5) RS code: $8^3 = 512$ codewords $= 2^9$

- Each symbol can be represented as 3 bits, a codeword has $n = 7$ symbols = 21 bits and $k = 3$ data symbols = 9 bits.
- The (7,3,5) RS code can be considered as a (21,9) binary code.
- $t = 2$ symbols - since 5 bit errors may cover 3 symbols, corrects any burst error of 4 bits or less.

Example 8-5 t=3 GF(64)

- n = 64-1 = 63
- α a root of the primitive polynomial x^6+x+1

$$g(x) = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)(x-\alpha^5)(x-\alpha^6)$$
$$= x^6 + \alpha^{59}x^5 + \alpha^{48}x^4 + \alpha^{43}x^3 + \alpha^{55}x^2 + \alpha^{10}x + \alpha^{21}$$

- (63,57,7) RS code
- $64^{57} = 8.96\text{x}10^{102}$ codewords
- $64^{63} = 6.16\text{x}10^{113}$ vectors
- sphere volume is $9.94\text{x}10^9$ so the spheres fill about 14.5% of the vector space

GF(7) Example

- RS codes can be constructed over any finite field
- Consider $q = 7$ so that $n = q\text{-}1 = 6$, and $t = 2$
- First find a primitive element in GF(7)

ø(6) = 2 so two primitive elements

$3^1=3 \quad 3^2=2 \quad 3^3=6 \quad 3^4=4 \quad 3^5=5 \quad 3^6=1 \longrightarrow$ 3 is primitive

$b=1 \qquad g(x) = (x-3^1)(x-3^2)(x-3^3)(x-3^4)$

$\qquad\qquad = (x-3)(x-2)(x-6)(x-4)$ (6,2,5) RS code

$b=2 \qquad g(x) = (x-3^2)(x-3^3)(x-3^4)(x-3^5)$

$\qquad\qquad = (x-2)(x-6)(x-4)(x-5)$ (6,2,5) RS code

- One can pick any group of consecutive roots

$$
\begin{aligned}
g(x) &= (x-3^1)(x-3^2)(x-3^3) \\
&= (x-3)(x-2)(x-6) && \text{(6,3,4) RS code} \\
&= x^3+3x^2+x+6
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= (x-3^2)(x-3^3)(x-3^4) \\
&= (x-2)(x-6)(x-4) && \text{(6,3,4) RS code} \\
&= x^3+2x^2+2x+1 = g^*(x) && \text{self reciprocal}
\end{aligned}
$$

$$
\begin{aligned}
g(x) &= (x-3^1)(x-3^2)(x-3^3)(x-3^4)(x-3^5) \\
&= (x-3)(x-2)(x-6)(x-4)(x-5) && \text{(6,1,6) RS code} \\
&= x^5+x^4+x^3+x^2+x+1 = g^*(x) && \text{self reciprocal}
\end{aligned}
$$

Properties of RS Codes

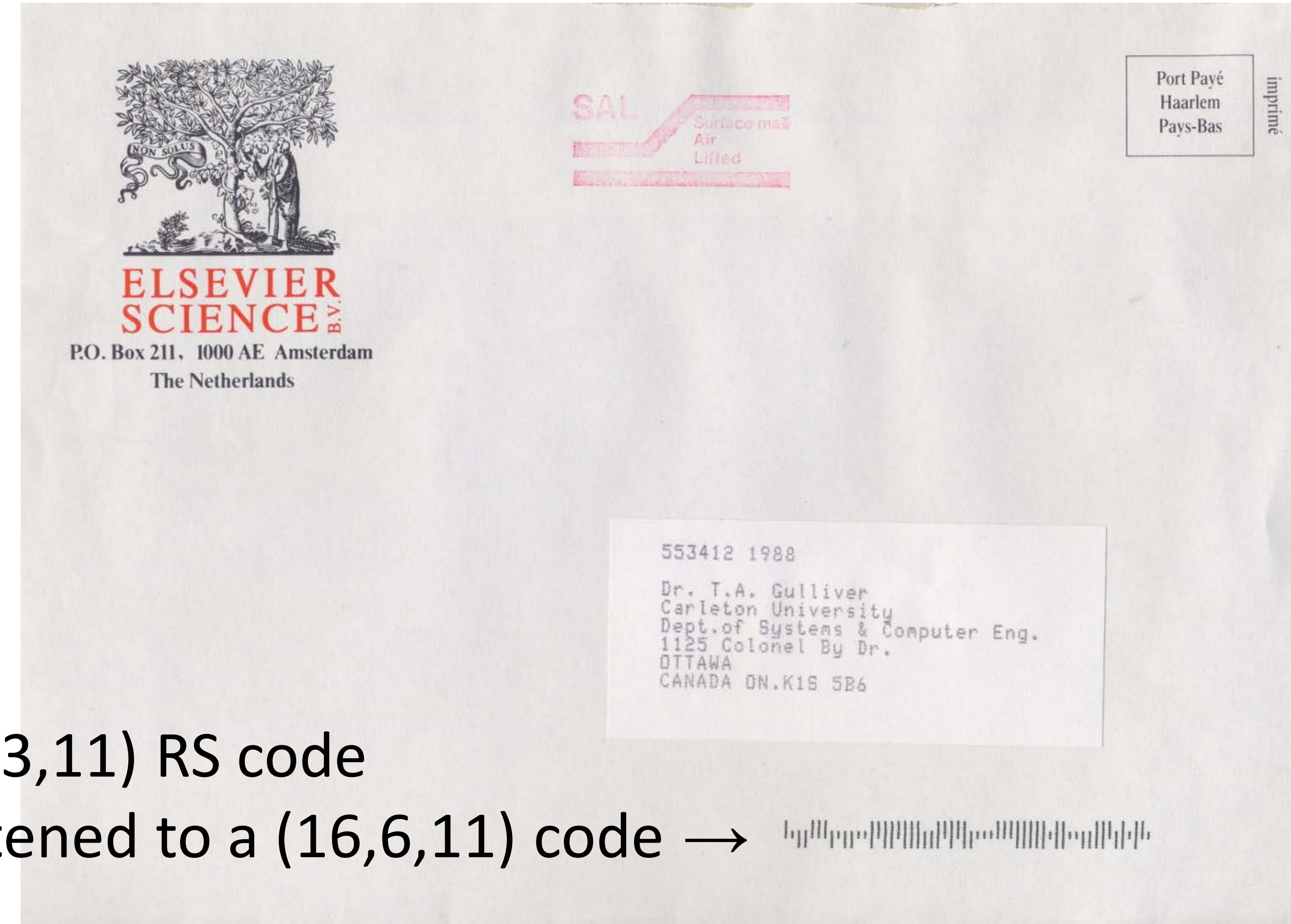
- The dual code of an RS code is also MDS
  - C (6,2,5) code over GF(7)
  - $C^{\perp}$ (6,4,3) code over GF(7)
- Since RS codes are cyclic codes, they can always be put in systematic form $x^{n-k}m(x)+d(x)$
- A shortened RS codes is MDS

$$(n,k,n\text{-}k+1) \longrightarrow (n\text{-}u,k\text{-}u,n\text{-}k+1) \quad (6,4,3) \longrightarrow (5,3,3)$$

- A punctured RS code is MDS

$$(n,k,n\text{-}k+1) \longrightarrow (n\text{-}u,k,n\text{-}k\text{-}u+1) \quad (6,4,3) \longrightarrow (5,4,2)$$

Example: Bar Codes over GF(64)



(63,53,11) RS code
shortened to a (16,6,11) code ⟶

Extended RS Codes

- An (n,k) RS code over GF(q) with $n = q$-1 can be extended twice to a $(q+1,k)$ MDS code
- There is a technique for constructing such codes which are cyclic
- A very few RS codes can be triply extended to obtain an MDS code
  - $k = 3$ or n-$k = 3$ and $q = 2^m$
  - In this case $n = q+2$

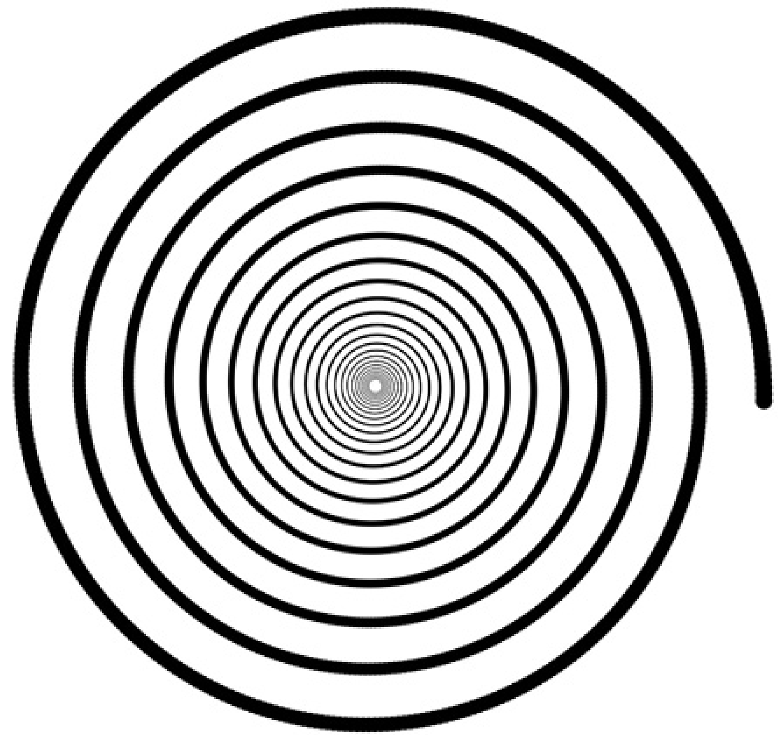
Example: NASA/JPL Code

- q = 256, n = q-1 = 255
- (255,223,33) RS code over GF(2^8)

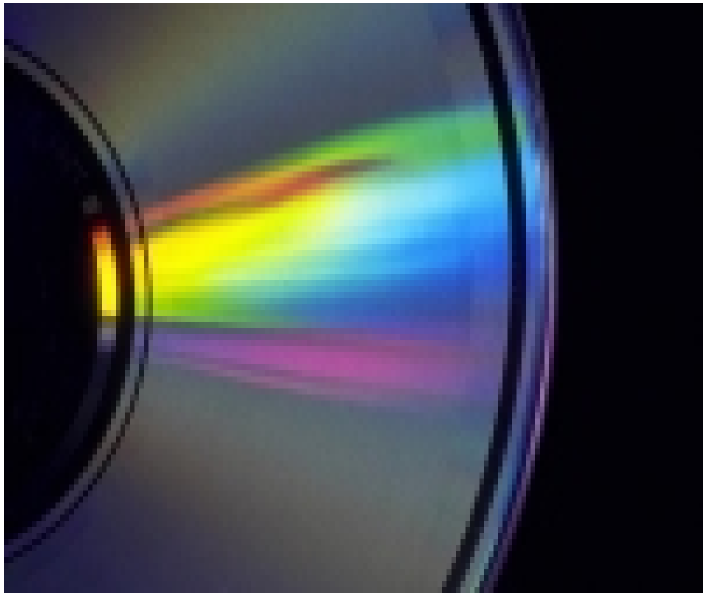
$$\frac{\text{\# of codewords} \times \text{volume}}{\text{size of vector space}} = 2.78 \times 10^{-14}$$

Example: Compact Discs

- 44.1 kHz sample rate
- 16 bit stereo samples
- $2 \times 16 \times 44100 = 1.41$ Mbps
- Original CD capacity: 74 minutes of audio or 650 MB of data
- Data stored on a spiral, not concentric circles
  – length 5.38 km
  – velocity 1.2 m/s



Kees Schouhamer Immink (1946 -)



Sources of Error

1) Defects caused during disc production
    - inferior disc pits and bubbles during disc formation
    - defects in the aluminum film and a poor reflective index
2) Defects caused in handling
    - fingerprints and scratches
    - dust
3) Variations and disturbances during playback
    - disturbance of the servo mechanism
4) Jitter - time variation of the signal
5) Interference

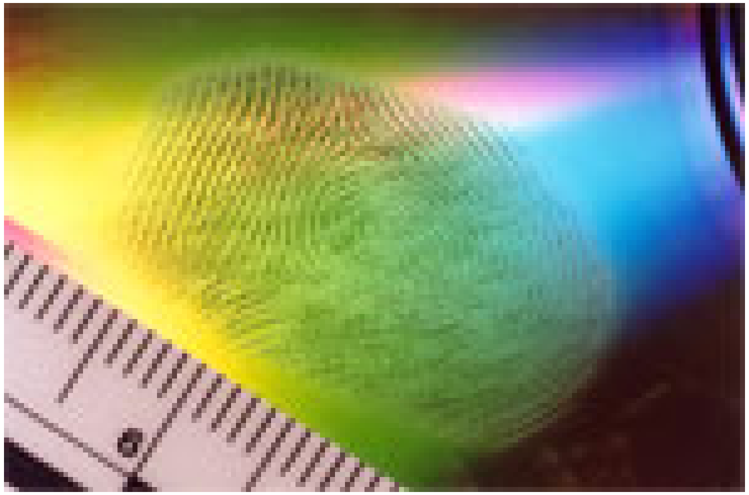
(1)-(3) cause burst errors

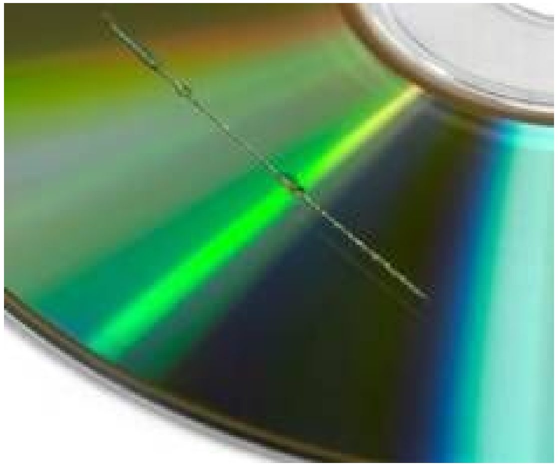
(4) and (5) cause random errors

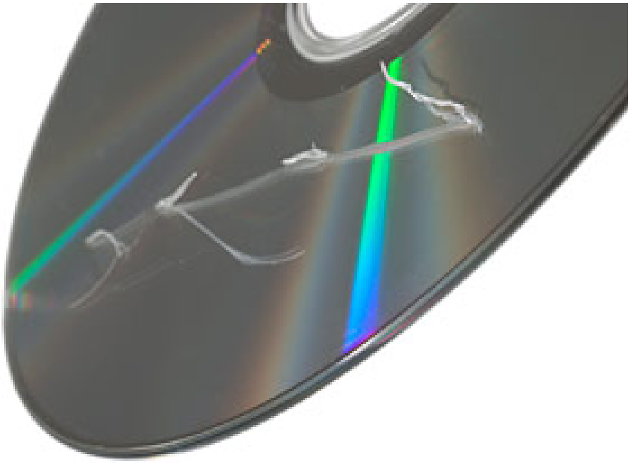
Causes of Disc Errors

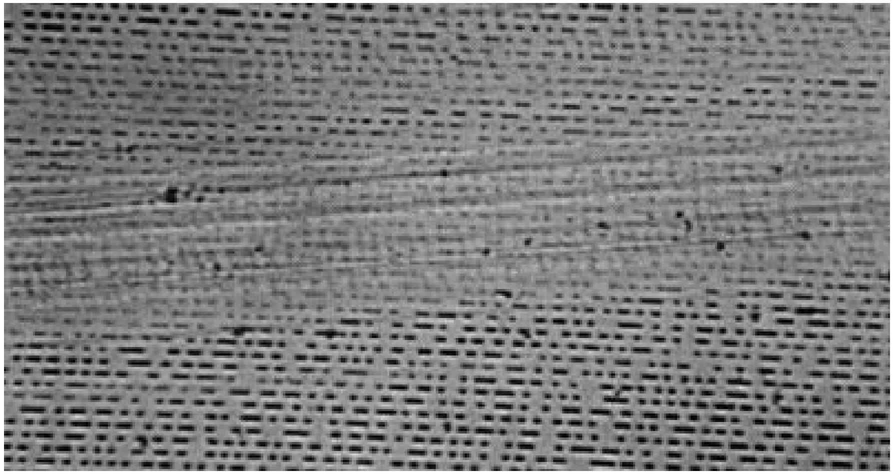
- Fingerprints cause 43% of errors
- General wear and tear causes 25% of errors
- Player-related issues cause 15% of errors
- User-related issues cause 12% of errors
- Manufacturer errors cause 2% of errors

Causes of Disc Errors









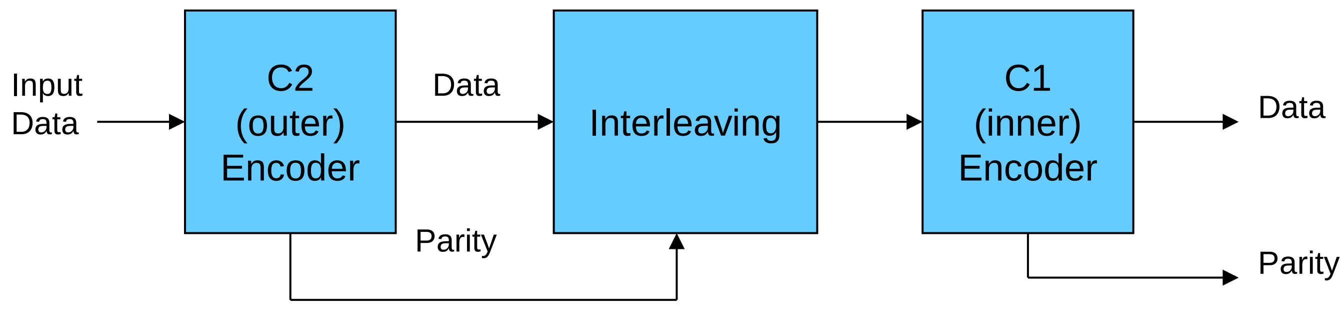
Error Correction

- Reed-Solomon code
  - (255,251,5) code over GF(2^8)
- Shortened to a (28,24,5) outer code
- These codewords are interleaved to reduce the effects of burst errors
- (32,28,5) inner code
- Overall code rate is

$$\frac{24}{28} \times \frac{28}{32} = 0.75$$

CIRC Encoder

- CIRC – Cross Interleaved Reed-Solomon Code



- Interleaving disperses the codewords so they are not contiguous on the disc
- mitigates long burst errors associated with scratches and fingerprints
  - Maximum correctable burst error length
    - 4000 bits = 2.5 mm

Encoding Algorithm

- Samples are split into two 8 bit symbols
- Six samples from each channel are grouped to obtain 24 symbols
- Four outer RS code parity symbols are generated to give a frame of 28 symbols
- Symbols are interleaved over 109 frames
- Four inner RS parity symbols are generated to give 32 symbols
- These frames are also interleaved

Control and Error Correction

- Skips are caused by physical disturbances
  - Wait for disturbance to subside
  - Retry
- Read errors caused by disc/servo problems
  - Detect error
  - Choose location for retry
  - Retry, if it fails interpolate if applicable

Interpolation

- Used when decoding fails
- Fill missing audio data using adjacent data
  - time or channel
- Only valid for audio CDs

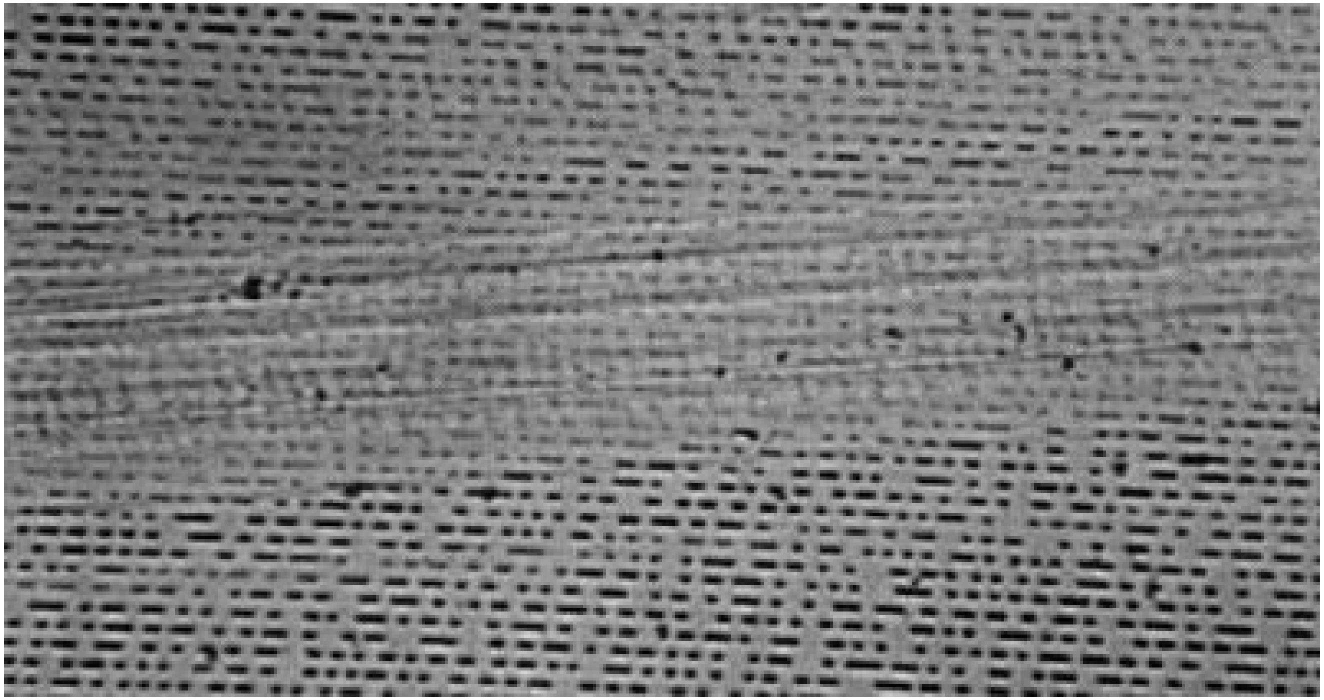
Decoding RS Codes

1. Compute the syndromes
2. Determine the error locator polynomial $\Lambda(x)$
3. Determine the error magnitudes from $\Lambda'(x)$ and $\Omega(x)$

$$\Omega(x) = [1 + S(x)]\Lambda(x)$$

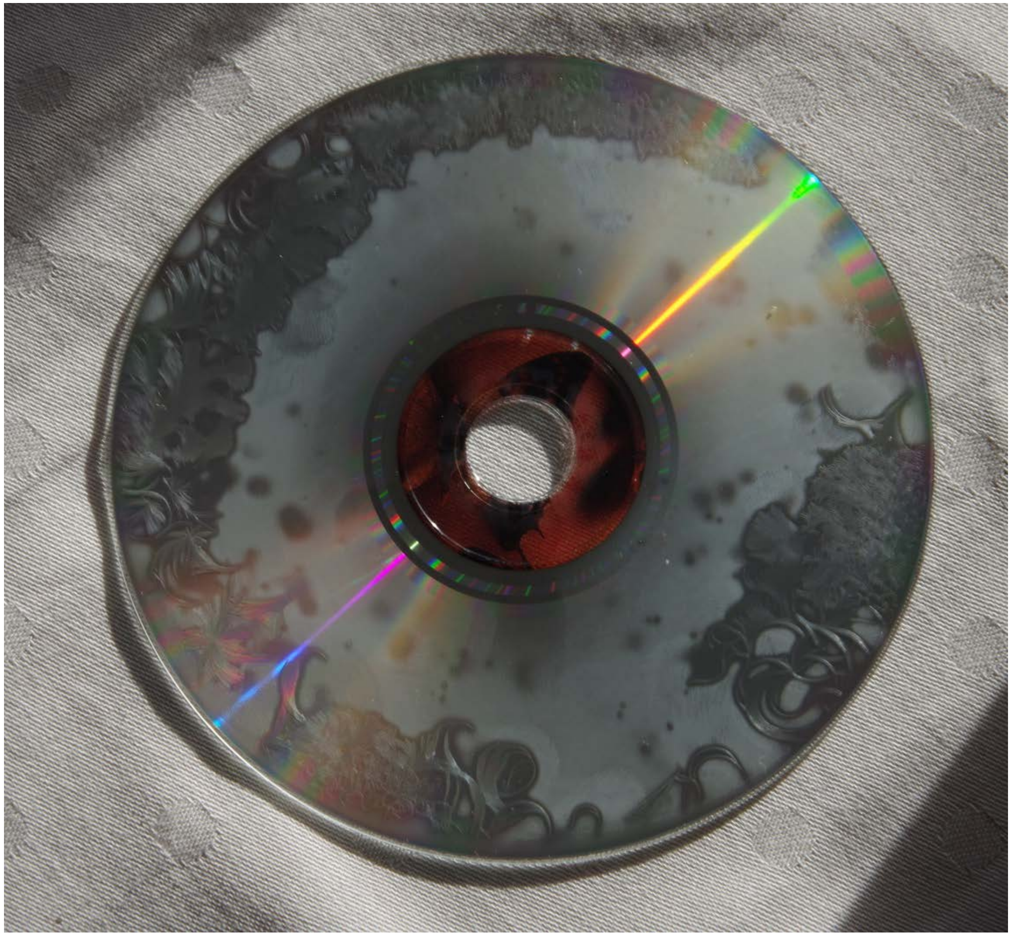
4. Evaluate the error locations and the error values at those locations.

CD Errors due to a Ball Point Pen



A Highly Corroded Disc

- Two minutes can still be played.



Audio Data Format