ELEC 250:

LINEAR CIRCUITS I

COURSE OVERHEADS

These overheads are adapted from the Elec 250 Course Pack developed by Dr. Fayez Guibaly.

Introduction

Chapter 1
Circuit Variables and Elements

Chapter 2
Resistive Circuits

Chapter 3
Network Theorems

Chapter 4
Node Voltage Analysis

Chapter 5
Mesh Current Analysis

Chapter 6
Energy Storage Elements
(Inductance and Capacitance)

Chapter 7
First-Order RC and RL Circuits

Chapter 8
Second-Order RC and RL Circuits

Chapter 9
Phasors

Chapter 10
AC Analysis Using Phasors

Chapter 11
AC Power
Sections 10.1 - 10.7 (Text)

Chapter 12
Series and Parallel Resonance

Chapter 13
Mutual Inductance

Chapter 14
Balanced Three-Phase Circuits

Chapter 15
Operational Amplifiers

Introduction

Course Purpose

- The introduction of the basic elements used in analog electric circuit and some of the core analysis methods.
  - We do not cover the transistor (basis for digital circuits)
  - We also do not cover Laplace transform (3rd year - simplifies frequency analysis)
- For Elec and Ceng students this course is the basic building block for many courses in 3rd and 4th year. For example:
  - Digital Signal processing (3rd and 4th years)
  - Transistor Circuits (3rd year)
  - Digital Design Courses (3rd and 4th years)
  - Power (4th year)
  - *etc*.

If you learning this material now it will make 3rd and 4th year easier.

- For Mech students this course is the core introduction to electric circuits and circuit theory
    - Use for understanding/employing sensors (*i.e.,* stress sensors change resistance with applied stress)
    - For connecting electronic control (*e.g.* microcontrollers) to mechanical systems.
    - Mechatronics (4th year option)
    - Pure mechanical systems rarely exist - almost everything is connected back to a computer, to electronic sensors, to electronic controllers, *etc*.
    - Industrial mechanical systems are high voltage high current systems - safety.
    - *etc*.
- If you want to do well in the course:
    - **Don't sit at the back!**
        - The room is large (~100 students in a 300-seat lecture hall)
        - Siting 50+ feet from the black board and overheads is not going to improve the learning experience.
        - The lights in the back of the lecture hall will be turn off - if the light above you is off you are sitting too far back.

- **DO PROBLEMS, DO PROBLEMS, DO PROBLEMS**
  - Don't just copy solutions (for any engineering course, you learn best by doing)
  - More importantly, you figure out what you don't know or don't quite understand by trying problems.
- **Read the text.**
- **Attend the lectures** (even at 8:30 am)
  - A core focus of the lectures is to show you how to solve problems.
  - In the lectures we talk about where the common mistakes are made and how to avoid them.
- **Attend the tutorials**
  - To review and sometimes get a different 'take' on the material and to clarify the fine points.
  - 5% of your course mark will come from single question quizzes in each tutorial.

- The course is structured to provide 3 different ‘takes’ on the material
  - text,
  - lectures and lecture notes,
  - and the tutorials).

If you don’t “get it” through one of them then you may “get it” through one of the others.

- **Try problems:**
  - Engineering is not a field where pure memorization works - understanding the concepts and theories and, more importantly, how and when to apply them is what is important.
  - It is not unusual for there to be more than one valid path to arrive at a solution (or multiple paths that look valid) - but, usually one path is better than the others *for the particular problem*.
  - Knowing what this better path is and why is what distinguishes good engineers from poor ones.

  - Real-world engineering mistakes tend to arise due to either: (a) invalid assumptions, and/or (b) error prone solution paths (*i.e.*, poor solution paths).

  - The only way to gain the understanding and knowledge (and intuition) required to do good engineering is by doing the work of working through problems.

- **If you do not understand something - ASK**

  - in the lectures,

  - in the tutorials,

  - in the lab,

  - during the office hours,

  - by e-mail,

  - *etc*.

<u>For checking correctness:</u> usually, within the context of the course material we cover, independent approaches exist which allow you to check you answers (or at least part of your answers) - for the circuits we deal with there is only one correct answer.

***This is a very good skill to master - real-world engineering has no solution sets**.*

- **Marking**
  - Standard Engineering Grading Scheme (see handout)
    - Assignments 10% (one assignment per chapter)
    - Lab 10%
    - Midterm 30% (June 26th)
    - Tutorial Quizzes 5%
    - Final 45%
- **Course Web Site:** http://www.ece.uvic.ca/~sneville/Teaching/Elec250
  - Office hours and contact information and all lecture notes are on the web site.
  - **IMPORTANT:**
    - All Elec 250 e-mail must have "*Elec 250:*" as the beginning of the subject line.
    - E-mail must come from uvic.ca domain to bypass spam filters.

Course Outline

Chapter 1: Circuit Variables and Circuit Elements
Chapter 2: Resistive Circuits
Chapter 3: Network Theorems
Chapter 4: Node Voltage Analysis
Chapter 5: Mesh Current Analysis
Chapter 6: Energy Storage Elements (Inductance and Capacitance)
Chapter 7: First-Order RC and RL Circuits
Chapter 8: Second-Order Circuits
Chapter 9: Phasors
Chapter 10: AC Analysis Using Phasors
Chapter 11: AC Power
Chapter 12: Series and Parallel Resonance
Chapter 13: Mutual Inductance
Chapter 14: Balanced Three-Phase Circuits
Chapter 15: Operational Amplifiers

Format:

- Theory will be presented on the overheads
- Examples will then be worked on the board.
    - Solutions to lecture note examples will only be provided as part of the lectures
    - They will not be posted on the web.
- The overheads only present an overview of the material (*i.e.*, the key concepts)
    - They are not intended to be a substitute for reading the text book.
    - Test material will be inclusive of what is covered both in the text book and on the lecture slides.
- In class interaction is expected:
    - If you have a question **- ASK!**
    - If you do not understand something - **ASK!**
    - If you are not following something then most likely the person next to you isn't either - **ASK!**

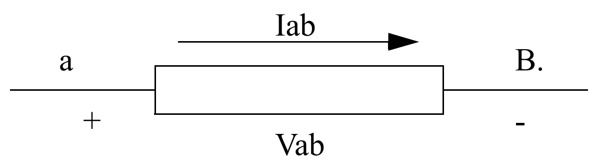
0.1. Basic Symbols

Quantity	Symbol	Units
Time	t	seconds (s)
Resistance	R	Ohms (Ω)
Capacitance	C	Farads (F)
Inductance	L	Henries (H)
Voltage	V, v or v(t)	Volts (V)
Current	I, i or i(t)	Amperes (A)

- Convention:
  - Upper case symbols are constants
  - Lower case symbols are variables (or time varying parameters)

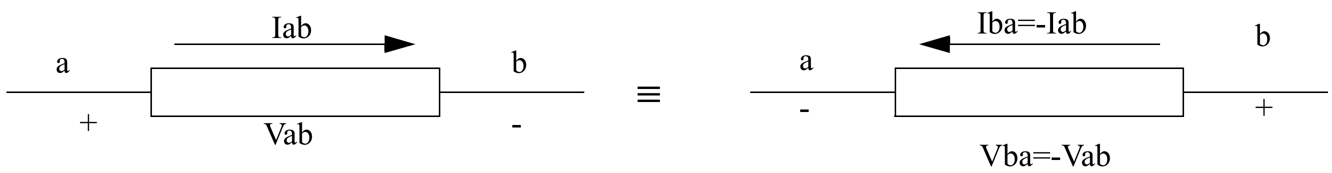
0.2. Circuit Elements and their Circuit Symbols:

- Generic Passive Circuit Element (*Consumes Power*)



  - Arrow shows direction of current flow.
  - +/- show direction of the voltage drop.
  - Note the convention is that current is the flow of *positive* charge.
  - Actual electron flow is opposite to the current flow
  - *i.e.,* in a battery the electrons flow from the negative terminal through the circuit to the positive terminal *but* the convention for the *current,* I, is that it flows from the positive terminal to the negative one (*i.e.,* the arrow points the opposite way from how the electrons move).
  - B. Franklin defined current as the flow of positive charges.

- Note subscripts denote assumed *Voltage drop* between terminal *a* and *b* (*i.e.,* $V_a > V_b$) (*i.e.*, terminal *a* is at a higher potential energy than terminal *b*)



- If the *assumed voltage drop* turns out to be in the wrong direction then the Voltage will be negative when we solve for it. (*i.e.,* swapping the subscript order is implies multiplying by -1)

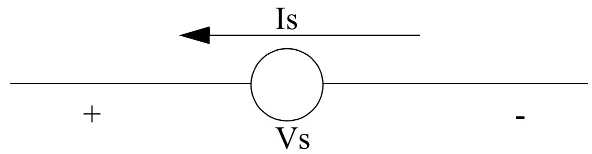
- As long as we are consistent in how we label things the math will work out right.

- Equivalent holds for assumed Current directions

$$V_{ab} = -V_{ba}$$

$$I_{ab} = -I_{ba}$$

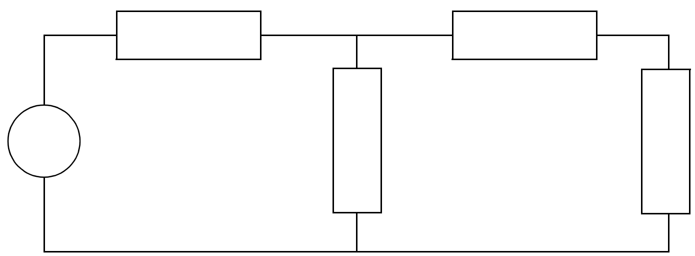
- Generic Active Circuit Element (*Produces Power*)



  - Note the convention is that current flows from the *negative* terminal out the *positive* terminal in active elements.

  - Active elements (*within this course*) will always be labelled with respect to either the direction of their current flow or the voltage across them.

0.3. Interconnection of Circuit Elements (Wires)



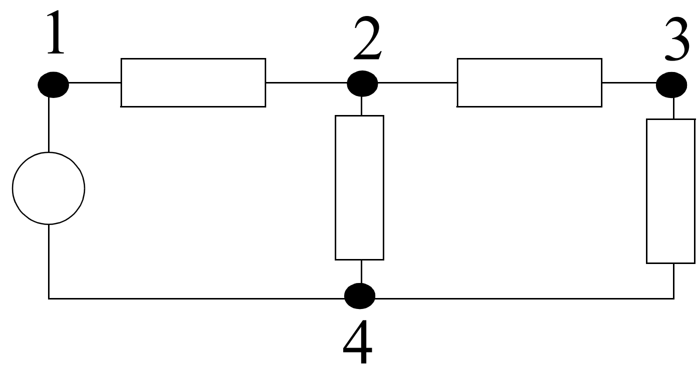
  - Wires connect electrical elements to form circuits

  - Assume *ideal* wires (*i.e.,* no parasitic resistances, inductances, or capacitances due to the wires - in high speed (*i.e.,* microwave) circuits this assumption is invalid and cannot be used - theory of waveguides in 4th year ECE)

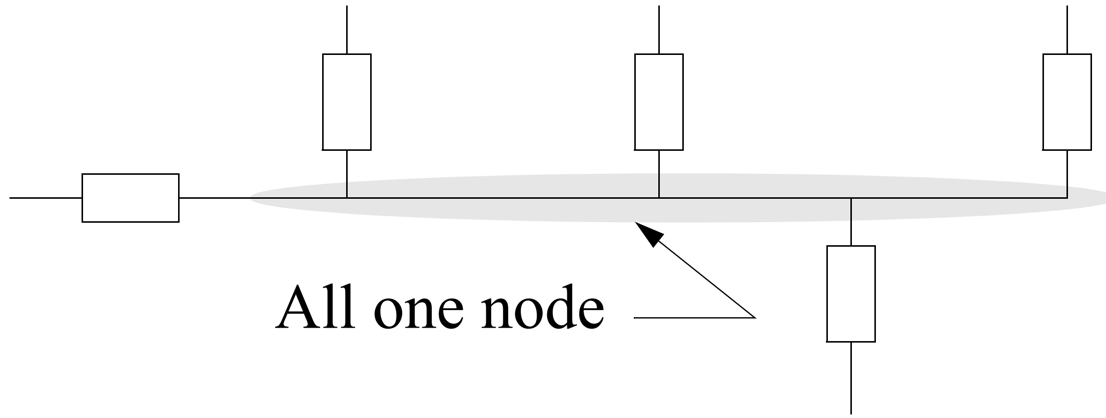
- Certain remote sensing applications with very long wires this assumption is also invalid (MECH - e.g. tethered underwater vehicles)

- May have multiple power sources in a circuit as well as multiple passive elements which consume power (from the principle of Conservation of energy the net power supplied to a given circuit must equal the net power consumed).

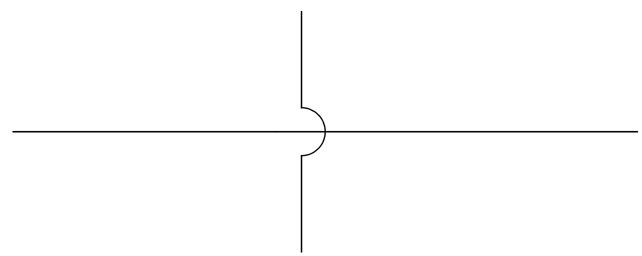
- *Nodes* are where the wires intersect (or where wires intersect with elements)



- A **node** splits into *branches* to connect other circuit elements (as long as the wire(s) do not cross through another circuit element they are still part of the same node)



- If two wires are not connected then this is show as,



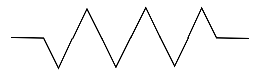
- Otherwise it is always assumed that the wires connect.

- *Path* - An ordered transversal from node to node. (*i.e.,* in the circuit above (node 1, node 2, node 3) is a path in the circuit

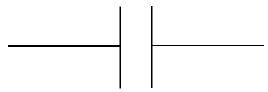
- *Loop (closed path)* - a path which ends at the same node where it began. (*i.e.,*in the circuit above (node 1, node 2, node 4, node 1) is a loop.)

0.4. Basic Passive Elements (*Consume Power*)

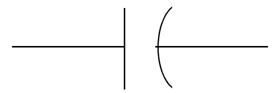
- Resistor Symbol (Chapter 2)



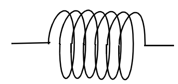
- Capacitor (Chapter 6)



or

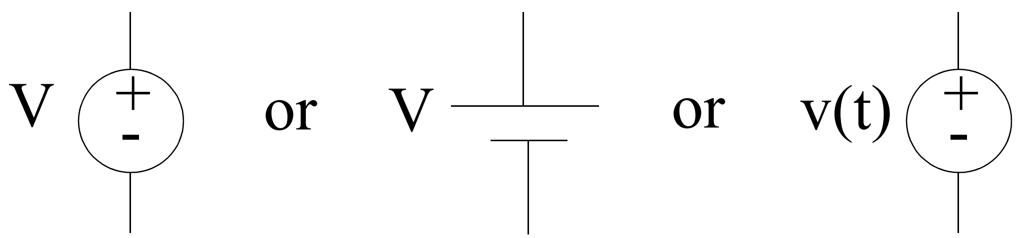


- Inductor (Chapter 6)

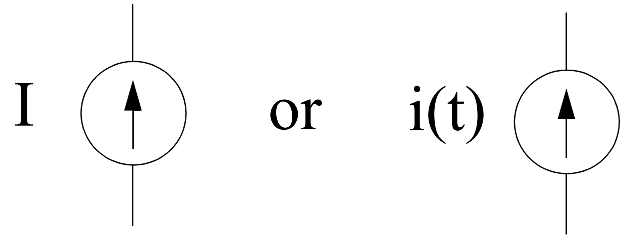


0.5. Power Sources (*Supply Power*)

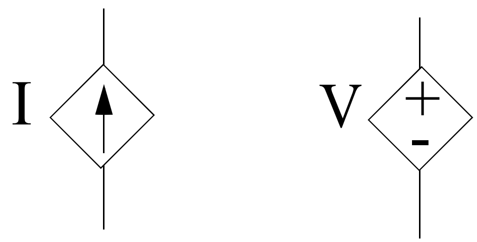
- Voltage Sources (Independent)



- Current Sources (Independent)



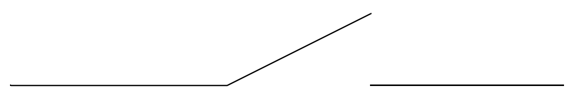
- Dependant Current and Voltage Sources (*i.e.,* value is a function of another parameter(s) (current or voltage) in the circuit)



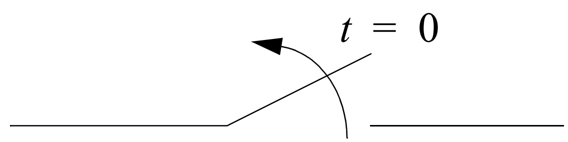
- Transistors are examples of dependent current sources (covered in Elec 330 and 380).

0.6. Other Elements

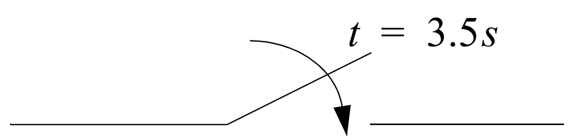
- Switch (connects or disconnects parts of a circuit)



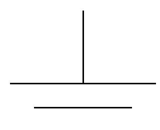
- Switch opening at $t = 0$



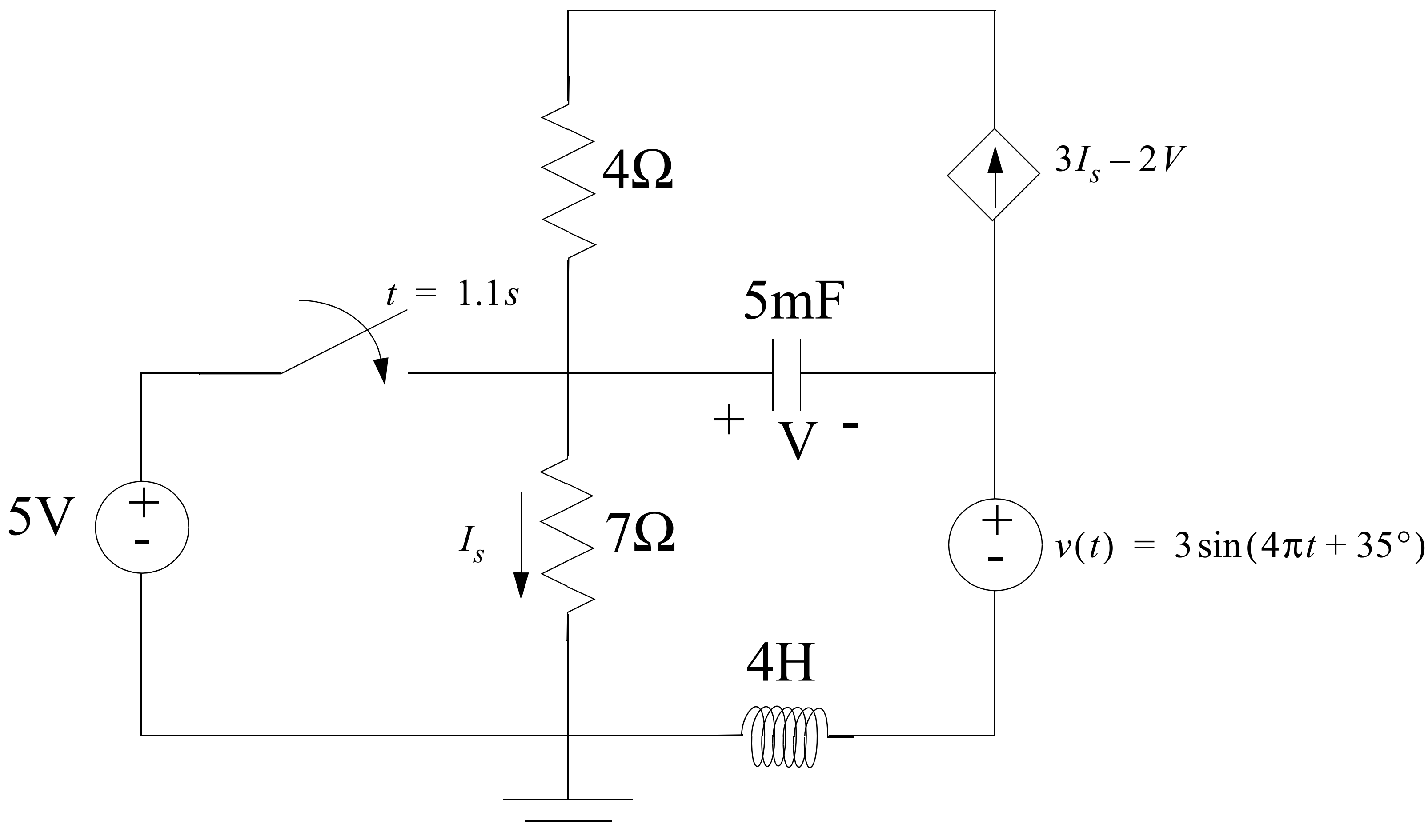
- Switch closing at $t = 3.5s$



- Ground (defined as $V = 0$)



- Example of a circuit using these elements (and one which you will be able to analyze by the end of the course)



Elec 250 focusses on circuit analysis.
Later courses build on this core for circuit synthesis
(i.e., how to design circuits that solve specific problems).

Real World Example:

- Simple Analog Amplifier Schematic:

  - Unfortunately, most interesting circuits required material from 3rd and 4th year to understand how they work

  - But, this understanding builds on the framework put in place in Elec 250.

Chapter 1
Circuit Variables and Elements

1.1 Electric Current

- Electric current is due to the flow of electric charges
  - measure of charge is the Coulomb (unit C)
  - q, Q is the symbol for charge
  - one electron has a charge of -1.6021×10^{-19}C
  - the movement of charge over time is current, I, or i for time varying current
  - Units *Amperes (A)*
  - By convention, current is defined as the flow of *positive* charges over time
  - Formally,

$$i = \frac{dq}{dt}$$

Ex. 1.1 If a charge of flowing into an element is given by

$$q(t) = 10^{-3}(1-e^{-5t})C$$

Then the current through the element is given by

$$i(t) = \frac{dq}{dt} = 5\times 10^{-3}e^{-5t}A$$

- Given current we can also find the total charge which passes through the device through integration.

$$q_{total} = \int_{t_1}^{t_2} i\,dt$$

Ex. 1.2 The current flowing in a wire is 10A. What is the total charge flowing through the wire between the times $t = 3s$ and $t = 9s$

$$q_{total} = \int_{3}^{9} 10\,dt = 60C$$

- Current Flow

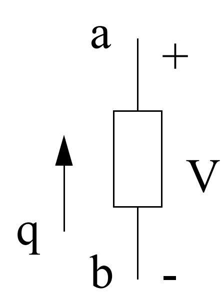
    - current flow in a wire is given by its amplitude and direction (both must be specified)

1.2 Voltage

- Work must be done to move an electric charge from one location to another

    - The amount of work to move 1 positive unit of charge (+1 C) is defined as voltage (V)

    - Energy is supplied to move the charge - hence, we say that the charge has gained energy.



- When charge q moves from point b to a, then an amount of work (W) must be done on the charge q.

- Energy gained by the charge must be supplied by a means external to the circuit (i.e. a current source or voltage source - for example a battery)

$$V_{ab} = V_a - V_b = \frac{W}{q} = V$$

and,

$$V_{ba} = -V_{ab} = -V$$

Ex. 1.3 The voltage across a passive element is $V_{ab} = 15V$. How much work must be done to move a charge of $q = 10^{-10}C$ from point *a* to *b* and vice versa?

$$W_{ab} = q \times V_{ab} = 15 \times 10^{-10} = 1.5 \times 10^{-9}J$$

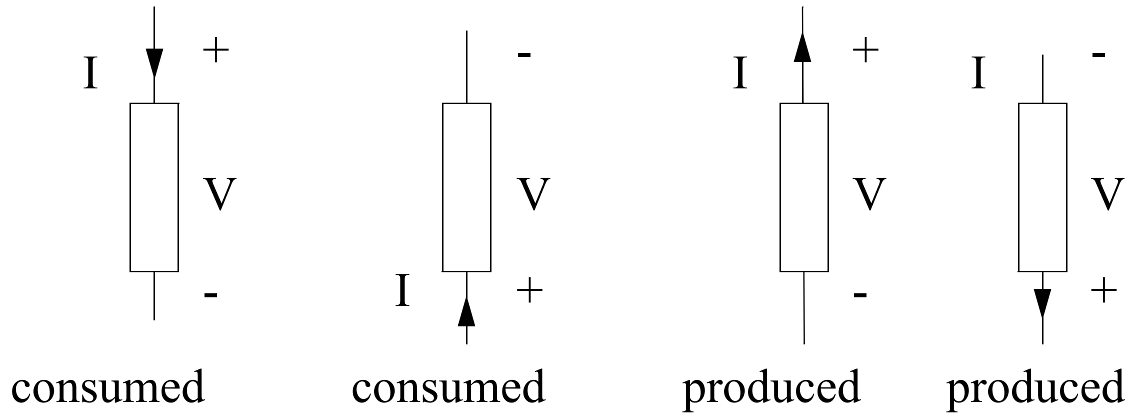
$$W_{ba} = q \times V_{ba} = -15 \times 10^{-10} = -1.5 \times 10^{-9}J$$

- When q moves from *a* to *b* energy is lost due to moving the charge through the (passive) element.

1.3 Energy and Power

- Assume a *positive* charge moves through an element from its *negative* end to its *positive* end.
  - According to our definition of voltage, the charge **gains** energy
  - The element gives energy to the charge, which carries it to the rest of the circuit
  - We say that an element **generates** or **supplies** energy to the circuit if the *current leaves its positive end*.
- Assume a *positive* charge moves through an element from its *positive* end to its *negative* end
  - According to our definition of voltage, the charge **losses** energy
  - The element gains energy from the charge, taking it from the rest of the circuit

- We say that the element **absorbs** or **consumes** energy from the circuit if the *current enters its positive end*.



- The energy (w) absorbed by an element is given by

$$w = qV$$

- Measured in *Joules (J)*

- **Power** (p) consumed by the element equals the rate of change of energy over time

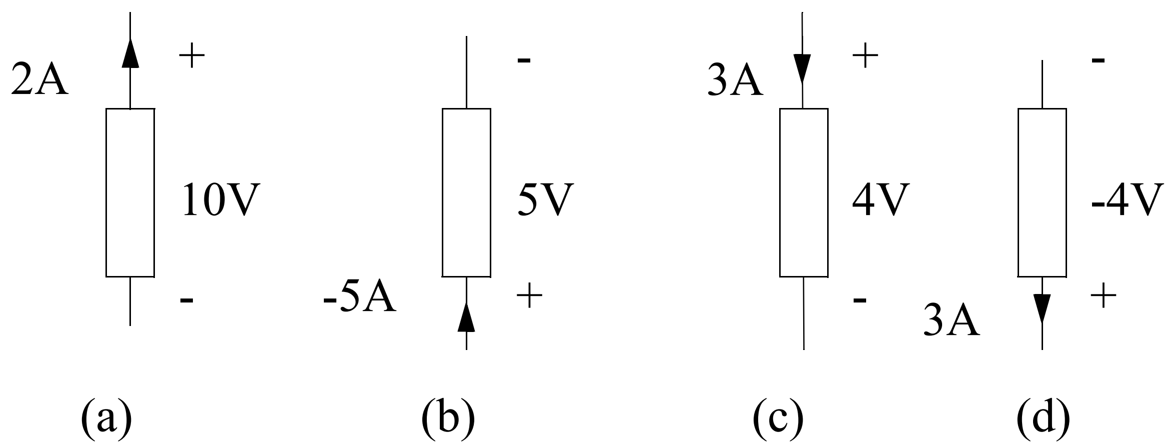
$$p = \frac{dw}{dt} = v\frac{dq}{dt} = iv$$

- We can also obtain the energy consumed (over an interval $[t_1, t_2]$) by an element if we know the power absorbed by integrating the above equation.

$$\Delta w = w(t_2) - w(t_1)$$

$$= \int_{t_1}^{t_2} p\,dt = \int_{t_1}^{t_2} iv\,dt$$

Ex. 1.4 Find the power consumed by each of the elements



Ex. 1.5 The power absorbed by an element is given by

$$p(t) = \begin{cases} 5W & 0 \le t < 10s \\ 0 & t > 10 \end{cases}$$

- Find the function for the energy absorbed by the element

  - For $t < 0$ $w(t) = \int_{-\infty}^{0} p(\tau)d\tau = \int_{-\infty}^{0} (0)d\tau = 0$ J

  - For $0 \le t \le 10s$ $w(t) = w(0) + \int_{0}^{t} p(\tau)d\tau = 0 + \int_{0}^{t} (5)d\tau = 5t$ J

  - For $t > 10s$ $w(t) = w(10) + \int_{10}^{t} p(\tau)d\tau = w(10) = 50$ J

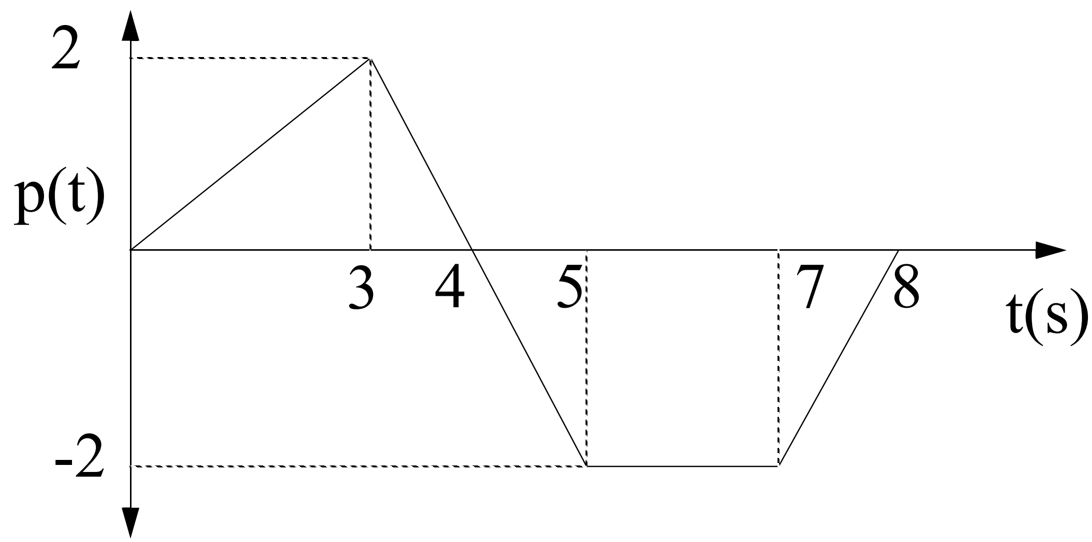
1.4 Passive and Active Elements

- Elements which we will study are divided into 2 classes: **passive** and **active**
  - **Passive Element**: an element which cannot deliver more energy to the circuit than what has been supplied to it by the circuit. (i.e. Resistors, Inductors, Capacitors)
  - **Active Element**: an element which delivers more energy to the circuit than what the circuit supplies to it. (i.e. voltage sources, current sources)
  - By the conventions for current flow, it follows that for an element to be a passive element it must satisfy

$$w(t) = \int_0^t iv\,d\tau \geq 0$$

    - Assuming that $w(0) = 0J$
    - This is a reasonable assumption since for electric circuits we can always assume that no power supplies were connected to a circuit prior to some initial time t_0

Ex. 1.6 The power absorbed by an element is given by



- The total energy absorbed by the element is the area under the curve, hence

$$w = \frac{2\times 3}{2} + \frac{2\times 1}{2} - \frac{2\times 1}{2} - 2\times 2 - \frac{2\times 1}{2} = -2\ \text{J}$$

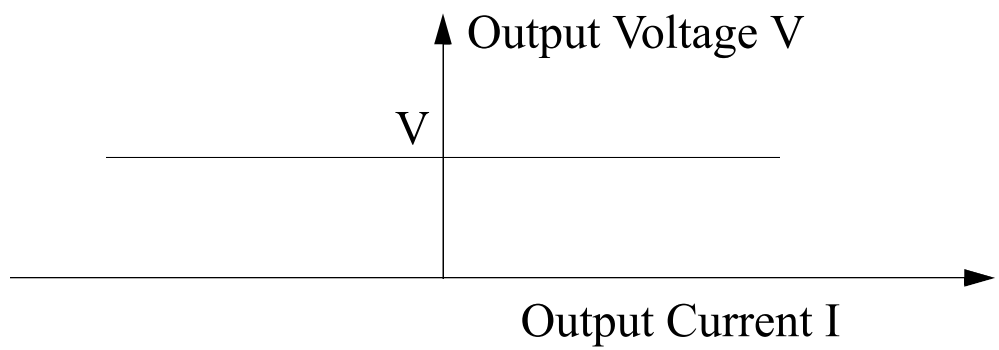
- The total energy is negative, hence the element is *active* over the interval $[0, 8]$

1.5 Power Sources

- There are two types of power sources (supplies): **voltage sources** and **current sources**
  - Batteries are examples of **constant**, or **dc** (direct current), voltage sources

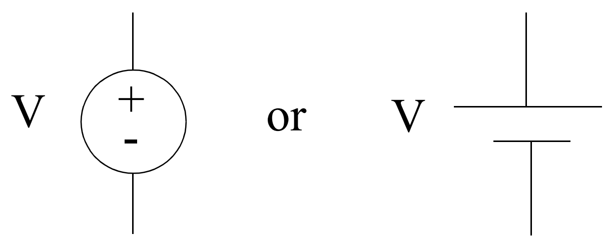
- Household electric power outlets are examples of **alternating**, or **ac** (alternating current), voltage sources

- For an **ideal** voltage source the voltage is independent of the current through the voltage source



- For *real* voltage sources the output voltage will decay as the output current is increased due to parasitic resistances and other limiting effects, hence, this *ideal* assumption will not hold for real voltage sources.

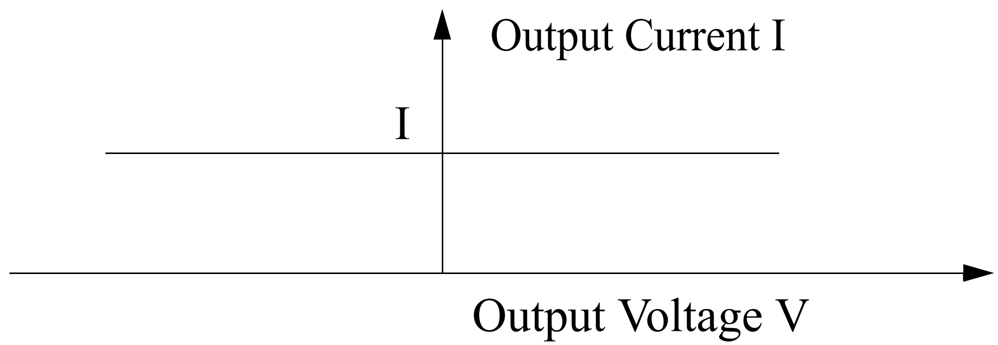
- The circuit symbol(s) for voltage sources are



- Transistors are examples of **current sources** (transistors are the basis of computer circuits and operational amplifiers)

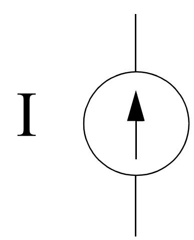
    - Current sources can also be dc or ac

    - For an *ideal* current source its current is independent of the voltage across the current source



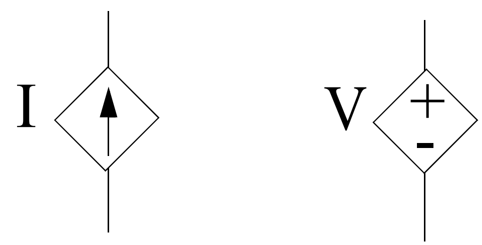
    - For *real* current sources the output current will decay as the output voltage is increased due to parasitic resistances and other limiting effects, hence, this *ideal* assumption will not hold for real current sources.

    - The circuit symbol for a current source is



1.6 Dependent Sources

- There are also voltage and current sources whose output depends on the value(s) of other circuit parameters (such as a voltage or current elsewhere in the circuit)
  - These are called **dependent** or **controlled** sources
  - An operational amplifier is such a source
  - The circuit symbols for these sources are



  - Dependent sources will always have a label which specifies their functional dependency (in terms of other circuit parameters).

Assignment #1

Refer to Elec 250 course web site for assigned problems.

Due 1 week from today @ 5pm
(in Elec 250 Assignment drop box on 3rd floor EOW)

Chapter 2
Resistive Circuits

2.1 Resistors

- A resistor is the most basic circuit element (next to wires) where the voltage across it is related to the current flowing through it by **Ohm's Law**

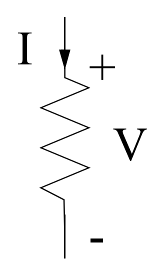
$$V = IR$$

where $R \geq 0$ is the **resistance**, which is measured in **ohms** (Ω)

- Ohm's Law can also be written in terms of conductance, $G = \frac{1}{R}$, which has units of siemens (S) (or sometimes referred to as mho(℧))

$$I = GV$$

- The circuit symbol for a resistor is



- The current always enters the positive lead of a resistor, hence, resistors always absorb energy (passive elements)

- The power consumed by a resistor is

$$p = VI = (IR)I = I^2R = \frac{V^2}{R} = GV^2 \text{ W (Watts)}$$

Ex. 2.1 Design an electric heater that generates 1kW when the applied voltage is 120 V. How must should the resistance of the heating element be and how much current will flow through it?

$$p = \frac{V^2}{R}$$

$$1000 = \frac{120^2}{R}$$

$$R = \frac{14,400}{1000} = 14.4\Omega$$

$$I = \frac{V}{R} = \frac{120}{14.4} = 8.33A$$

2.2 Wires, shorts, and open circuits

- Wires which connect circuit elements together have a small but non-zero resistance ($R \approx 0$). It is assumed that these parasitic resistances are much smaller than that of the actual circuit elements hence they can be neglected (note: for high frequency circuits this is not the case (i.e. microwave circuits))
  - Within this course the resistance due to the wires can be neglected and assumed to be zero.

2.2.1 Short Circuit

- A short-circuit occurs when the resistance to the current flow along a given path is zero.
  - Large current flows when $R \to 0$ even for a small voltage drop due to

$$I_{\text{short-circuit}} = \lim_{R \to 0}\left[\frac{V}{R}\right] = \infty$$

  - Example of a short-circuit is when the positive and negative leads of a voltage source are directly connected by a wire. This usually results in damaging the voltage source due to the large current which flows during the short-circuit.

2.2.2 Open-Circuit

- An open-circuit occurs when the resistance to the current flow along a given path is very large (i.e. $R \to \infty$).

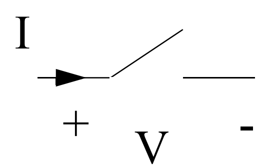
    - A very small current flows when $R \to \infty$ even for a large voltage drops since

$$I_{\text{open-circuit}} = \lim_{R \to \infty}\left[\frac{V}{R}\right] = 0$$

Example of an open-circuit is a battery which is unconnected to a circuit. The battery maintains its charge because in the open-circuit no current flows.

2.2.3 Ideal Switch

- The ideal switch is an open-circuit in the “open” (or off) state and a closed circuit in the “closed” (or on) state



    - The resistance from an ideal switch is nonlinear and is given by

$$R(on) = 0 \text{ and } R(off) = \infty$$

2.3 Kirchhoff's Current Law (KCL)

- Based on the **conservation of charge** - at all times, the sum of charges entering a node must exactly equal the sum of charges leaving the node (i.e. a node must always have a net zero charge on it)

  - Formally we can state this as

$$\sum_{i \in \{entering\}} I_i = \sum_{j \in \{leaving\}} I_j$$

The sum of currents entering a node equals the sum of currents leaving the node

  - Or, if we assume a *sign convention* that currents *entering* a node are *positive* and currents *leaving* a node are *negative* then

$$\sum_{i \in \{entering\}} I_i - \sum_{j \in \{leaving\}} I_j = 0$$

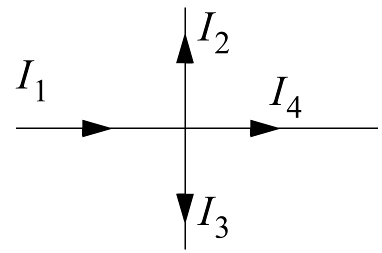
$$\sum_{\forall k} I_k = 0$$

  - **The algebraic sum of currents entering any node is zero.**

  - Note: we could have reverse our sign convention and assumed a current leaving the node is positive and entering a node is negative and arrived at the same result.

- Which convention we choose is arbitrary. We must keep the same convention throughout the circuit we are analyzing.

Ex. 2.2 Four currents enter/leave the node as shown below.



From KCL we can write

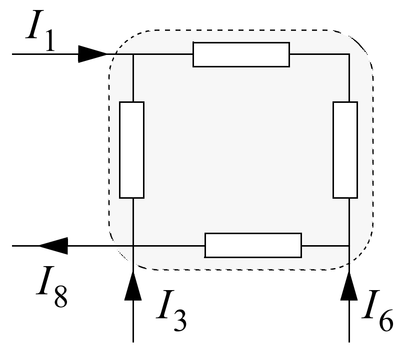
$$I_1 = I_2 + I_3 + I_4$$

or moving all terms to the left-hand side of the equation

$$I_1 - I_2 - I_3 - I_4 = 0$$

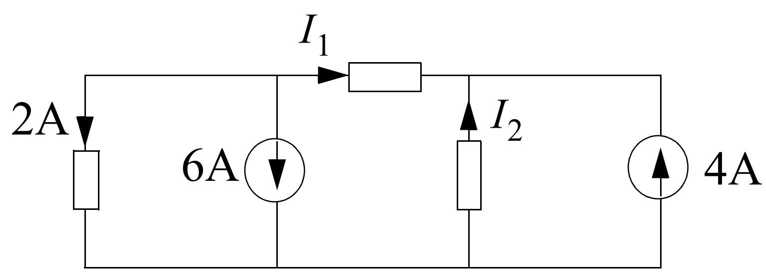
- KCL applies to **any closed boundary** within a circuit - even if that boundary encloses several nodes. (i.e. the sum of charges entering any closed boundary equals the sum of charges leaving the closed boundary due to the conservation of charge)

Ex. 2.3 In the portion of the circuit shown below the application of KCL to the closed boundary (shown by the dotted line) gives



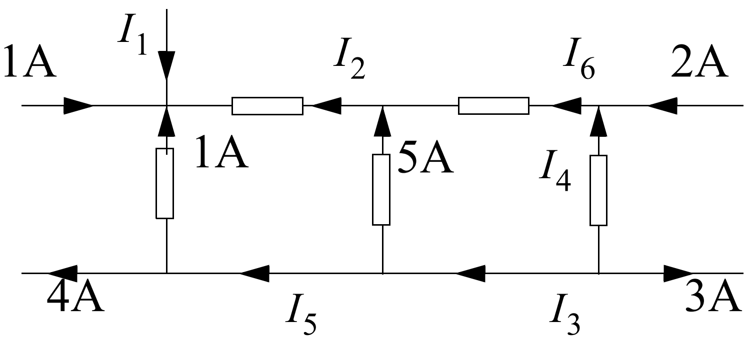
$$I_1 + I_3 + I_6 - I_8 = 0$$

Ex. 2.4 Find the unknown currents for the given circuit



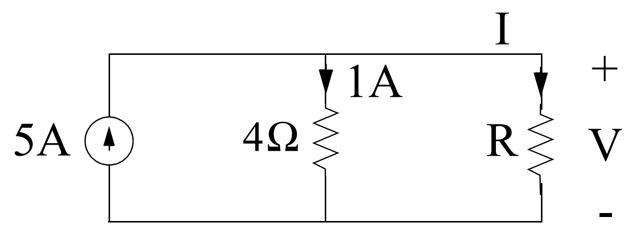
Solution:

Ex. 2.5 Find the unknown currents in the following circuit



Solution:

Ex. 2.6 Find I, V and R for the following circuit



Solution:

2.4 Kirchhoff's Voltage Law (KVL)

- Kirchhoff's voltage law is related to the conservation of energy which states that the sum of work done along a closed path must equal zero

- Formally we can state this as

$$\sum_{i \in \{rise\}} V_i = \sum_{j \in \{drop\}} V_j$$

The sum of voltage rises around a loop equals the sum of voltage drops around the same loop

- Or, using our convention that voltage is positive when we go from a lower to a higher potential (from - to +) and negative when we go from a higher potential to a lower one (+ to -) while traversing an element (from the definition of voltage)

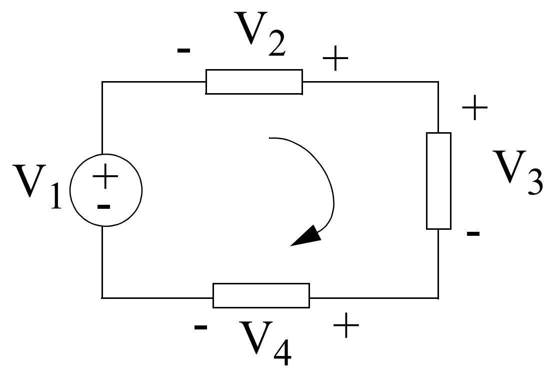
$$\sum_{i \in \{rise\}} V_i - \sum_{j \in \{drop\}} V_j = 0$$

$$\sum_{\forall k \in \{\text{closed path}\}} V_k = 0$$

The algebraic sum of voltages around any closed path (loop) is zero.

- Note: we could have reverse our sign convention.Which convention we choose is abritrary. But, we **must** keep the same convention throughout the circuit we are analyzing.

Ex. 2.7 Apply KVL to the following circuit



- Beginning with the voltage source and moving along the closed path in the direction specified by the arrow we have

$$V_1 + V_2 - V_3 - V_4 = 0$$

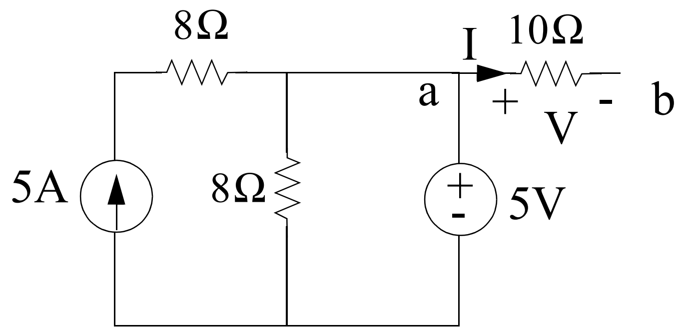
Ex. 2.8 If in the above example if $V_1 = 5V$, $V_2 = -3V$ and $V_4 = 6V$ then find V_3.

- From applying KVL we have

$$5 + (-3) - V_3 - 6 = 0$$

$$V_3 = 4V$$

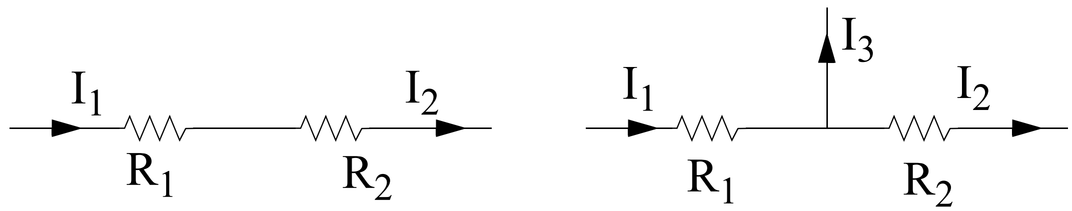
Ex. 2.9 Given the circuit shown below what is the voltage and current through the "hanging" 10Ω resistor?



Solution:

2.5 Series Resistors

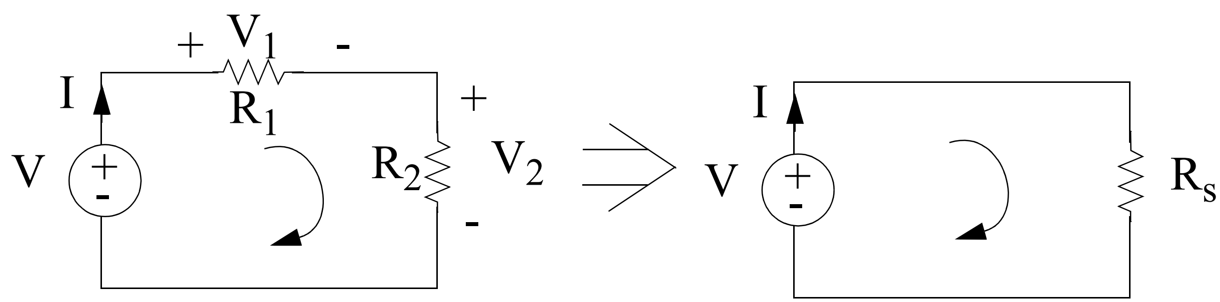
- Two resistors are said to be connected in *series* if they share one node and the *same current* flows through them.



Resistors in Series

Resistors **NOT** in Series

- Given two resistors in series we can ask the question: What is their equivalent resistance?
  - We can use KVL to answer this question



(a) (b)

- Applying KVL to circuit (a) we have

$$-V + V_1 + V_2 = 0$$

- Applying Ohm's law to each resistor

$$V = IR_1 + IR_2$$

$$I = \frac{V}{R_1 + R_2}$$

- Applying Ohm's law to the circuit (b) we have

$$V = IR_s$$

$$I = \frac{V}{R_s}$$

- If the current and voltage to both circuits are identical then for the circuits to be equivalent we must have

$$R_s = R_1 + R_2$$

Resistances sum when they are connected in series.

- The above case can be generalized to the case of n series-connected resistors. In that case the equivalent series resistance is given by

$$R_s = \sum_{i=1}^{n} R_i$$

2.5.4 Voltage Division

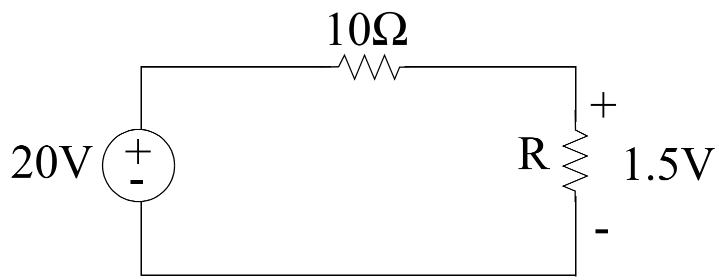
- Given the series-connected resistors we can also ask what is the voltage across each of the resistors. From circuit (a) above and the discussion which followed we have that

$$V_1 = IR_1 = \frac{R_1}{R_1 + R_2} V$$

$$V_2 = IR_2 = \frac{R_2}{R_1 + R_2} V$$

  - The voltage across each resistor is the same fraction of the source voltage as the given resistor is of the total series-connected resistance.
  - Series resistors therefore perform *voltage division*.

Ex. 2.10 What is the value of R in the circuit below?



- We can apply the voltage division formula from above

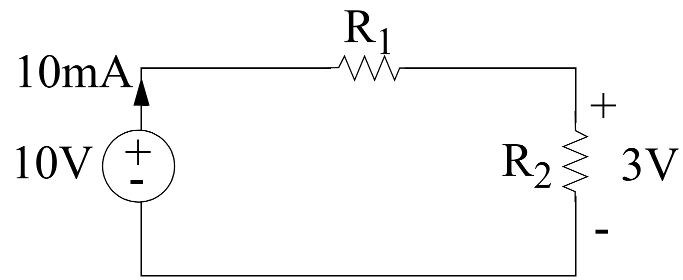
$$1.5V = \frac{R}{R+10}(20V)$$

$$1.5R + 15 = 20R$$

$$R = 0.81\Omega$$

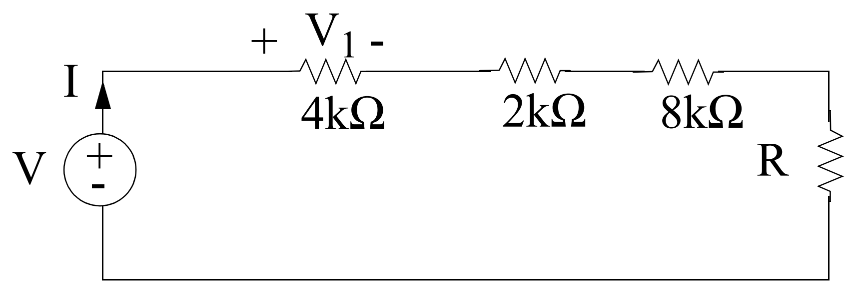
- Note: whenever we calculate for R it must always be *positive* - if it isn't then we've made a *math mistake* in the calculations

Ex. 2.11 Design a potential divider such that the output voltage is 3V when the input voltage is 10V and the current in the circuit is 10mA.



Solution:

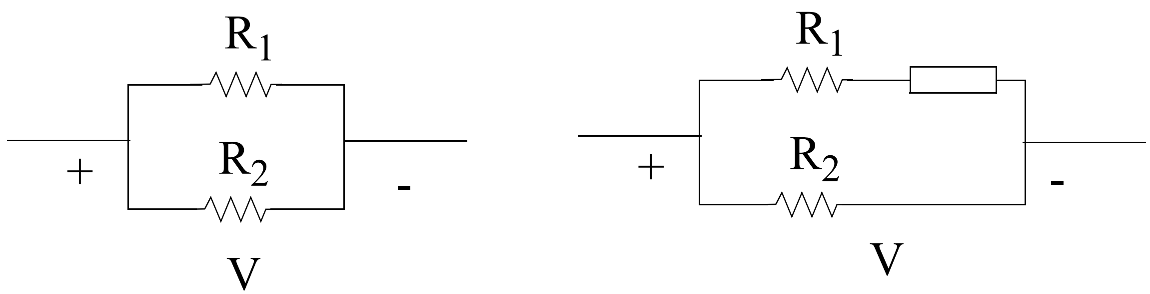
Ex. 2.12 For the following circuit, given that $V_1/V=0.125$ and the power delivered by the source is 8mW, find the values of R, V, V_1, and I.



Solution:

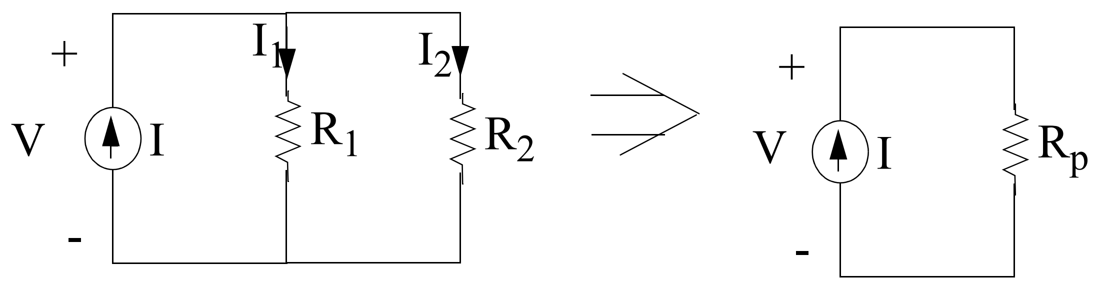
2.6 Parallel Resistors

- Two resistors are connected in *parallel* if they are both tied to the same two nodes such that they both have the *same* voltage across them.



Resistors in parallel Resistors NOT in parallel

  - We can ask, what is the equivalent circuit of two resistors in parallel?



(a) (b)

- Applying KCL and Ohm's law to circuit (a) we have

$$I = I_1 + I_2$$

$$I = \frac{V}{R_1} + \frac{V}{R_2} = G_1 V + G_2 V$$

$$V = \frac{I}{G_1 + G_2}$$

- Applying Ohm's law to circuit (b) we have

$$V = IR_p = \frac{I}{G_p}$$

- We say that circuits (a) and (b) are equivalent when the source current generates the same voltage V in both circuits. This occurs when

$$G_P = G_1 + G_2$$

  - or in terms of resistances

$$R_p = \frac{R_1 R_2}{R_1 + R_2}$$

- For *n* resistors in parallel these results can be generalized to give

$$G_p = \sum_{i=1}^{n} G_i$$

Ex. 2.13 An amplifier is connected to a 12Ω load, it is required to reduce that load to 8Ω using another resistor in parallel. What is the value of the parallel resistor?

$$8 = \frac{R \times 12}{R + 12}$$

$$R = 24\Omega$$

2.6.5 Current Division

- The currents through each of the resistors in circuit (a) above are given by

$$I_1 = VG_1 = \frac{G_1}{G_1 + G_2} I$$

$$I_2 = VG_2 = \frac{G_2}{G_1 + G_2} I$$

- In terms of resistances these equations become

$$I_1 = \frac{R_2}{R_1 + R_2} I$$

$$I_2 = \frac{R_1}{R_1 + R_2} I$$

  - Notice that the current through each resistor is proportional to the value of the conductance. It can be easily verified that the sum of these two currents is I.

- The above result can be generalized to *n* parallel-connected resistors

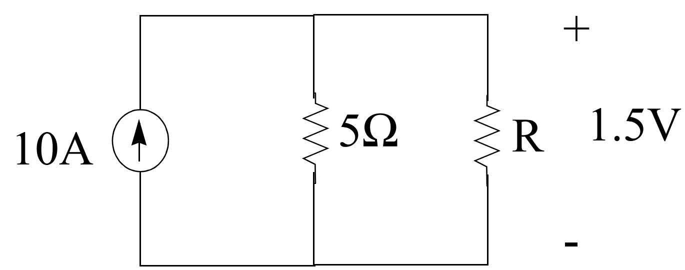
$$I_i = \frac{G_i}{G_p} I, \qquad 1 \le i \le n$$

where G_P is given by

$$G_p = \sum_{i=1}^{n} G_i$$

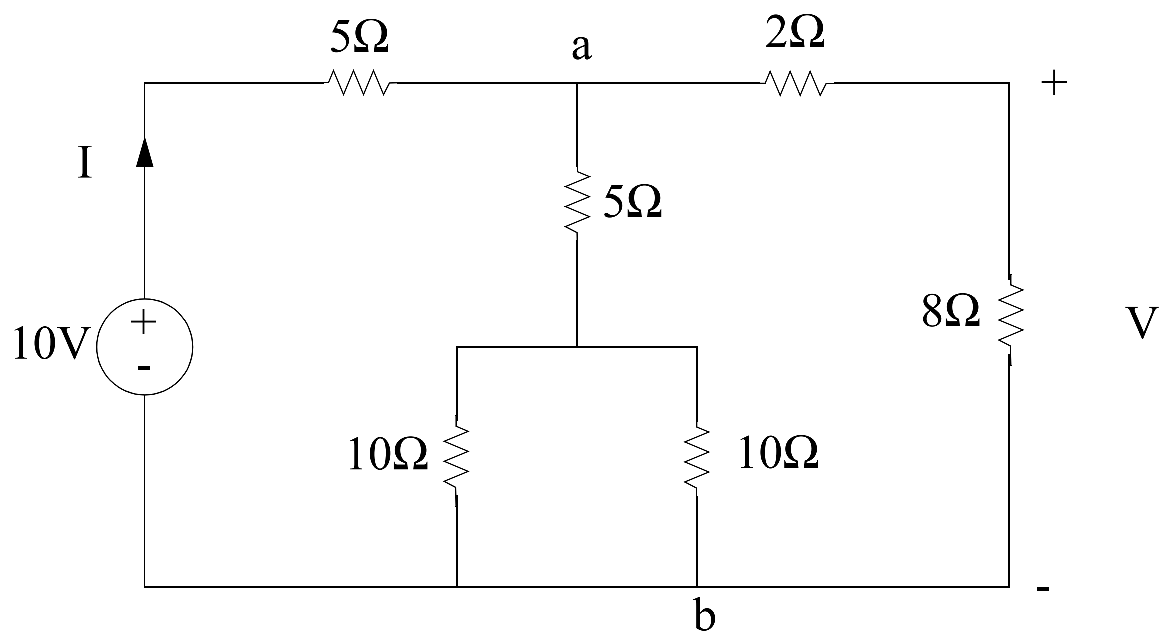
  - Parallel resistors therefore perform *current division.*

Ex. 2.14 What is the value of the load resistor R such that the value of the voltage across it is 1.5V?



Solution:

Ex. 2.15 Find V and I in the circuit below using series and parallel resistor simplification techniques.



Solution:

Assignment #2

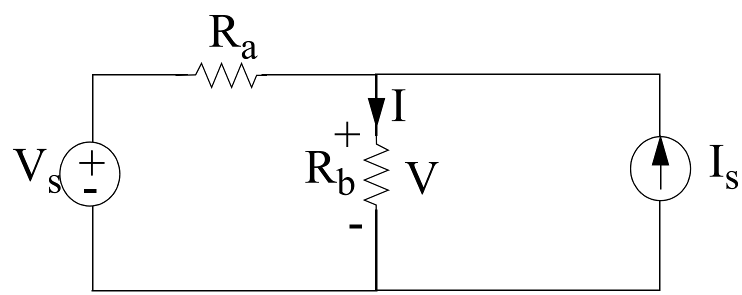
Refer to Elec 250 course web site for assigned problems.

Due one week from today @ 5pm in the Elec 250 Assignment drop box.

Chapter 3
Network Theorems

3.1 Linearity

- Most of the engineering systems which we deal with are *linear* systems. An electric circuit is linear if its circuit variables (currents through the elements and voltages across them) are linear functions of power sources that cause them.



- Using Ohm's Law, KVL and KCL from the previous chapter we have that

$$V_a = V_s - V \text{ (KVL around first loop)}$$

$$I_a = \frac{V_a}{R_a} = \frac{V_s - V}{R_a} \text{ (Ohm's law at } R_a\text{)}$$

$$V = IR_b \text{ (Ohm's law at } R_b\text{)}$$

$$I = I_a + I_s \text{ (KCL at node connecting } R_a \text{ and } R_b\text{)}$$

- So we can write an equation for I just in terms of the two sources

$$I = \frac{V_s - IR_b}{R_a} + I_s$$

$$I = \left[\frac{1}{R_a + R_b}\right]V_s + \left[\frac{R_a}{R_a + R_b}\right]I_s$$

$$I = K_1 V_s + K_2 I_s$$

where K_1 is a constant with dimensions of conductance and K_2 is a dimensionless constant.

- Similarly we can write the voltage V in terms of just the two sources

$$V = \left[\left[\frac{1}{R_a + R_b}\right]V_s + \left[\frac{R_a}{R_a + R_b}\right]I_s\right]R_b$$

$$V = \left[\frac{R_b}{R_a + R_b}\right]V_s + \left[\frac{R_a R_b}{R_a + R_b}\right]I_s$$

$$V = K_3 V_s + K_4 I_s$$

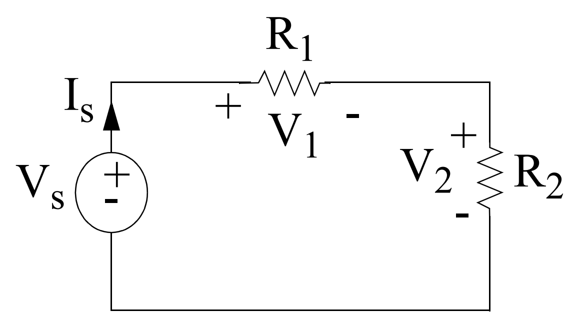
where K_3 is a dimensionless constant and K_4 is a constant with dimensions of resistance.

- Hence, in the above circuit the circuit parameters are *linear* functions of the power sources
  - By *linear circuits* we mean all such circuits in which all the circuit parameters are linear functions of the power sources.
- Linear circuits (and more generally linear systems) have two important properties which can be used to simplify their analysis: **proportionality** and **superposition**
  - All the circuits we will deal with in this course will be linear circuits.

3.1.1 Proportionality

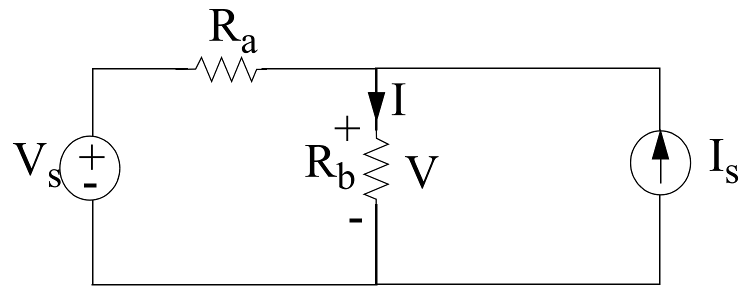
- The principle of proportionality states that within a linear circuit, if the value of a power supply is multiplied by a factor of a then all of the resulting circuit parameters are multiplied by the same factor a.

Ex. 3.1 In the following circuit the value of V_1 is given by $V_1 = \frac{R_1}{R_1 + R_2} V_s$. What is the value of V_1 if V_s is changed to $V_s' = aV_s$?



$$V_1' = \frac{R_1}{R_1 + R_2} V_s' = \frac{R_1}{R_1 + R_2}(aV_s) = aV_1$$

Ex. 3.2 Assume both source are doubled in value what is the resulting value of V?



- From previously we have that without the doubling of the sources

$$V = \left[\frac{R_b}{R_a + R_b}\right]V_s + \left[\frac{R_a R_b}{R_a + R_b}\right]I_s$$

- Hence, by proportionality if we now double the value of both sources

$$V' = \left[\frac{R_b}{R_a + R_b}\right](2V_s) + \left[\frac{R_a R_b}{R_a + R_b}\right](2I_s) = 2V$$

3.1.2 Superposition

- The super position principle is stated as

The value of a circuit variable is the sum of the values of that variable that are produced by each of the power sources acting alone.

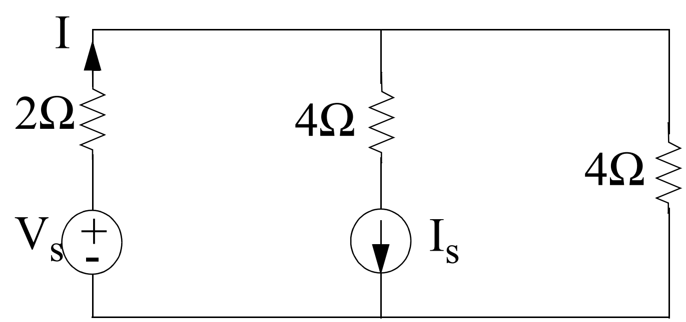
- This means that we can simplify our analysis by looking at how each power supply acts individually on the circuit and then summing the results to get the total effect of all the power supplies on the circuit.

- To look at how each power source acts alone we must disable the other power sources

  - To disable **voltage sources** we replace them with **short-circuits**.

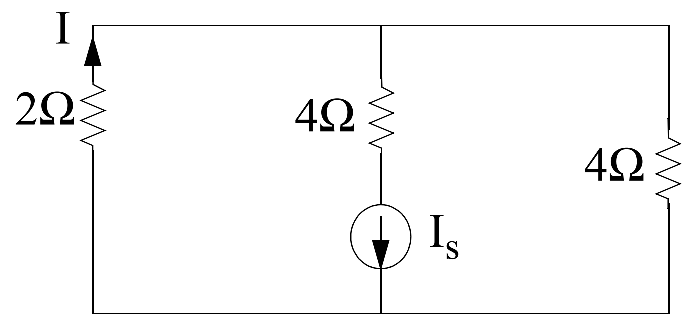
  - To disable **current sources** we replace them with **open-circuits**.

Ex. 3.3 Use superposition to solve for the I in the following circuit



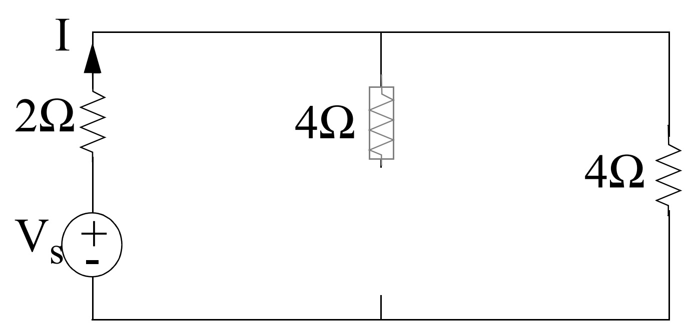
  - First step is to decompose the circuit into a circuit with the voltage source acting alone and one with the current source acting alone.

- For the current source acting alone we have (replace voltage source by short-circuit)



$$I' = \frac{4}{4+2} I_s = \frac{2}{3} I_s \text{ (current divider)}$$

- For the voltage source acting alone we have (replace current source by open-circuit)

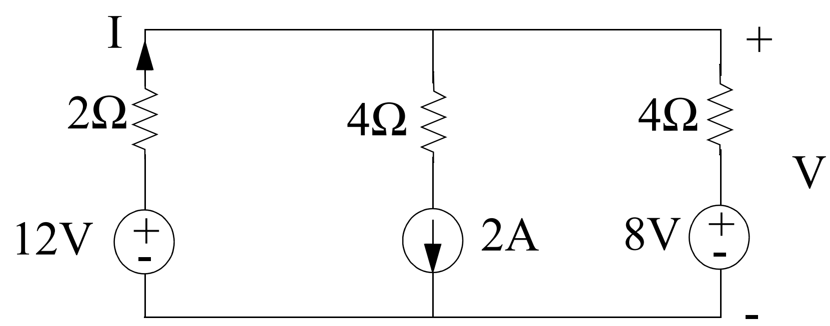


$$I'' = \frac{V_s}{2+4} = \frac{V_s}{6} \text{ (Ohm's law)}$$

- Combining the two sources we have

$$I = I' + I'' = \frac{2}{3}I_s + \frac{1}{6}V_s$$

Ex. 3.4 Find V and I using superposition



Solution:

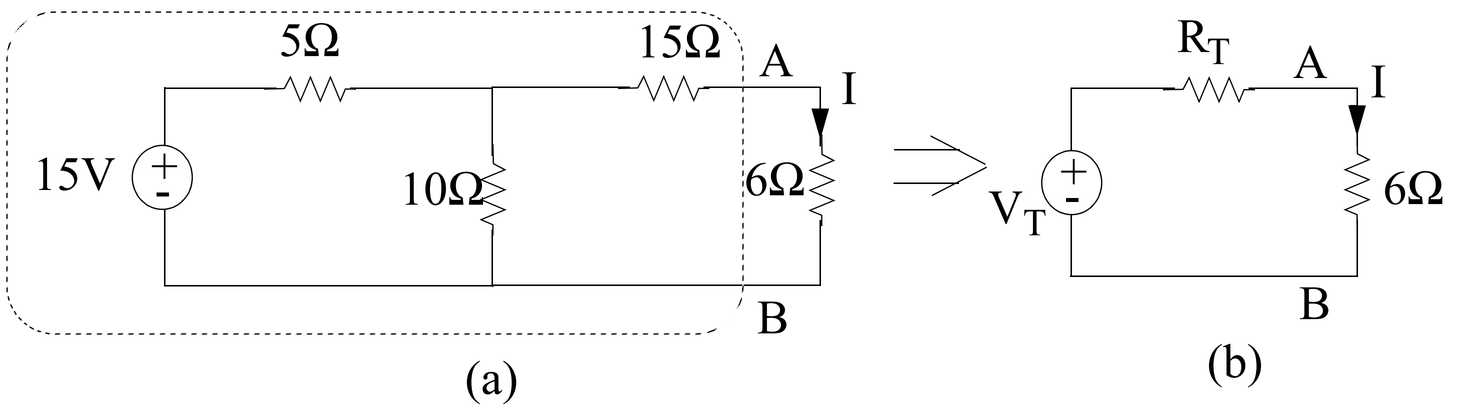
3.2 Thevenin's Theorem

- In many situations, we are interested in finding a certain node voltage or a certain branch current in a circuit. If the circuit is complex, then we would like a means of simplifying the circuit to just those terms we are interested in. Thevenin's theorem allows us to simplify the circuit to just two elements.

  - Thevenin's theorem states:

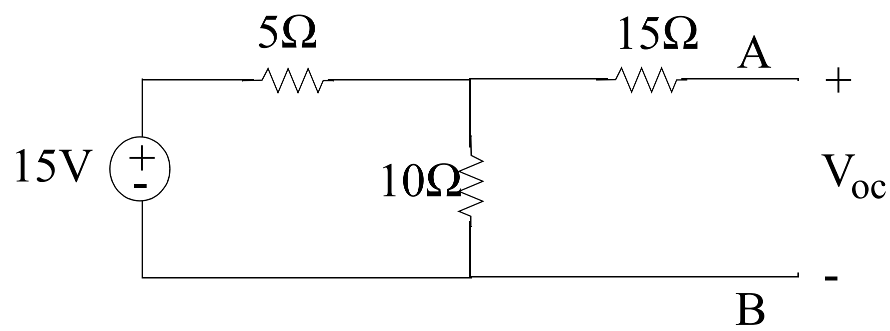
    Any linear circuit can be represented at a given pair of nodes by an equivalent circuit consisting of a single voltage source in series with a single resistor.

Ex. 3.5 In the following circuit, Thevenin's theorem can be used to replace the circuit on the left of the 6Ω load resistor with the circuit shown in (b)



- Circuit (a) can be replaced by circuit (b) where V_T is the Thevenin voltage and R_T is the Thevenin resistance.

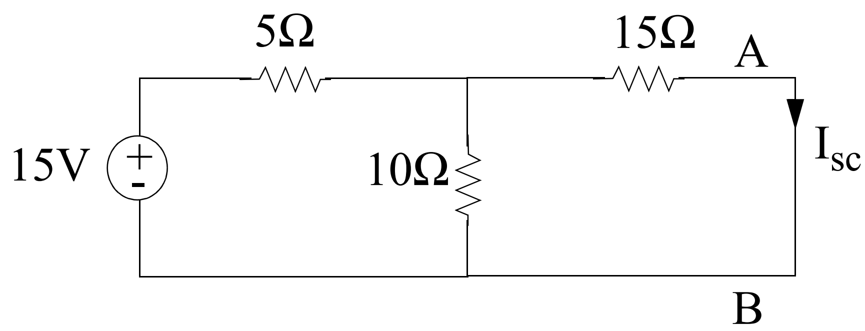
- The value of V_T is found by replacing the load resistor (6Ω) with an open circuit and calculating the open circuit voltage V_{oc}.



$$V_{oc} = \frac{10}{10+5}15V = 10V$$

$$V_T = V_{oc} = 10V$$

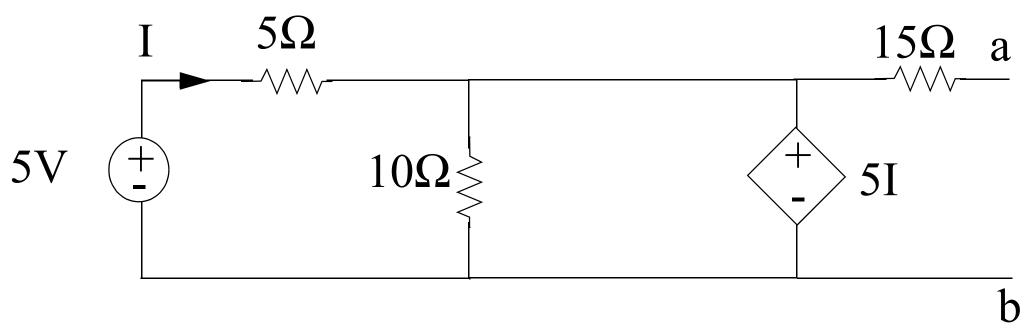
- The value of R_T is found by replacing the load resistor with a short-circuit and calculating the short-circuit current I_{sc}.



$$I_{sc} = \frac{(15V)}{5 + 10//15}\left(\frac{10}{10+15}\right) = 0.545A$$ where // means “in parallel” (i.e. $R_1//R_2 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$)

$$R_T = \frac{V_{oc}}{I_{sc}} = \frac{10V}{0.545A} = 18.3\Omega$$

Ex. 3.6 Find the Thevenin equivalent circuit between terminals a and b



Solution:

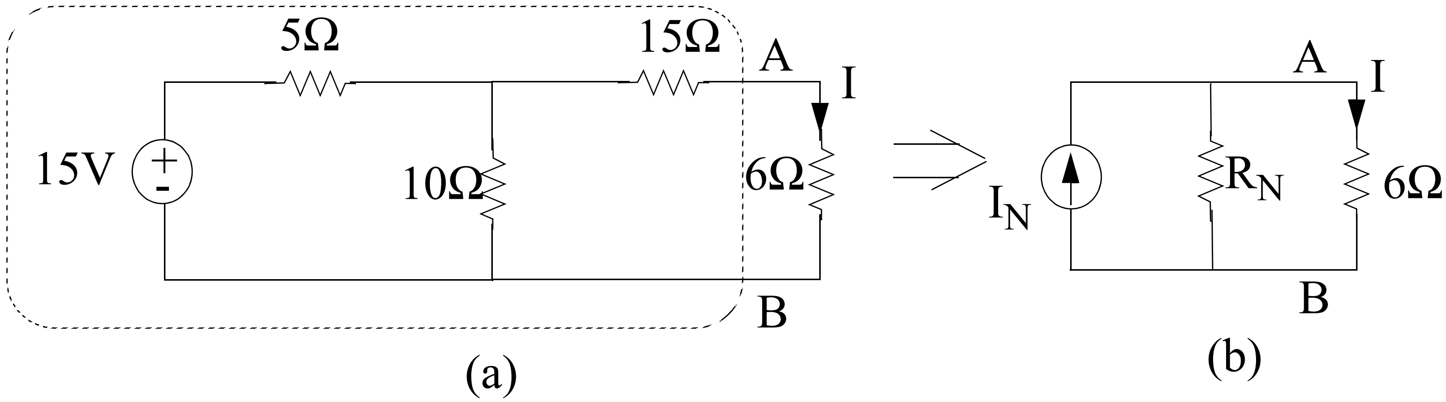
3.3 Norton's Theorem

- Norton's theorem also allows us to simplify a circuit to just two elements, but this time in terms of a current source and a parallel resistor instead of the voltage source and series resistor given by Thevenin's theorem.

  - Norton's theorem states

    Any linear circuit can be represented at a given pair of nodes by an equivalent circuit consisting of a single current source in parallel with a single resistor.

Ex. 3.7 In the following circuit, Norton's theorem can be used to replace the circuit on the left of the 6Ω load resistor with the circuit shown in (b)



- To solve for I_T and R_N we find the open-circuit voltage V_{oc} and the short-circuit current I_{sc} (as we did in the Thevenin's theorem)

- From the previous section we have for this circuit that

$$V_{oc} = \frac{10}{5+10}(15V) = 10V$$

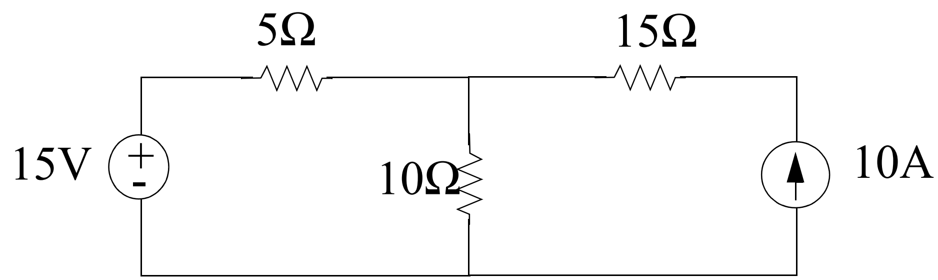
$$I_{sc} = \frac{10}{10+15}\left(\frac{15V}{5+10//(15)}\right) = 0.545A$$

$$R_T = \frac{V_{oc}}{I_{sc}} = 18.3\Omega$$

- For Norton's equivalent circuit (b) we have that

$$I_N = I_{sc} \text{ and } R_N = R_T$$

Ex. 3.8 Find Norton's equivalent circuit at the 10Ω load resistor

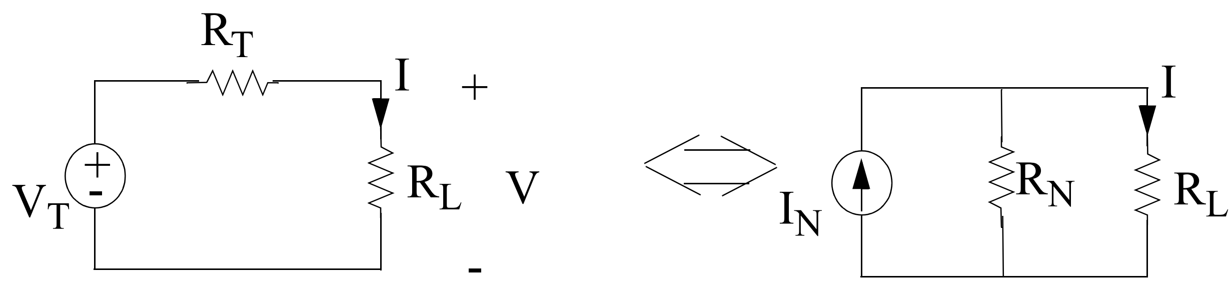


Solution:

3.4 Source Transformation

- From the previous two sections we have seen the ability to transform any linear circuit into either its Thevenin or Norton equivalents.

  - Obviously, we can always move between Norton and Thevenin equivalents



Thevenin's Equivalent Circuit Norton's Equivalent Circuit

  - Applying KVL to Thevenin's equivalent circuit

$$V = V_T - R_T I$$

    - The current at the load resistor is

$$I = \frac{V_T - V}{R_T} = \frac{V_T}{R_T} - \frac{V}{R_T}$$

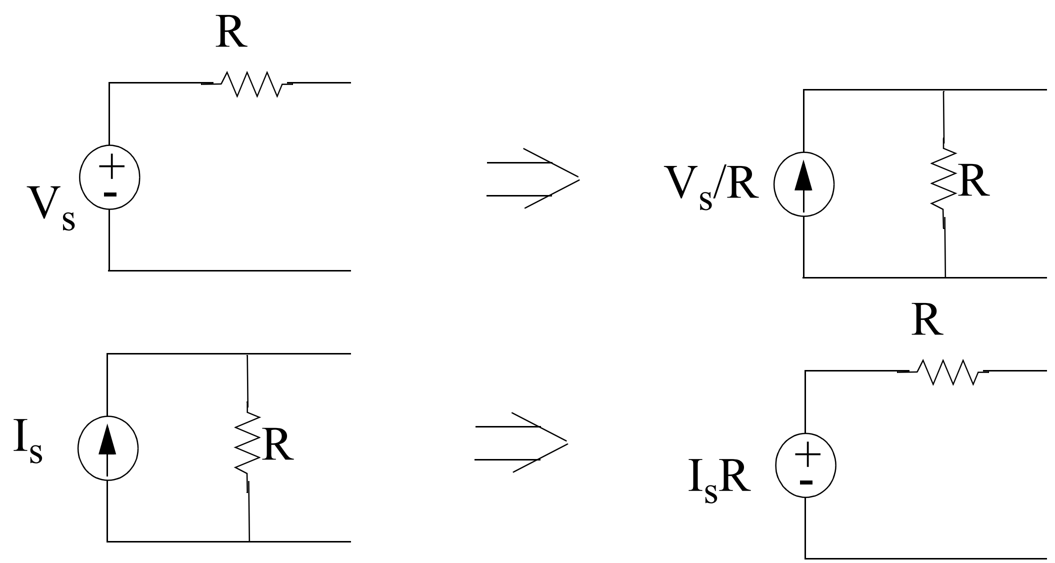
- Applying KCL to Norton's equivalent circuit gives

$$I = I_N - \frac{V}{R_T}$$

- Combining the above results for the two equations for I we have that

$$V_T = I_N R_T$$

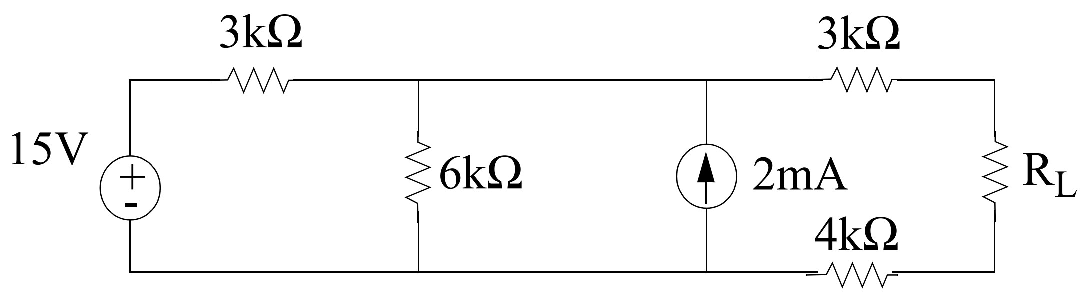
- Using this result we have a means of transforming a voltage source in series with a resistor to a current source in parallel with the resistor and vice versa



- Note the relationship between the direction of the polarity of the voltage sources and the direction of the current in the current sources.

- We can use source transformations to simplify circuits prior to solving them if there are sources which meet the requirements shown above.

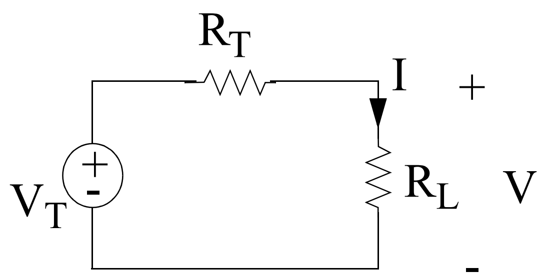
Ex. 3.9 Solve for the voltage across the load resistor for the following circuit.



Solution:

3.5 Maximum Power Transfer

- Sometimes we want to find the maximum power that can be delivered to a load from a given circuit. Norton and Thevenin equivalent circuits can help us to answer that question (remember that any linear circuit can be placed into its Norton or Thevenin equivalent)
    - Powering stereo speakers would be an obvious case were we would like to achieve maximum power.
    - Consider the following a circuit were the circuit driving the load has been replaced by its Thevenin equivalent.



    - Assume that the load resistor can be varied with the objective of getting the maximum power we can out of the circuit (i.e. we will match the load resistor to the circuit to maximize the power output across the load resistor)

- Applying KVL to the circuit we get

$$V_T = I(R_T + R_L)$$

$$I = \frac{V_T}{R_T + R_L}$$

- From Chapter 1 we have that

$$P = I^2 R_L = \frac{V_T^2 R_L}{(R_T + R_L)^2}$$

- We want to maximize P which will occur at the value of R_T for which $\frac{dP}{dR_T} = 0$

$$\frac{dP}{dR_L} = V_T^2 \left[\frac{R_T - R_L}{(R_T + R_L)^3}\right] = 0$$

- Therefore maximum power will be delivered to the load when

$$R_L = R_T$$

- At this value of R_L the output current and voltage will be

$$V_{max} = \frac{V_T}{2}$$

$$I_{max} = \frac{V_T}{2R_T}$$

- And the maximum power from the circuit is

$$P_{max} = \frac{V_T^2}{4R_T}$$

- In terms of the Norton equivalent circuit the maximum power will be when $R_L = R_T$ and the load voltage, load current, and maximum power will be given by

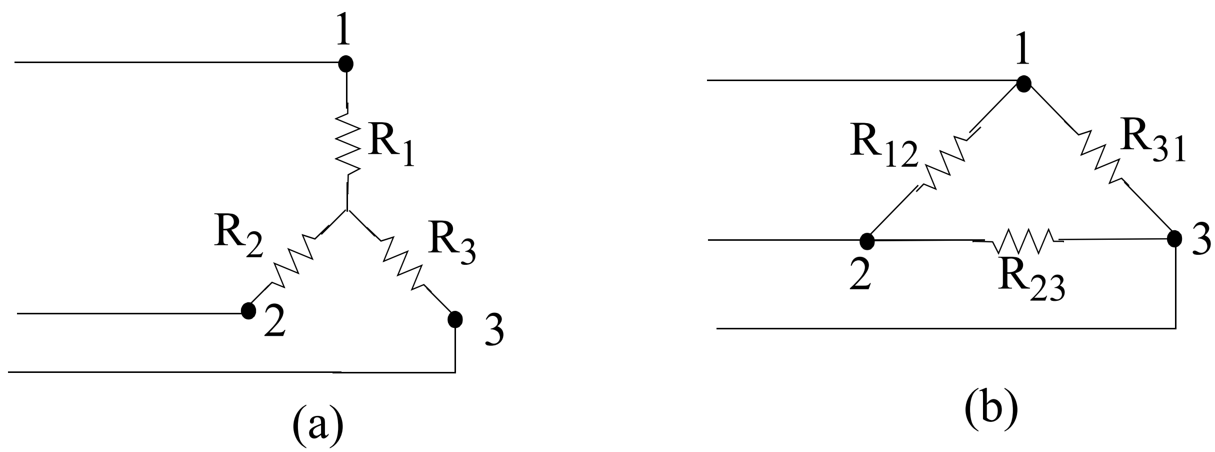
$$V_{max} = \frac{I_N R_T}{2}$$

$$I_{max} = \frac{I_N}{2}$$

$$P_{max} = \frac{I_N^2 R_T}{4}$$

3.6 Wye-Delta (Y-D) Transformations

- The three resistors shown in circuits (a) and (b) below cannot be reduced to a single resistor since none of them is in series or in parallel.



  - Circuit (a) is termed a Wye (Y) configuration
  - Circuit (b) is termed a Delta (Δ) configuration
  - Sometimes a circuit can be simplified by converting between these two configurations - where such a conversion preserves the voltages and currents seen by the circuit connected to nodes 1, 2, and 3.

3.6.3 Wye to Delta Transformation

- By using KCL and KVL for the Y and Δ connected circuits and assuming that all of the circuits parameters (currents and voltages) connected to the 3 leads in both circuits are identical we can find the transformation to convert a Y circuit to a Δ circuit
  - after fairly lengthy algebra we will arrive at

$$R_{12} = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

$$R_{23} = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

$$R_{31} = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

  - If it is assumed that $R_1 = R_2 = R_3 = R_Y$ and $R_{12} = R_{23} = R_{31} = R_\Delta$ then we have that

$$R_\Delta = \frac{3R_Y^2}{R_Y} = 3R_Y$$

3.6.4 Delta to Wye Transformation

- To transform a D connection to a Y connection we can use the following formulas

$$R_1 = \frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}}$$

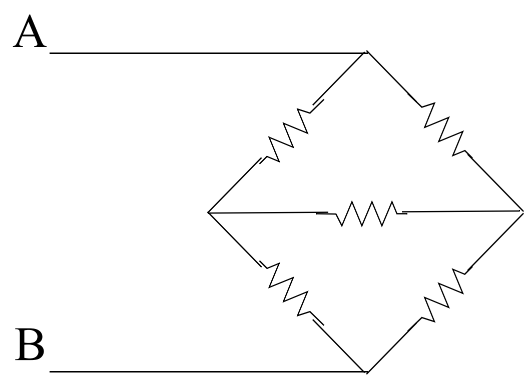
$$R_2 = \frac{R_{23}R_{12}}{R_{12}+R_{23}+R_{31}}$$

$$R_3 = \frac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}}$$

  - If it is assumed that $R_1 = R_2 = R_3 = R_Y$ and $R_{12} = R_{23} = R_{31} = R_\Delta$ then we have that

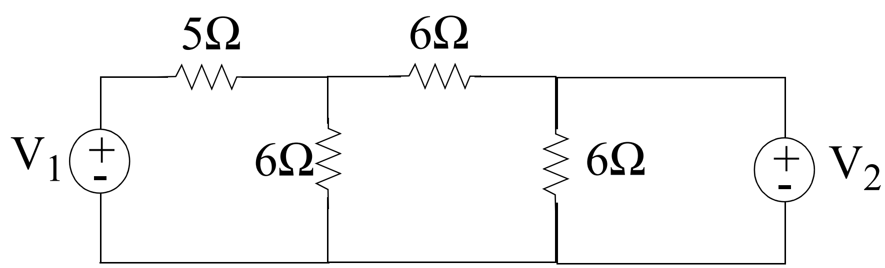
$$R_Y = \frac{R_\Delta}{3R_\Delta^2} = \frac{1}{3}R_\Delta$$

Ex. 3.10 Assume all the resistances are equal to 6Ω. Find the equivalent resistance between terminals A and B.

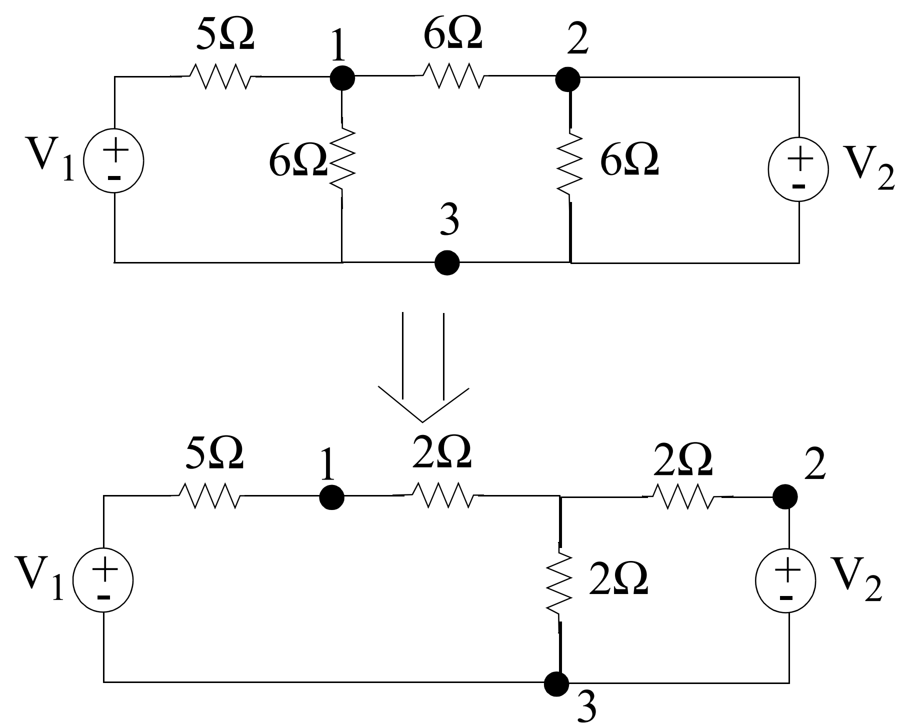


Solution:

Ex. 3.11 Find the current entering V_1 by replacing the Δ connected resistors with a Y connection (Assuming $V_1 = 10V$, $V_2 = 5V$)



Solution:



Assignment #3

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

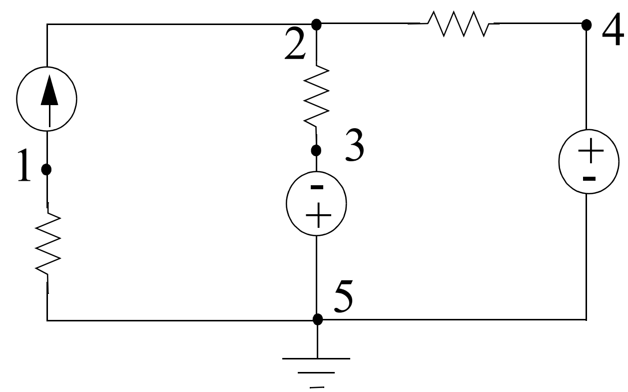
Chapter 4
Node Voltage Analysis

4.1 Motivation

- In the proceeding two chapters we have seen KVL, KCL and Ohm's law and learned how to apply them to solve for circuit parameters
  - our current method of solving the problems though is slow, fairly tedious, and error prone.
  - we would like a more systematic method of solving for the circuit parameters which is less *ad hoc*.
  - As we have seen, we tend to end up with the need to solve a system of simultaneous equations -> matrix algebra.
  - Is there a way we can be more systematic about writing these equations?
    - Nodal analysis is one such approach

4.2 Node Voltage

- A node in a circuit is a point connecting two or more circuit elements (as defined in the Introduction)
  - We can choose any one of the nodes within the circuit as the **reference node** (i.e. we ground this node such that its voltage is 0V)
  - If the circuit already has a ground element then we use that as the reference node



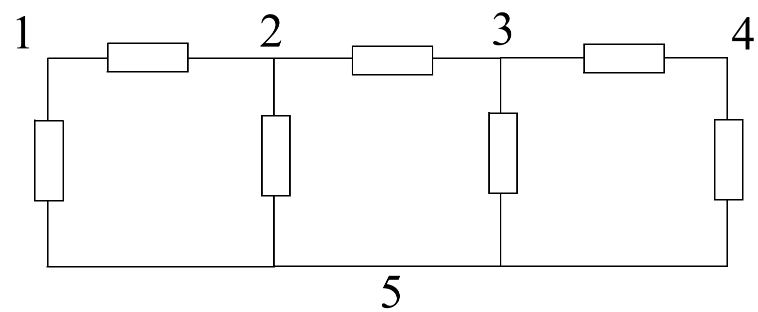
  - The circuit above has 5 nodes, of which node 5 has been made the reference node (note: the choice of which node to make the reference node is arbitrary, but a good choice can simplify the circuit equations)
  - **Node voltage** - the voltage of the given node relative to the reference node, using the convention that the reference node is at 0V.

- In the above circuit, V_{35} is therefore the voltage difference between node 3 and node 5.

- Generally, for voltage differences which include the reference node we drop the reference node from the subscript.

$$V_{35} = V_3 - V_5 = V_3 - (0) = V_3$$

Ex. 4.1 Find the voltages across all of the elements given the node voltages: $V_1 = 3V$, $V_2 = 5V$, $V_3 = -2V$, $V_4 = 4V$, and node 5 is the reference node.



Solution:

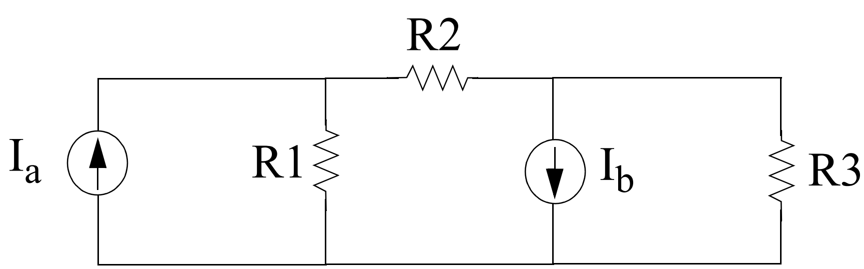
4.3 Nodal Analysis

- For a circuit containing *n* nodes we can obtain *n-1* nodal equations by applying KCL at each node.
  - We can then solve these equations via matrix algebra
  - Once we know the node voltages we can easily compute any other circuit parameter we wish to know

4.3.1 Circuit with only current sources

- Nodal analysis is easiest to understand for circuits with only current sources.

Ex. 4.2 Solve for the node voltages in the following circuit



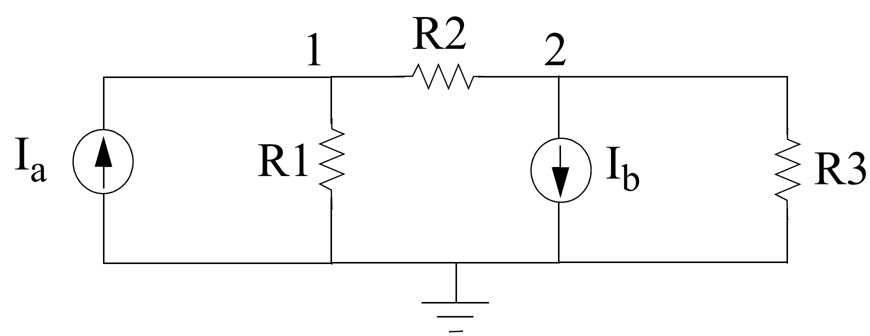
- Step 1: Label the nodes.

- Step 2: Assign the reference node. Generally, it helps to assign the reference node to the node connected to the most elements (i.e. node 5 in Ex. 4.1)

- Step 3: Write the KCL equation at each of the nodes. We need to assume a sign convention to be able to do this.

Convention: Currents entering a node are assumed to be negative and currents leaving a node are assumed to be positive.

- The convention is arbitrary. We could choose the opposite convention. But we must be consistent for all the equations we write for the given circuit.



- At node 1 we have,

$$-I_a + I_1 + I_{12} = 0$$

- In terms of node voltages the currents can be written as

$$-I_a + \frac{V_1}{R_1} + \frac{V_{12}}{R_2} = 0$$

$$-I_a + \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_2} = 0$$

$$-I_a + G_1 V_1 + G_2(V_1 - V_2) = 0$$

- Note the above equation has 3 terms (one for each of the branches in node 1) and is written in terms of the current source, the node voltages, and the conductances.

- For node 2 we have,

$$G_2(V_2 - V_1) + G_3 V_3 + I_b = 0$$

- We now have 2 equations in two unknown which we can solve using matrix algebra

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_1 + G_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_a \\ -I_b \end{bmatrix}$$

- The above equations have been put into the form $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_1 + G_2 \end{bmatrix}$$

$$\boldsymbol{v} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} I_a \\ -I_b \end{bmatrix}$$

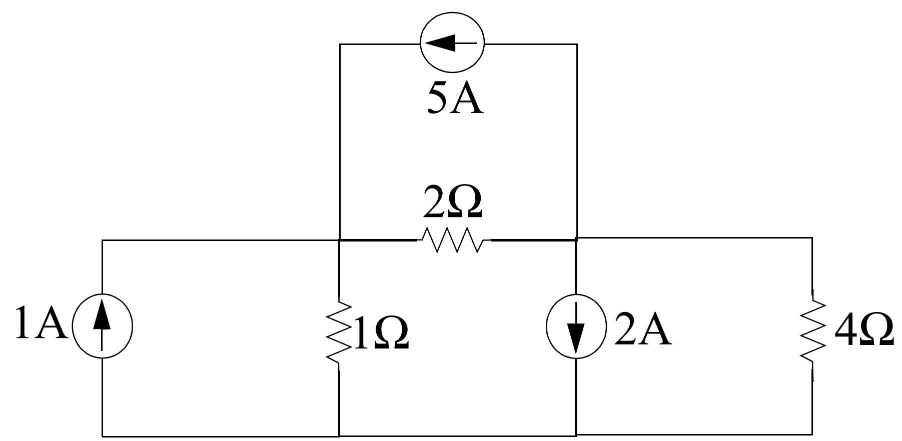
- The solution to $\boldsymbol{v}$ can be found by $\boldsymbol{v} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

- Within the course you are encouraged to use Matlab to solve the assigned problems (since hand solving matrix equations is tedious and very error prone)

- Assuming values for $I_a = 5A$, $R_1 = 3\Omega$, $R_2 = 4\Omega$, $R_3 = 7\Omega$ and $I_b = -2A$ then in matlab the above solution can be found by typing

```
unix prompt% matlab
>> A = [1/3+1/4) -1/4; -1/4 1/4+1/7];
ans =
      0.5882   -0.25
      -0.25    0.3929
>> b = [5 2]';
ans =
      5
      2
>> A\b
ans =
      14.7857
      14.5000
```

Ex. 4.3 Use nodal analysis to find the node voltages of the circuit below



Solution:

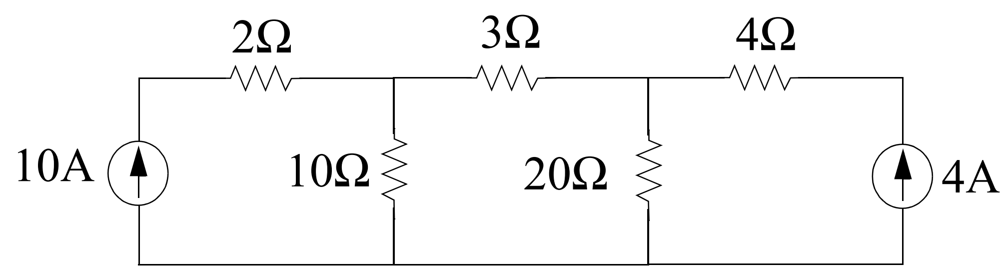
4.3.2 Node Equations from Inspection

- Obviously, we can directly write the nodal equations in matrix form provided that
  - 1. The circuit in question has no voltage sources. Only current sources are allowed if we are to write the matrix equations directly.
  - 2. There must be no dependent sources.
  - If these conditions hold then
    - The diagonal elements a_{ii} of $\boldsymbol{A}$ equal the sum of all conductances connected to node i.

- The elements a_{ij} equal the negative of the conductances connecting node i to node j.

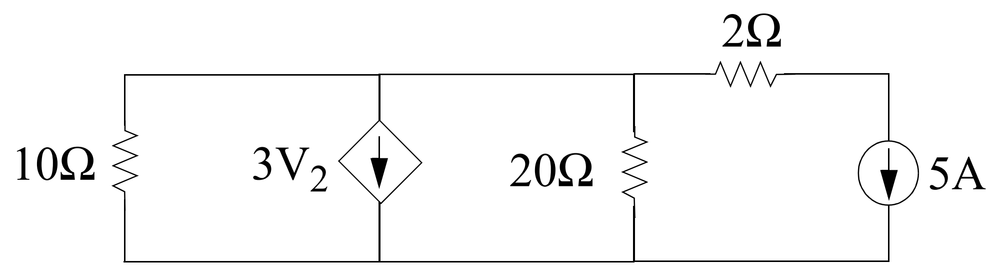
- Element b_i of $\boldsymbol{b}$ is the algebraic sum of all currents connecting to node i, where positive current is then the current enters the node.

Ex. 4.4 Find the node voltages by inspection for the following circuit



Solution:

Ex. 4.5 Perform nodal analysis on the circuit below



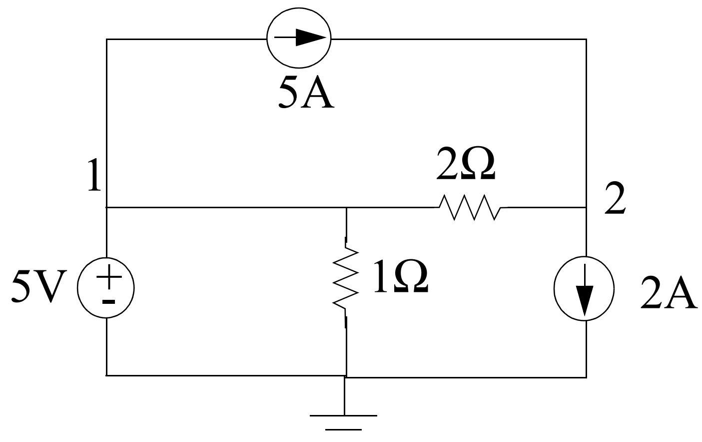
- Note the dependent source. This means the inspection method will not work so we must write out the equations and then put them into matrix form.

Solution:

4.3.3 Node Analysis with voltage sources

- Nodal analysis is simplified when voltage sources are present since they either (a) directly give one of our node voltages, (b) allow us to write a simple equation for the voltage difference between two nodes.

  - Case (a) occurs when the voltage source is connected to the reference node.



  - In this case we already are given one of the node voltages

$$V_1 = 5V$$

  - So we only have one of the node voltages left to solve for,

- Write the node voltage equation at node 2 to get V_2

$$-5+2+\frac{1}{2}(V_2-V_1) = 0$$

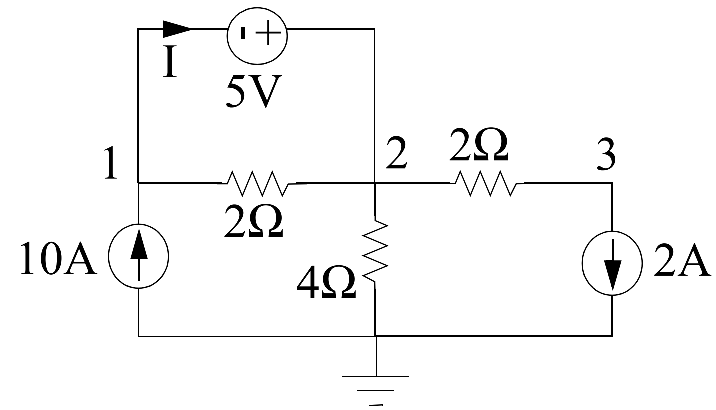
$$V_2 = 11V$$

- We can now solve for any of the other circuit parameter which may be of interest.

Ex. 4.6 In the above circuit solve for the power produced by V_1

Solution:

- Case (b) occurs when the voltage source is connected to two non-reference nodes.



- In this case we can also quickly write one of the voltage equations

$$V_2 - V_1 = 5$$

$$V_2 = V_1 + 5$$

- but the current I in the circuit posses a problem since it depends on the values of the other circuit elements. (i.e. it does not depend on the value of the voltage source which it flows through)

- We have quickly eliminated one variable (or found a simple expression for it) but we have gained another variable, I, which we need to solve for.

- There are two approaches to this new problem: a simple, slow approach, and the supernode method

4.3.4 Simple, Slow Approach (for voltage sources not connected to the reference node)

- We need to assume a current entering the voltage source. It doesn't matter which direction we choose for the current, but we need to include this current in our node voltage equations.

  - We have 3 unknowns (V_1, V_2, V_3) so we will need 3 equations.

  - From the voltage source connecting nodes 1 and 2 we have that

$$V_2 = V_1 + 5 \text{ (Eq 1)}$$

  - From node 1 we have that

$$-10 + \frac{1}{2}(V_1 - V_2) + I = 0$$

  - From node 2 we have that

$$\frac{1}{4}V_2 + \frac{1}{2}(V_2 - V_1) + \frac{1}{2}(V_2 - V_3) - I = 0$$

  - If we combine the above two equations we can eliminate I

$$-10 + \frac{1}{2}(V_1 - V_2) + \frac{1}{4}V_2 + \frac{1}{2}(V_2 - V_1) + \frac{1}{2}(V_2 - V_3) = 0$$

- Simplifying

$$\left(\frac{1}{2}+\frac{1}{4}\right)V_2-\frac{1}{2}V_3 = 10 \text{ (Eq. 2)}$$

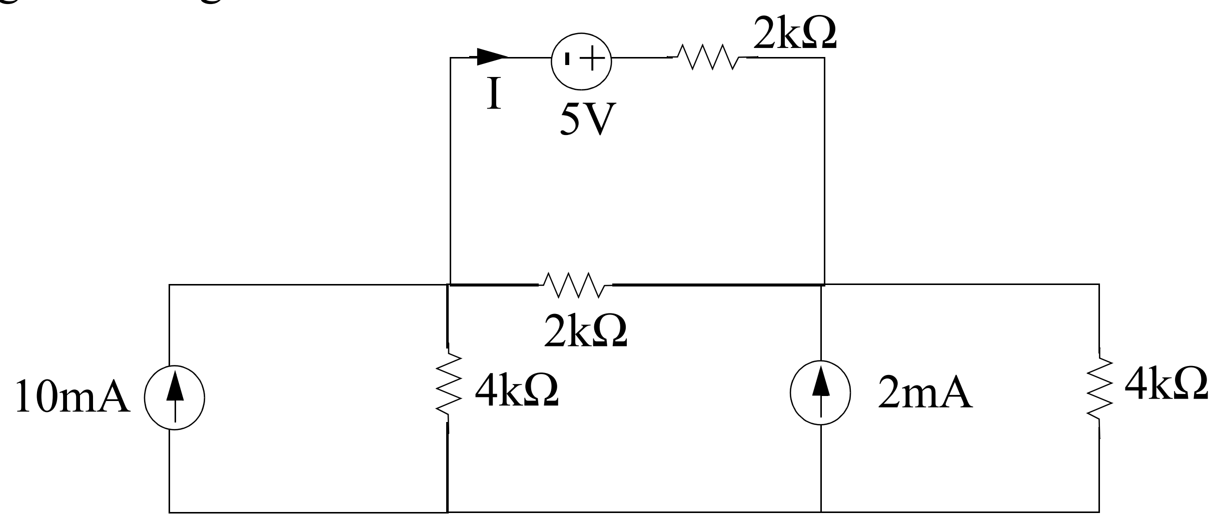
- Now we need just 1 more equation which we can get by KCL for node 3

$$\frac{1}{2}(V_3-V_2)+2 = 0 \text{ (Eq. 3)}$$

- With 3 equations in our 3 unknowns we can now apply matrix algebra to get the solution

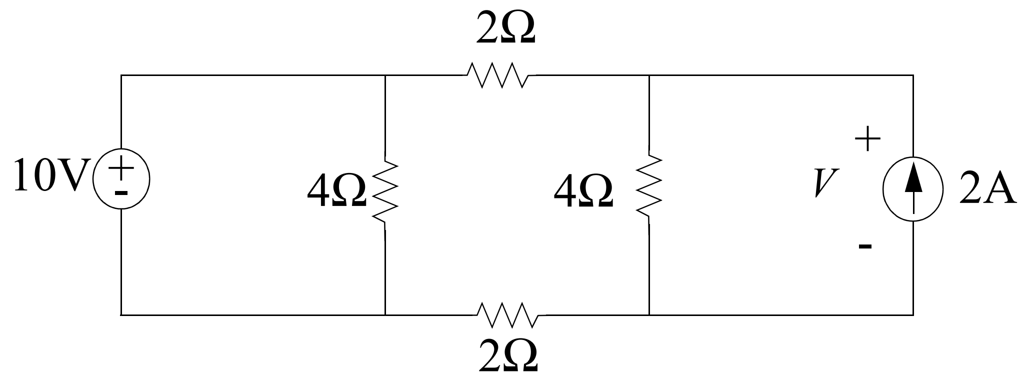
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0.75 & -0.5 \\ 0 & -0.5 & 05 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -2 \end{bmatrix}$$

Ex. 4.7 Write down the node equations for the circuit below and find the current entering the voltage source



Solution:

Ex. 4.8 Find V using nodal analysis

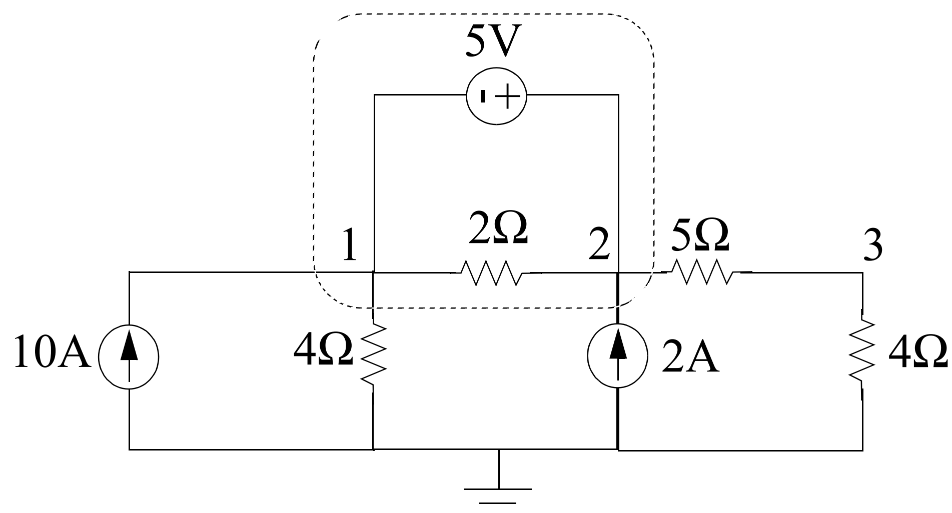


Solution:

4.3.5 Supernode Method

- The above approach results in an extra circuit variable in the equations which we must cancel out.
    - We know that KCL applies to any node that we define, including over closed boundaries (Chapter 2).
    - So we can simplify the analysis by enclosing the voltage source within a closed boundary and writing the KCL equation for this supernode.

Ex. 4.9 Applying supernode method



- Write down the equation relating V_1 and V_2 (given by voltage source)

$$V_2 - V_1 = 5V$$

- Write the KCL equation from the supernode formed by nodes 1 and 2 (just as we have done for the simple nodes before except we need to account for all currents crossing the supernodes boundary)

$$-10 - 2 + \frac{1}{4}V_1 + \frac{1}{5}(V_2 - V_3) = 0$$

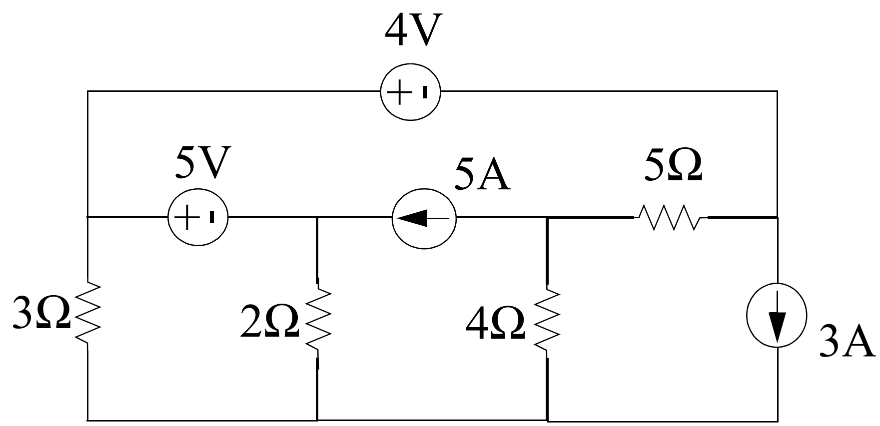
- Write the KCL equation for node 3

$$\frac{1}{4}V_3 + \frac{1}{5}(V_3 - V_2) = 0$$

- We now have directly obtained the 3 equations in 3 unknowns and can write the matrix equation

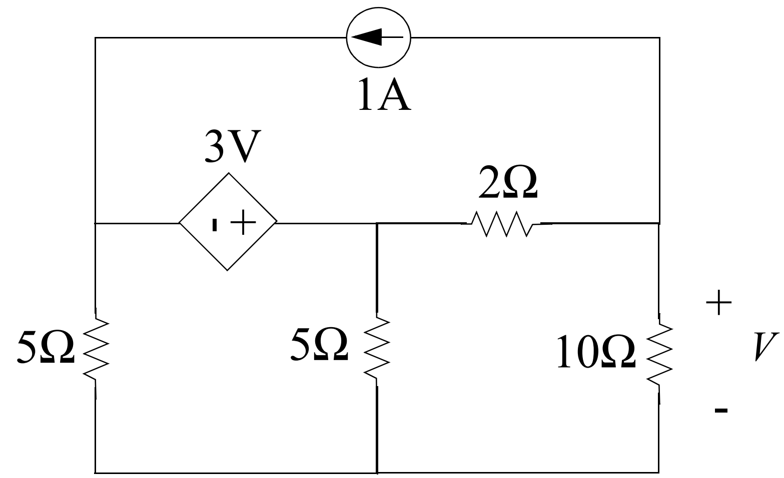
$$\begin{bmatrix} -1 & 1 & 0 \\ 0.25 & 0.5 & -0.2 \\ 0 & -0.2 & 0.45 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 0 \end{bmatrix}$$

Ex. 4.10 Find the current through the 5V source



Solution:

Ex. 4.11 Find the node voltages of the following circuit



Solution:

Assignment #4

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 5
Mesh Current Analysis

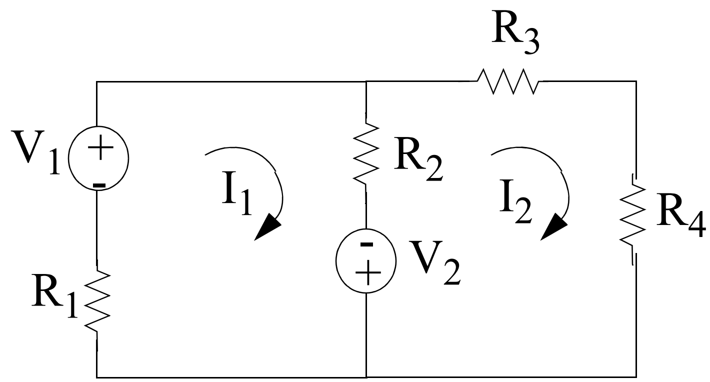
5.1 Motivation

- In the previous chapter we used KCL at each of the nodes to systematically generate the set of simultaneous node voltage equations (which we then solved with matrix algebra)
- In this chapter we use KVL around meshes (closed loops) to systematically generate a set of simultaneous mesh current equations.

5.2 Mesh Current

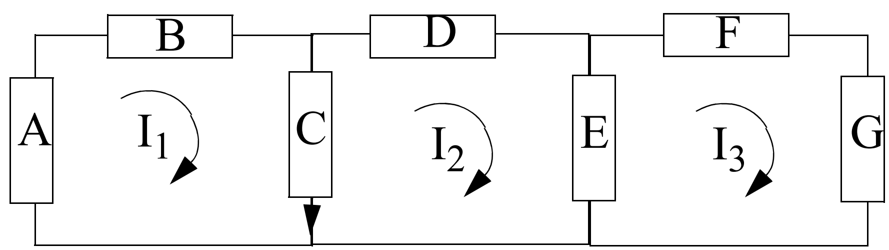
- A **mesh** is a closed path containing two or more circuit elements
- The **mesh current** is the current that flows around this closed path (mesh)
- If a branch is shared by two meshes, then the current in the branch is the algebraic sum of the two mesh currents.

Ex. 5.1 The circuit below contain 2 meshes.



- If we know the mesh currents then we can find all of the other circuit parameters

Ex. 5.2 For the given circuit find the current flowing through each of the elements if the mesh currents are $I_1 = 3A$, $I_2 = 5A$, and $I_3 = 7A$



- Note that the current through shared branches is the algebraic combination of the mesh currents which the branch shares.

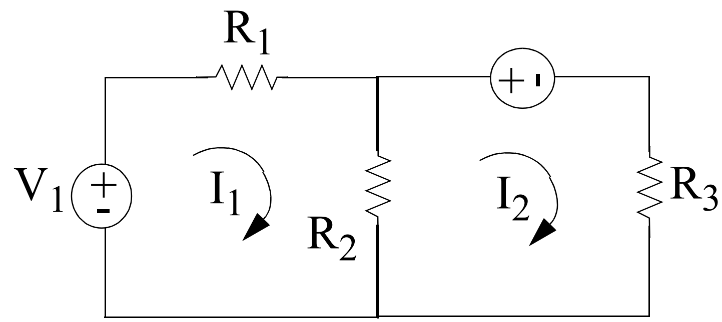
Solution:

- For circuits containing n meshes we have n unknowns and we can generate n simultaneous mesh equations based on applying KVL around each mesh. We can then solve these equations using matrix algebra.

5.3 Mesh Analysis

- Mesh analysis consists of applying KVL around each mesh in our circuit.

- Consider the following circuit



- This circuit has two meshes and two mesh currents I_1 and I_2

- Applying KVL around the first mesh

$$V_1 - R_1 I_1 - R_2(I_1 - I_2) = 0$$

$$(R_1 + R_2)I_1 - R_2 I_2 = V_1$$

- Note we are following our convention

    - moving from the negative to positive terminals of a voltage source (in the direction of the assigned current flow) produces a voltage rise which we've defined as positive.

    - moving from the positive to negative terminals of a resistor produces a voltage drop which we've defined as negative.

- Note that the current through the shared branch (resistor R_2) is the algebraic sum of the two mesh currents with the sign of the mesh currents based on the direction which they travel through the shared branch.
  - I_1 moves through the branch from top to bottom and since we are writing KVL for mesh 1 it is assumed to be positive.
  - I_2 moves through the shared branch in the opposite direction (from the bottom to the top), hence it produces a voltage on R_2 which is in the opposite direction as the voltage across R_2 caused by I_1.
  - Hence, the voltage drop across R_2 is given by $R_2(I_1 - I_2)$.
- Writing KVL for mesh 2

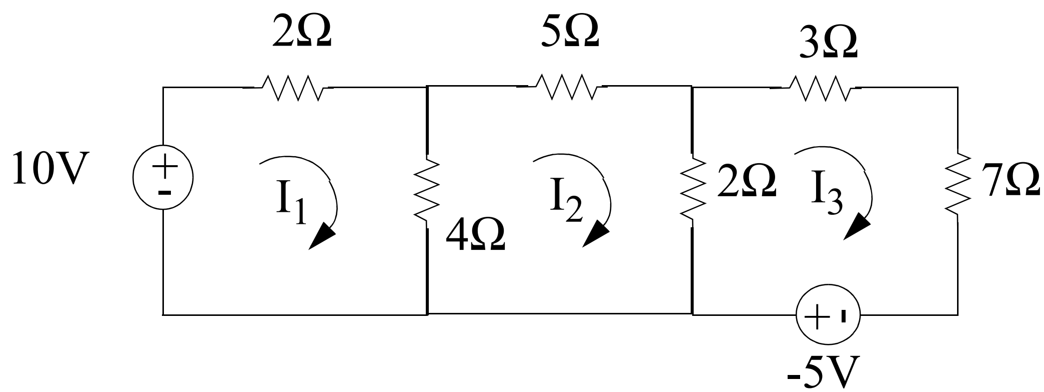
$$-R_2(I_2 - I_1) - V_2 - R_3 I_2 = 0$$

$$-R_2 I_1 + (R_2 + R_3) I_2 = V_2$$

- This gives us two simultaneous mesh equations which we can solve using matrix algebra

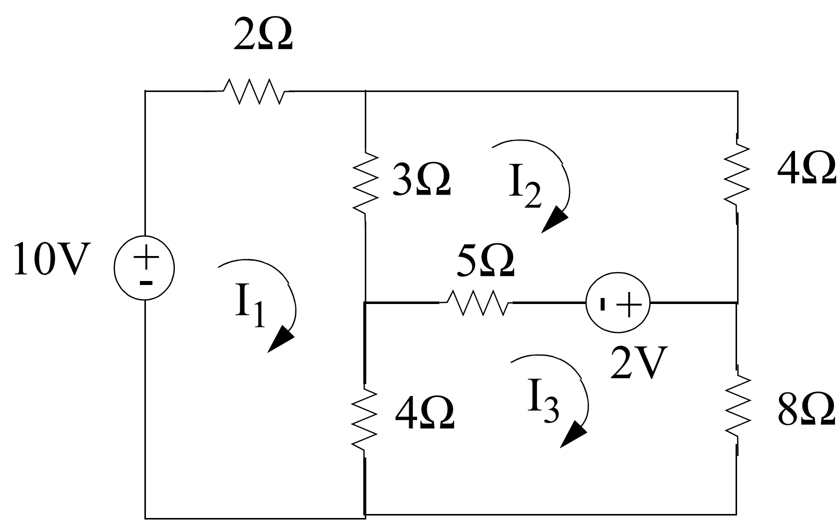
$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Ex. 5.3 Use mesh analysis to find the mesh currents for the circuit below



Solution:

Ex. 5.4 Use mesh analysis to find the mesh currents for the circuit shown below

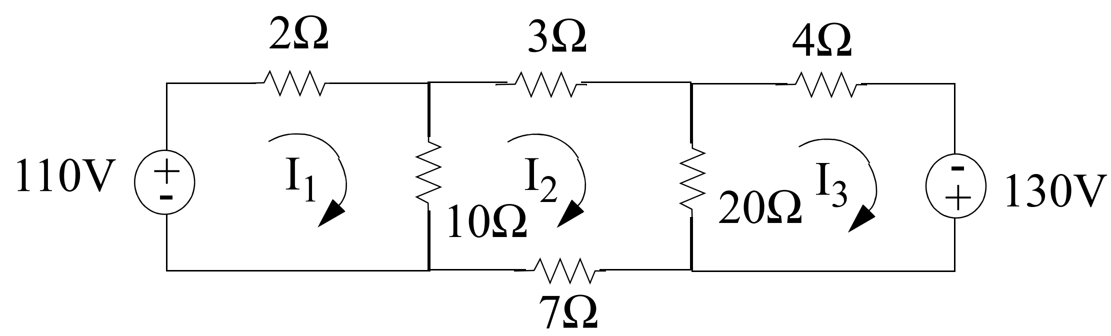


Solution:

5.4 Mesh Equations by Inspection

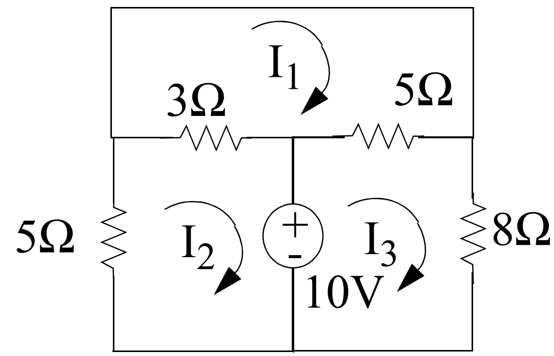
- From the previous examples we can form the matrix A and the vector b from inspection according to the following process
    - The circuit should not have any current sources.
    - The circuit should not have any dependent voltage or current sources.
    - The diagonal elements a_{ii} of $\boldsymbol{A}$ equal the sum of all of the resistances around mesh i.
    - The element a_{ij} equals the negative of the resistance shared by meshes i and j.
    - Element b_i of $\boldsymbol{b}$ is the algebraic sum of all of the voltages in mesh i, where positive voltage is when it is in the direction of the assumed mesh current.
- Note that the matrix $\boldsymbol{A}$ will be symmetric due to the rule for assigning the elements a_{ij}.
- Note that this process does NOT apply if there are current sources in the circuit.

Ex. 5.5 Find the mesh currents in the following circuit by inspection.



Solution:

Ex. 5.6 Find the mesh currents in the circuit below by inspection.

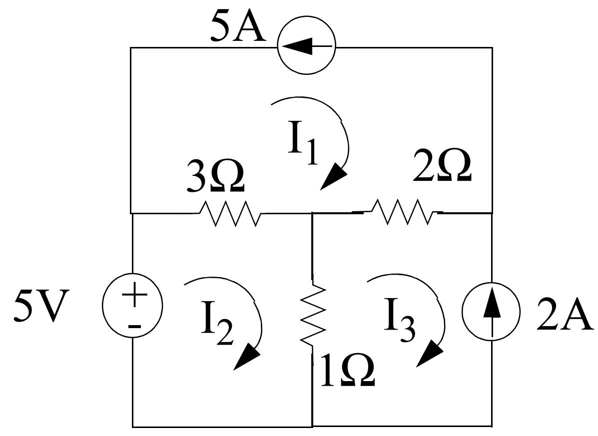


Solution:

5.5 Mesh Analysis with Current Sources

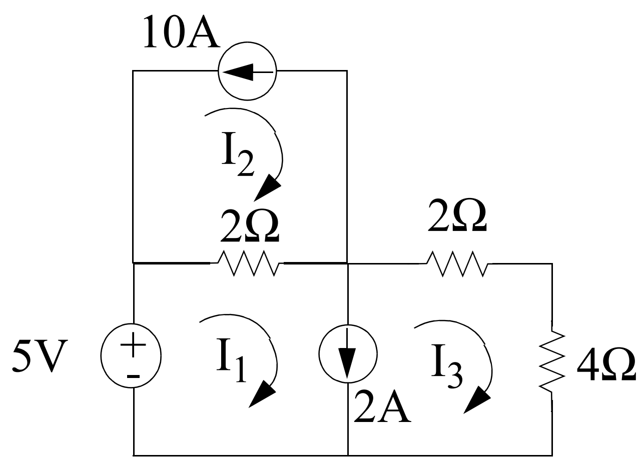
- As in node voltage analysis with voltage sources, there are two cases which may be present in mesh current analysis with current sources

- Each current source belongs to only one mesh.



(a)

- At least one of the current sources belongs to two meshes.



(b)

- For circuit (a) (the case where the current sources belong to only one mesh)

  - For the two current sources we can easily write the mesh equations

$$I_1 = -5A$$

$$I_3 = -2A$$

  - This gives us two of the three mesh currents, so we just need to write a mesh equation for mesh 2

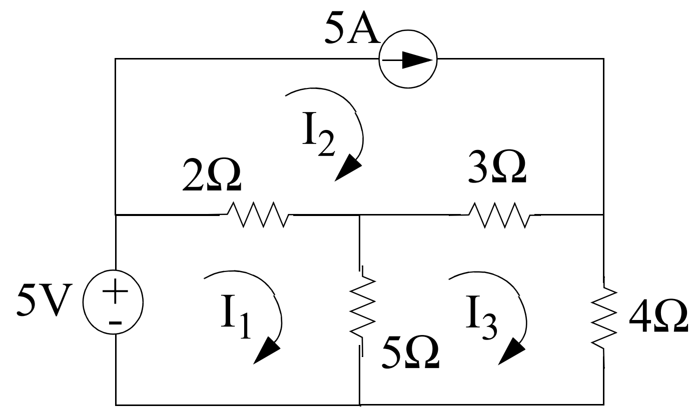
$$3(I_2 - I_1) + 1(I_2 - I_3) = 5V$$

$$4I_2 - (-15) - (-2) = 5$$

$$I_2 = -\frac{12}{4} = -3A$$

- So for a circuit where the current sources each belong to only one current mesh, we can trivially solve for those mesh currents.

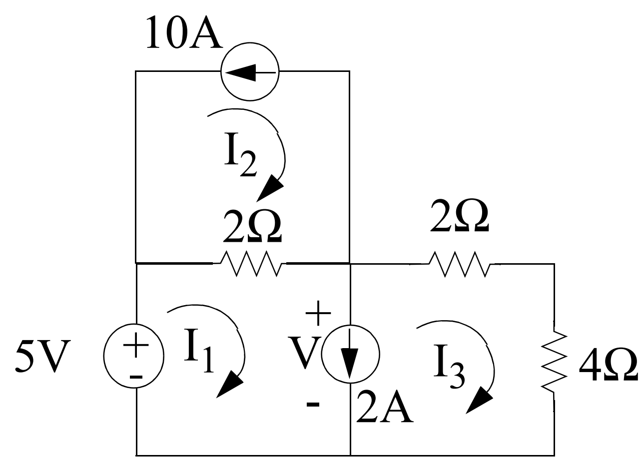
Ex. 5.7 Find the mesh currents for the circuit shown below. What is the voltage across the current source? What is the power delivered by each of the power sources?



Solution:

5.6 Solving for Current Sources Belonging to Two Meshes

- If the current source belongs to two meshes then we can immediately write an equation for the current source in terms of the two mesh currents.
  - From circuit (b) from above (reproduced below) we have that



(b)

$$I_1 - I_3 = 2A$$

  - However the voltage drop V across the 2A current source is unknown and must be taken into account in order to apply KVL to mesh 1 and mesh 3.
  - There are two methods of dealing with the voltage drop across the shared current source: 1) a simple but slow method or 2) the supermesh method.

5.6.1 Simple but slow method

- To illustrate this method we will solve the circuit (b) above.
  - The technique for handling this situation involves the following steps:
    1. Assume a voltage drop V across the current source that is shared by the two meshes. The direction of the voltage drop is not important. The magnitude of that voltage will be determined later by application of KVL
    2. Write down an equation relating the shared current source to the two mesh currents.

$$I_1 - I_3 = 2$$

    3. Write KVL equations for meshes 1 and 3

$$5 - 2(I_1 - I_2) - V = 0$$

$$V - (2 + 4)I_3 = 0$$

    4. Add these two equations to eliminate the voltage drop across the shared current source

$$5 - 2(I_1 - I_2) - (2 + 4)I_3 = 0$$

    5. Write KVL for the rest of the meshes, which in this example is mesh 3

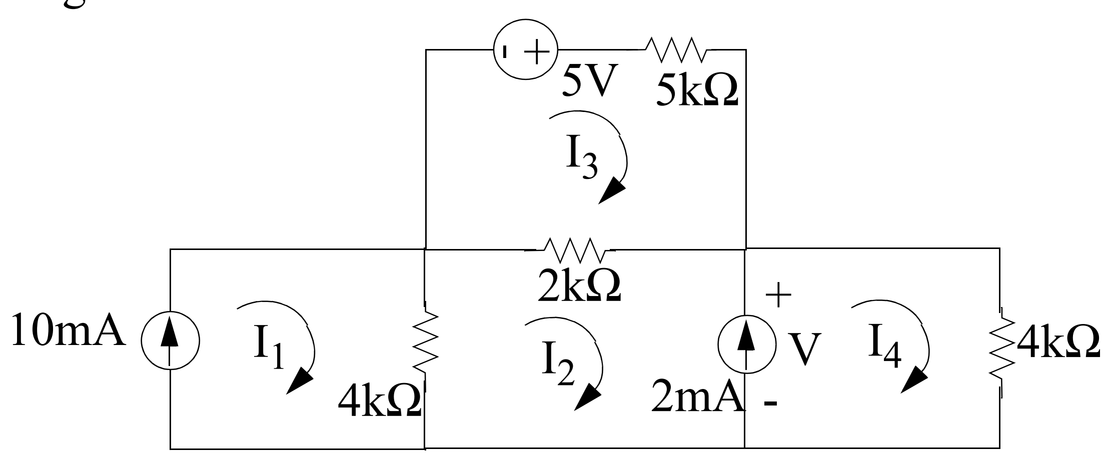
$$I_2 = -10A$$

6. Now we have a system of 3 equations in 3 unknowns, thus we can write the matrix equation

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -6 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -10 \end{bmatrix}$$

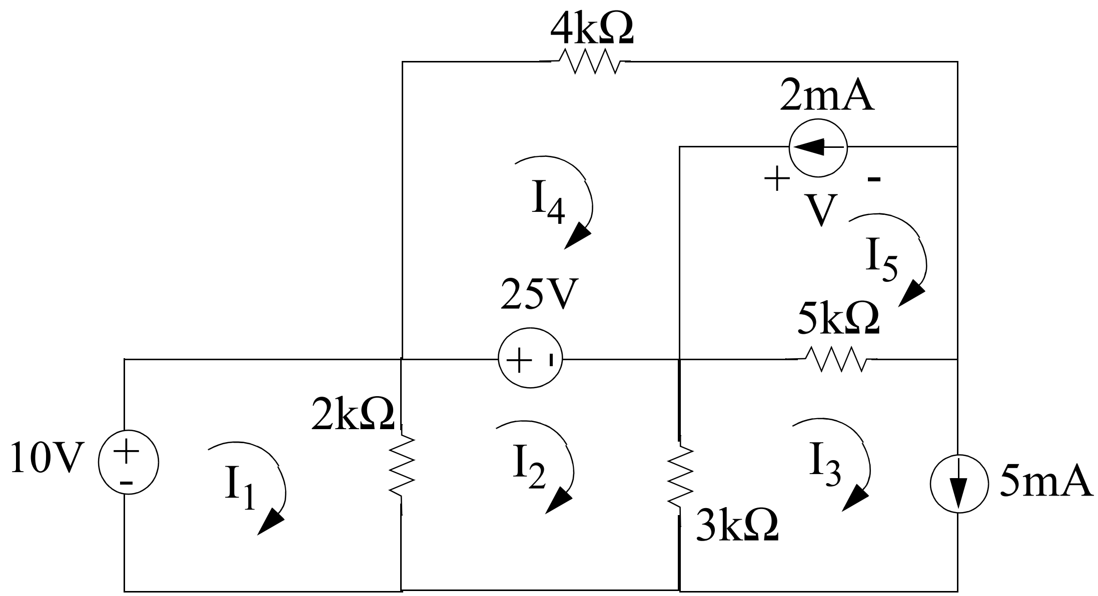
- After finding the values for all of the mesh currents, we can find the value of V by applying KVL to either mesh 1 or mesh 3.

Ex. 5.8 Find the mesh currents and the voltages across the current sources for the following circuit.



Solution:

Ex. 5.9 Find the mesh currents for the following circuit.



Solution:

5.6.2 Supermesh Method

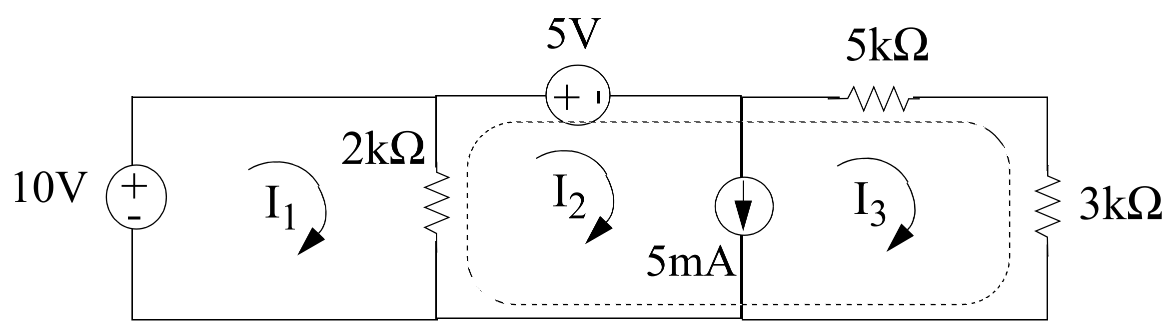
- In the previous chapter we used supernodes to simplify the process of obtaining our node voltage equations when we had voltage sources

- We can use a similar approach to obtain all of the mesh equations directly.

  - To do this we need to form a supermesh which includes the shared current source

  - We then perform KVL around the supermesh - remember from Chapter 2 that KVL applied to any closed boundary (loop) so we can apply it to the supermesh which we have constructed

Ex. 5.10 Find the mesh currents using the supermesh method.



Solution:

1. Write down the equation relating the current source and the 2 mesh currents which share it

$$I_2 - I_3 = 5$$

2. Write KVL for the supermesh formed by combining meshes 2 and 3

$$2I_1 - 2I_2 - 5 - 8I_3 = 0$$

3. Write KVL for the remaining mesh (mesh 1)

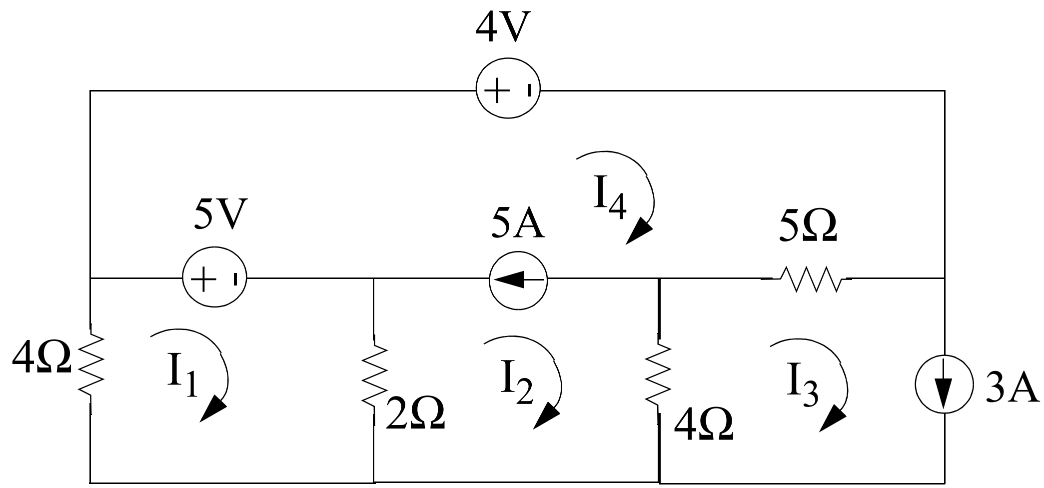
$$10 - 2I_1 + 2I_2 = 0$$

4. We now have a system of 3 simultaneous equations in 3 unknowns and we can write the matrix equation

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -8 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -10 \end{bmatrix}$$

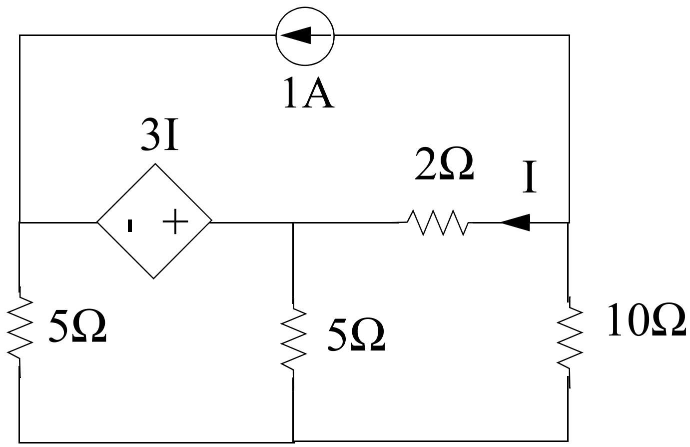
5. We can solve for the mesh currents as $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

Ex. 5.11 Find the voltage across the 5A current source using mesh analysis for the following circuit.



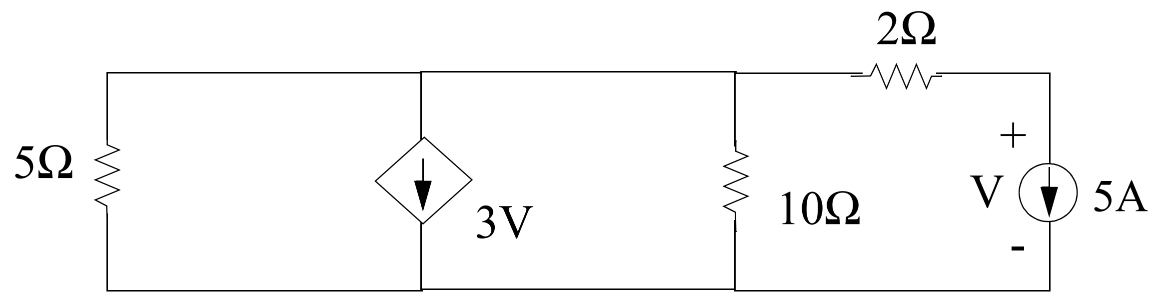
Solution:

Ex. 5.12 Find the mesh currents in the following circuit.



Solution:

Ex. 5.13 Find the mesh currents in the following circuit.



Solution:

Assignment #5

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 6
Energy Storage Elements
(Inductance and Capacitance)

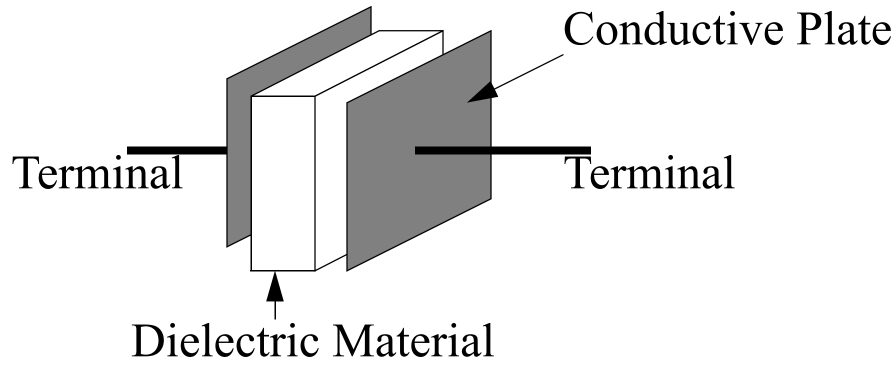
6.1 Introduction

- To this point we have just looked at circuits containing resistors and power sources.
- There are two other important elements, which respectively store electrical and magnetic energy
  - Capacitor - stores electrical energy.
  - Inductor - stores magnetic energy.
- These two elements are essential for building useful circuits such as electric power supplies, computer memories, receiver tuners, and electrical signal filters.

6.2 Capacitors

- A capacitor is a two terminal device which is capable of storing electrical energy.

- In general a capacitor is constructed by two plates separated by an insulating (dielectric) material.



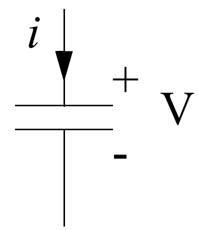
- The electrical energy stored in a capacitor results from the electric field located between the capacitor plates

- When a voltage is applied across the capacitor, equal but opposite electric charges appear on the capacitor plates in proportion to the voltage applied across it.

$$q = Cv$$

- where C is the **capacitance** and is measured in farads (F).

- The electrical symbol for a capacitor is



- The current through the capacitor is obtained by differentiating the above equation with respect to time (remember from Chapter 1 that $i = \frac{dq}{dt}$)

$$i = C\frac{dv}{dt}$$

Ex. 6.1 If the voltage across a capacitor changes with time as $10t\frac{V}{s}$ what is the current passing through the capacitor assuming $C = 4\mu F$?

Solution:

- According to the charge-voltage equation of a capacitor as the voltage v increases with time, the charge on each of the capacitor plates also increases.

- This is consistent with a current entering the positive lead of the capacitor as shown above.

- We can express $v(t)$ in terms of $i(t)$ by integration

$$v(t) = \frac{1}{C}\int_{-\infty}^{\tau} i d\tau = \frac{1}{C}\left[\int_{-\infty}^{0} i d\tau + \int_{0}^{\tau} i d\tau\right] = v(0) + \frac{1}{C}\int_{0}^{\tau} i d\tau$$

- where the initial voltage $v(0)$ is related to the initial charge on the capacitor $q(0)$ as

$$v(0) = \frac{q(0)}{C}$$

Ex. 6.2 If the current passing through a capacitor changes with time as $i(t) = 10e^{-50t} mA$. What is the voltage across the capacitor assuming $C = 4\mu F$ and $v(0) = 2V$

Solution:

Ex. 6.3 Find the expression for the current through the capacitor if the voltage across a capacitor has the waveform

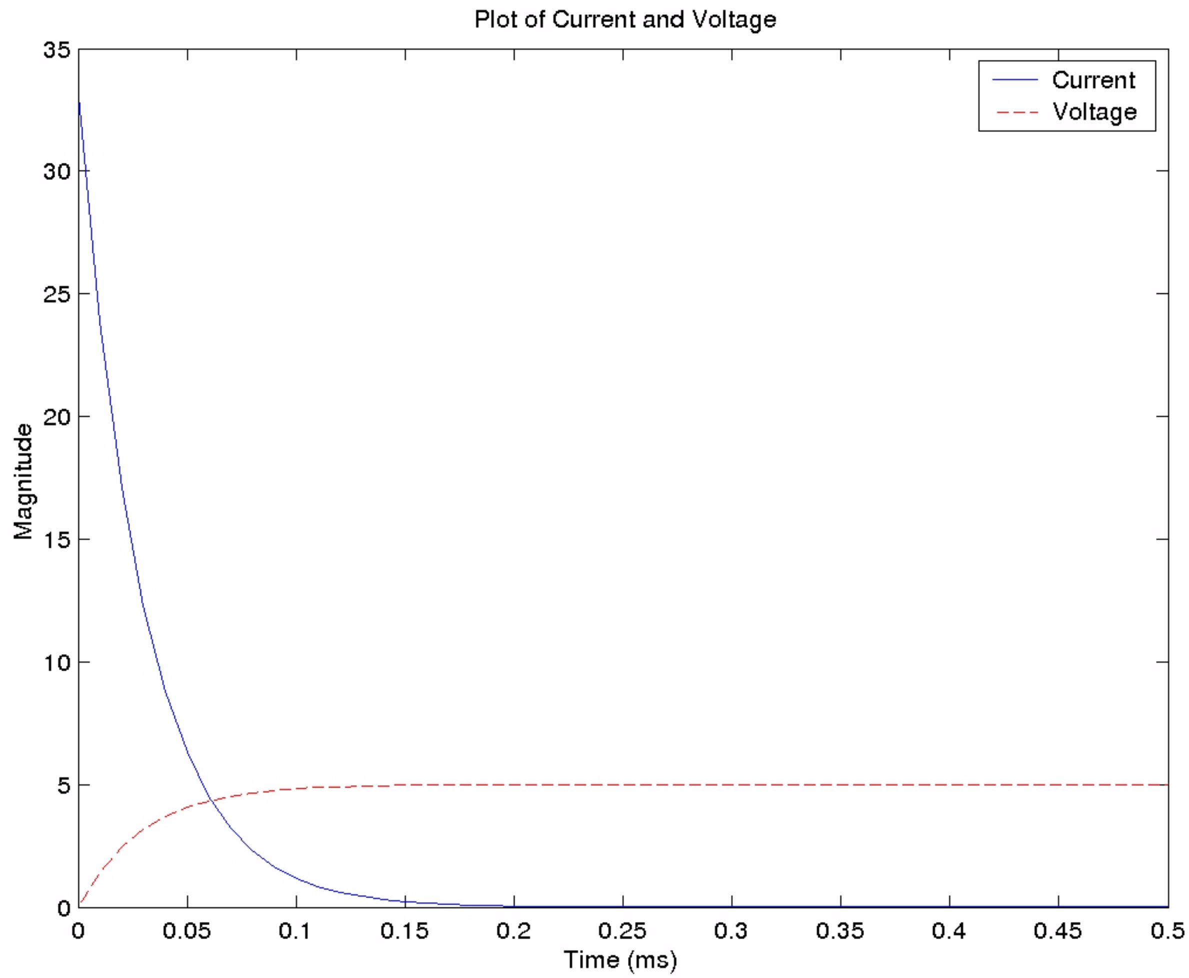
$$v(t) = \begin{cases} 0 & \text{for } t < 0 \\ a\left(1 - e^{-\frac{t}{\tau}}\right) & \text{for } t \geq 0 \end{cases}$$

Solution:

- Since the voltage across a capacitor is a function of the charges which have accumulated on its plates, the current flowing through a capacitor cannot change instantaneously even if there is a sudden change in the voltage across the capacitor (it will always take some time to dissipate the charges on the capacitor's plates).

- Sudden large magnitude changes in the voltage across a capacitor will result in sharp current pluses through the capacitor.

- The figure below shown the current versus voltage plot for the case where $C = 0.2H$, $a = 5A$, and $\tau = 30ms$ for the above example. Note that the sudden change of the voltage at $t = 0s$ results in a sharp pulse in the current.



6.2.1 Energy Stored in a Capacitor

- The energy stored in a capacitor at time t is given by

$$w(t) = \int_{-\infty}^{t} iv d\tau = \int_{-\infty}^{t} Cv\frac{dv}{d\tau}d\tau = C\int_{-\infty}^{t} vdv$$

  - Thus we have

$$w(t) = \frac{1}{2}Cv^2(t)$$

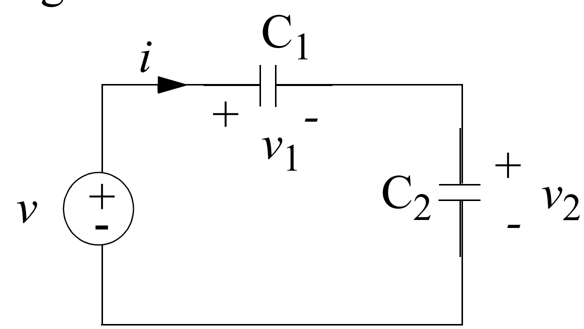
  - where $v(-\infty) = 0$.
  - Since $w(t) \geq 0$ the capacitor is a passive element.

Ex. 6.4 The energy stored in a $2\mu F$ capacitor is 15 mJ. Find the voltage across the capacitor and the stored charge.

Solution:

6.2.2 Capacitors Connected in Series

- As in the case for resistors, we would like to know the equivalent capacitance of two capacitors which are connected in series.
    - Consider the following circuit



- Applying KVL around the circuit gives

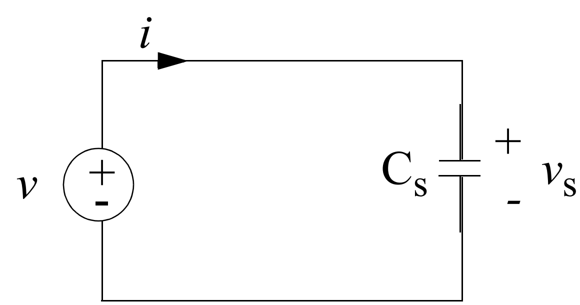
$$v = v_1 + v_2$$

- Applying $i = C\frac{dv}{dt}$ for capacitors, we get

$$v = \left[v_1(0) + \frac{1}{C_1}\int_0^t i\,d\tau\right] + \left[v_2(0) + \frac{1}{C_2}\int_0^t i\,d\tau\right]$$

$$v = [v_1(0) + v_2(0)] + \left[\frac{1}{C_1} + \frac{1}{C_2}\right]\int_0^t i\,d\tau$$

- Consider the following circuit



- Applying the current-voltage relationship for capacitors

$$v = v(0) + \frac{1}{C_s}\int_0^t i\,d\tau$$

- Hence, the two circuits above are equivalent (i.e. the voltage source supplies the same current to both circuits) when

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$v(0) = v_1(0) + v_2(0)$$

- The two capacitors are replaced by a single capacitor whose value is C_s and whose initial voltage equals the sum of the initial voltages of the series capacitors.

• The above results can be generalized for the case of n series connected capacitors to give

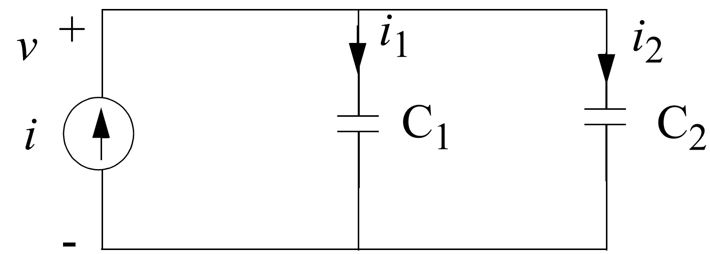
$$\frac{1}{C_s} = \sum_{j=1}^{n} \frac{1}{C_j}$$

$$v(0) = \sum_{j=1}^{n} v_j(0)$$

6.2.3 Capacitors in Parallel

• As in the case for resistors, we would like to know the equivalent capacitance of two capacitors which are connected in parallel.

- Consider the following circuit



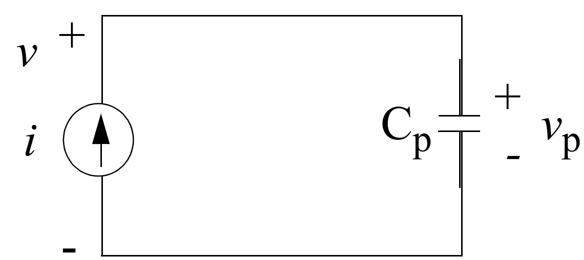
- Applying KCL at the top node gives

$$i = i_1 + i_2$$

- Applying the equation for the current through a capacitor

$$i = C_1\frac{dv}{dt} + C_2\frac{dv}{dt}$$

- Consider the following circuit



- Applying the equation for the current through a capacitor

$$i = C_p \frac{dv}{dt}$$

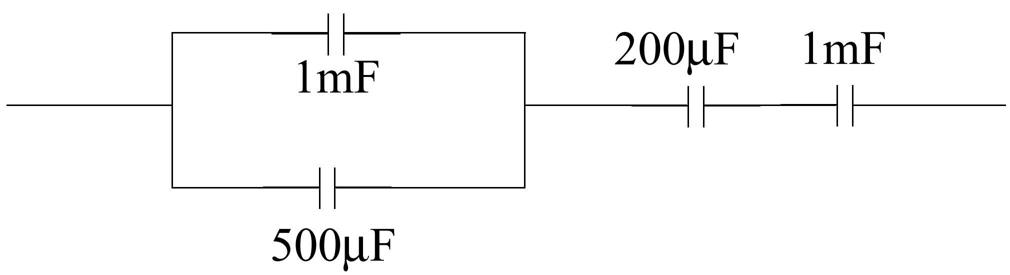
- The two circuits above are equivalent if

$$C_p = C_1 + C_2$$

- This result can be generalized to give the equivalent capacitance for a system of n parallel capacitors

$$C_p = \sum_{j=1}^{n} C_j$$

Ex. 6.5 Determine the equivalent capacitance for the following circuit



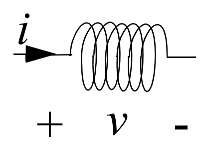
Solution:

6.3 Inductors

- An inductor is a two terminal device which stores magnetic energy.

- An inductor is constructed from a length of wire that is tightly wound into a spiral, which sometimes has a magnetic material as its core.

- The circuit symbol for an inductor is



- When current is passed though the inductor magnetic field lines are created in proportion to the magnitude of the applied current.

  - The total magnetic flux surrounding the inductor is given by

$$\lambda = Li$$

  - where L is the inductance (measured in Henry (H)) and λ is the magnetic flux.

  - Remember that passing a current through a wire generates a magnetic field, the coiled shape of an inductor causes it to store magnetic energy.

- According to Faraday's law, the voltage across the inductor is proportional to the rate of change of the magnetic lines of force surrounding the inductor, thus

$$v = \frac{d\lambda}{dt} = L\frac{di}{dt}$$

  - Alternatively, given v we can express i though integration

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v d\tau = \frac{1}{L}\int_{-\infty}^{0} v d\tau + \frac{1}{L}\int_{0}^{t} v d\tau = i(0) + \frac{1}{L}\int_{0}^{t} v d\tau$$

  - where $i(0)$ is the initial current on the inductor.

Ex. 6.6 The current in a 1H inductor changes linearly from 0 mA to 1 mA in 1 µs. Find the resulting voltage? (this is called the "bucking voltage")

Solution:

Ex. 6.7 The voltage across a 1 mH inductors changes as $v(t) = 120\cos 100t$. Find the current which flows through it?

Solution:

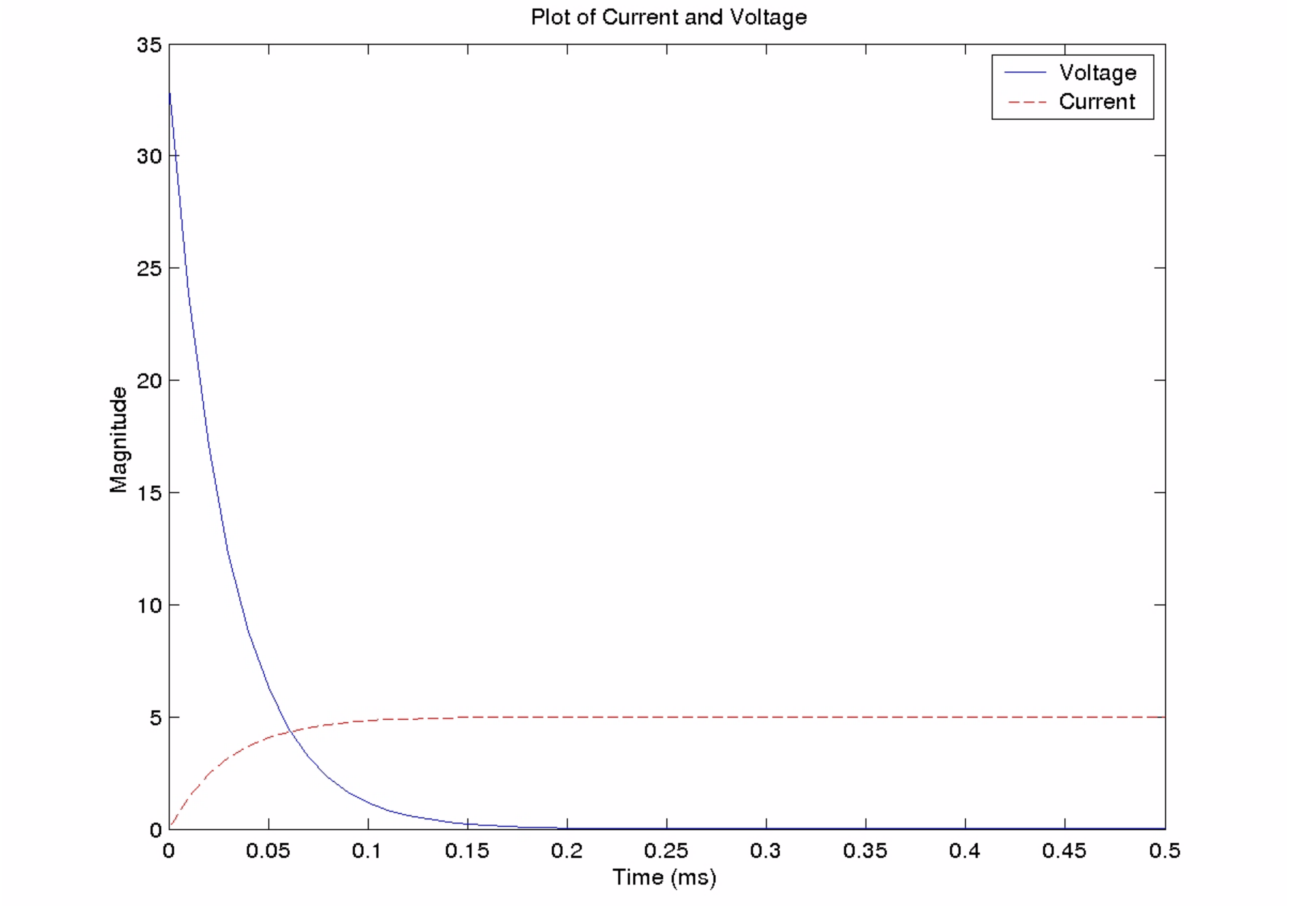
Ex. 6.8 The current through an inductor has the waveform

$$i(t) = \begin{cases} 0 & \text{for } t < 0 \\ a\left(1 - e^{-\frac{t}{\tau}}\right) & \text{for } t \geq 0 \end{cases}$$

- Find an expression for the voltage across the inductor.

Solution:

- The figure below shows the voltage and current waveforms for the case when $L = 0.2H$, $a = 5A$, and $\tau = 30ms$.



- Note that the sudden change in the current results in a sharp voltage pulse across the inductor

- Given that a sudden change in the current through an inductor causes a change in the magnetic flux through the inductor, we can conclude that the current through an inductor cannot change instantly since this would result in an infinite voltage pulse across the inductor.

6.3.1 Energy Stored in an Inductor

- The energy stored in an inductor ar time t is given by
  - thus we have that

$$w(t) = \int_{-\infty}^{t} iv d\tau = \int_{-\infty}^{t} i\left[L\frac{di}{d\tau}\right]d\tau = L\int_{-\infty}^{t} i di = \frac{1}{2}Li^2(t)$$

  - where the current $i(-\infty) = 0$.

$$w(t) = \frac{1}{2}Li^2(t)$$

  - Since $w(t) \geq 0$, the inductor is a passive element

Ex. 6.9 Find the energy stored in a 1mH inductor if the voltage across the inductor is given by $v(t) = 120\cos 100t$?

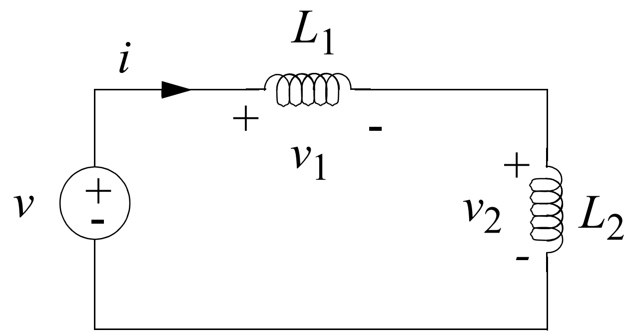
Solution:

Ex. 6.10 A voltage $v = 10\cos 50tV$ is applied across a 4mH inductor. Assume an initial state where the initial current in the inductor was 10A. (a) Obtain an expression for the current through the inductor. (b) Obtain an expression for the energy stored in the inductor. (c) At what times is the energy maximum?

Solution:

6.3.2 Series Inductors

- Consider the following circuit composed of two inductors connected in series



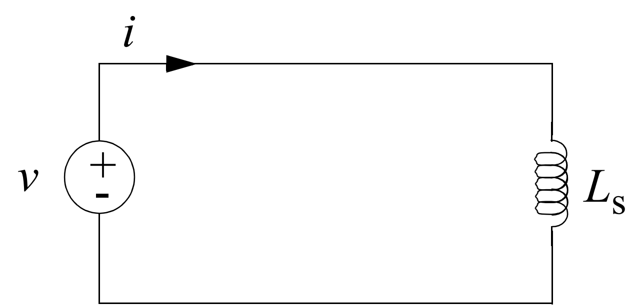
  - Applying KVL around the circuit gives

$$v = v_1 + v_2$$

  - Applying the equation for the voltage across an inductor gives

$$v = L_1\frac{di}{dt} + L_2\frac{di}{dt}$$

- Consider the equivalent circuit



- Applying the equation for the voltage across the inductor gives

$$v = L_s \frac{di}{dt}$$

- For the above two circuits to be equivalent then

$$L_s = L_1 + L_2$$

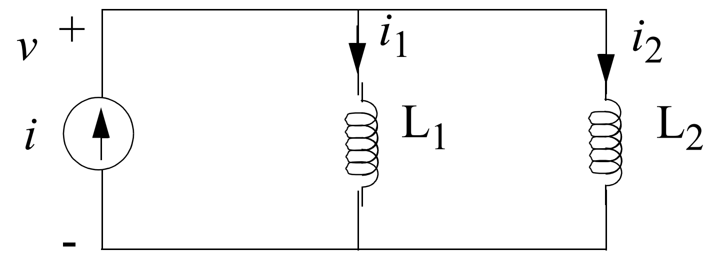
- This case can be generalized for n inductors connected in series to given

$$L_s = \sum_{j=1}^{n} L_j$$

6.3.3 Parallel Inductors

- As in the case for resistors, we would like to know the equivalent inductance of two inductors which are connected in parallel.

- Consider the following circuit



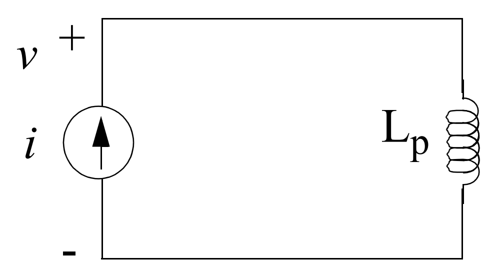
- Applying KCL at the top node gives

$$i = i_1 + i_2$$

- Applying the equation for the current through a inductor

$$i = \left[i_1(0) + \int_0^t \frac{1}{L_1} v d\tau\right] + \left[i_2(0) + \int_0^t \frac{1}{L_2} v d\tau\right] = [i_1(0) + i_2(0)] + \left[\frac{1}{L_1} + \frac{1}{L_2}\right]\int_0^t v d\tau$$

- Consider the following circuit



- Applying the equation for the current through a capacitor

$$i = i(0) + \frac{1}{L_p}\int_0^t v d\tau$$

- The two circuits above are equivalent if

$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2}$$

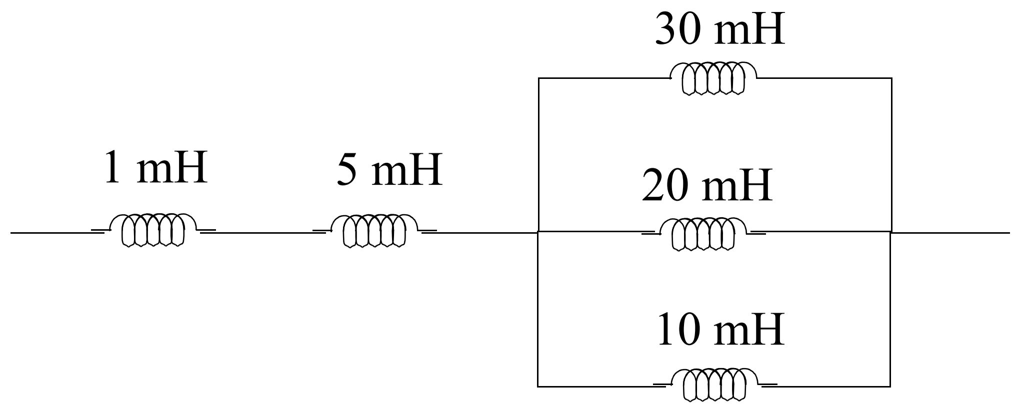
$$i(0) = i_1(0) + i_2(0)$$

- This result can be generalized to give the equivalent inductance for a system of n parallel inductors

$$L_p = \sum_{j=1}^{n} \frac{1}{L_j}$$

$$i(0) = \sum_{j=1}^{n} i_j(0)$$

Ex. 6.11 Determine the equivalent inductance for the following circuit



6.4 DC Behaviour of Capacitors and Inductors

- When the circuit is in dc steady state, the currents and voltages in all the circuit elements do not change with time, hence for every circuit element we have that

$$\frac{dv}{dt} = 0$$

$$\frac{di}{dt} = 0$$

  - Thus for a capacitor we get

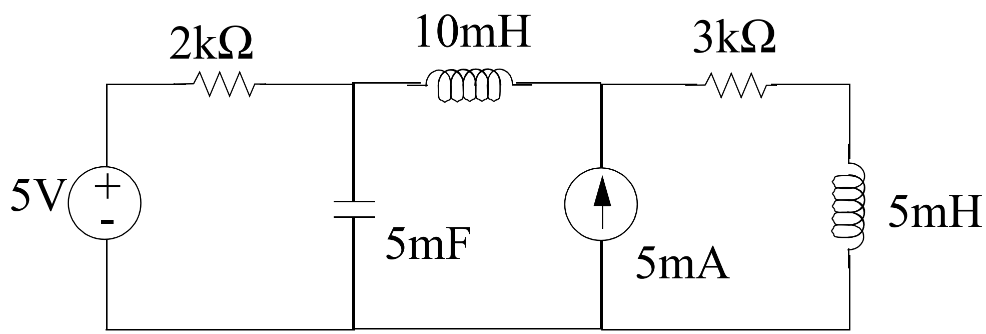
$$i = C\frac{dv}{dt} = C \cdot 0 = 0$$

- Thus the current in a capacitor is zero when the applied voltage across the capacitor does not change with time (dc steady state)
- A **capacitor** therefore acts as an **open-circuit** in **dc steady state**.
- For an inductor we get

$$v = L\frac{di}{dt} = L \cdot 0 = 0$$

- Thus the voltage across an inductor is zero when the current through the inductor does not change with time (dc steady state)
- An **inductor** therefore acts as a **closed-circuit** in **dc steady state**.

Ex. 6.12 For the circuit shown below find the voltages and currents in each circuit element when the circuit is in dc steady state.



Solution:

Assignment #6

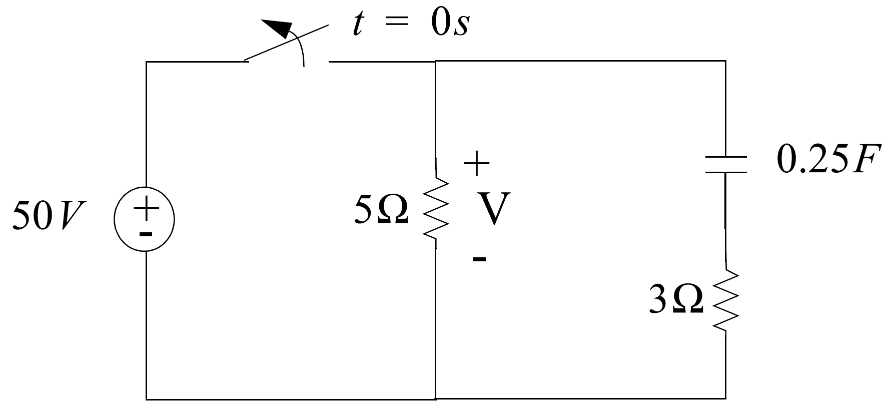
Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 7
First-Order RC and RL Circuits

7.1 Introduction

- Up to this point, we have looked only at circuits where there were no changes in the circuit during its operation
  - This is termed **steady-state** (i.e. the state of the system does not change)
  - Consider the following circuit were the switch is opened at $t = 0s$.



  - We cannot solve this type of circuit using only the circuit theorems we have learned up to this point since these theorems implicitly assume that the circuits are in steady-state.

- Steady-state circuit analysis is very important but a much larger class of useful circuits can be built if we allow the circuit to change during the course of its operation.
  - For example, computer circuits are based on the operation of large numbers of switches (implemented through transistors) which change state as the computer operates.
  - Camera flashes work by charging a capacitor to the level required to produced the "flash" and then switching the circuit to discharge the capacitor through the flash bulb.
  - Electronic devices with rechargeable batteries need to be able to switch from drawing power from the batteries to recharging the batteries based on whether or not the device is plugged into a power source. (Car batteries and alternators are another example of this situation)
  - Wireless communication circuits (such as cell phones, and cellular base stations) can change their operation based on noise levels within the communications channel, environmental temperature changes, etc.
- When we change a circuit during its operation **transient behaviours** are produced.
  - The **transients** are caused by the disturbance of the steady-state circuit and exist until the circuit settles down to its new steady-state behaviour.
  - The time that the circuit takes to settle down to this new steady-state is termed the time constant of the system and is denoted by τ

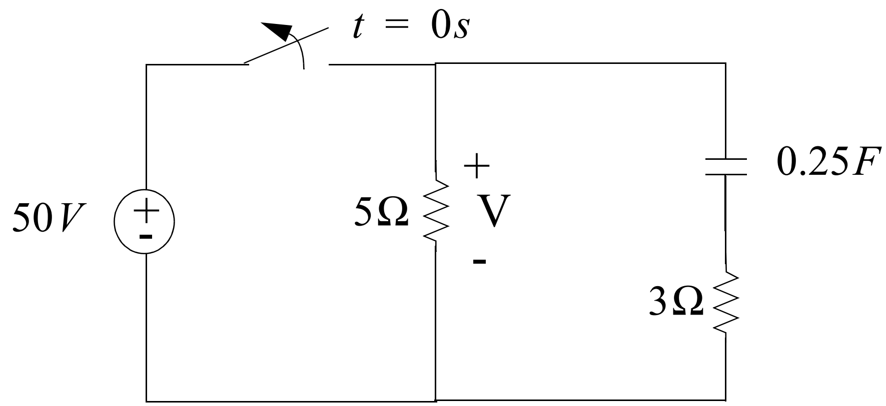
- Obviously, this time constant is of considerable interest to us as engineers when we design such circuits.

- Within this chapter we are concerned only with **first-order** circuits.
  - First-order means that the circuits have only one memory element.
  - From Chapter 6 we learned that capacitors store electrical charge and inductors store magnetic flux.
  - This storing of charge or flux represents a memory within the circuit (i.e. something that persists over a period of time).
  - The energy stored in these components cannot dissipate instantly.
  - Hence, within this course, first-order circuits refer to circuits with only one storage element (i.e. circuits which either have a single capacitor or inductor)
    - So when we say "the circuit has only one storage element" this means that the circuit is either capacitive or inductive NOT that the numerical count of the inductors (or capacitors) in the circuit is necessarily equal to one.
    - Remember, that for linear circuits we can always apply Thevenin or Norton's theorem to reduced a circuit, relative to the chosen terminals A and B, to an independent power source connecting to those terminals.

- We analyze first-order circuits by focusing on the parameters of the system which do not change suddenly when the state changes

  - We know from Chapter 6 that the current through and inductor and the voltage across a capacitor cannot change suddenly. (i.e. the stored energy in these elements cannot be dissipated instantly).

  - All the other circuit parameters can change instantly when the state of the circuit changes.



    - When the switch is opened, V changes instantly, but the voltage across the capacitor does not (i.e. $V_C(t=0+) = V_C(t=0-)$)

  - We use this property of energy storage elements to solve for the transients in circuits which are not at steady-state.

- These circuit parameters which do not change when the state of the circuit changes are called **state variables**.

Circuit Element	**State Variable**
Voltage Source	None
Current Source	None
Resistor	None
Capacitor	V
Inductor	I

7.2 Significance of State Variables

- We know that the current through an inductor and the voltage across a capacitor cannot change instantaneously
  - This means if the circuit undergoes a sudden change (i.e. we switch in (or out) a voltage (or current) source, or we switch in (or out) other circuit elements) then $V_C(t)$ and $I_L(t)$ **must** change in a continuous fashion (i.e. these signals cannot have discontinuities)
  - So at the **instant of change** the **only circuit variables** which can be assumed **not to have changed** are the **capacitor voltages** and the **inductor currents**.

- All other circuit variable have new values which we must determine without any reference to their past values. (i.e. they change discontinuously)

Circuit Element	**Voltage Across Element**	**Current Through Element**
Voltage source	*Constant*	Sudden change possible
Current source	Sudden change possible	*Constant*
Resistor	Sudden change possible	Sudden change possible
Capacitor	**Continuous function of time**	Sudden change possible
Inductor	Sudden change possible	**Continuous function of time**

- So our approach to solving these problems is:

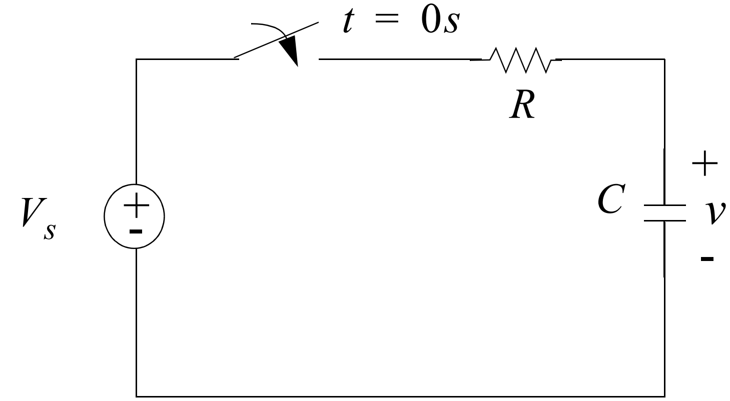
1. Solve for the state variables in the initial state

2. Solve for the transient at the state change via the state variables (since they are the only circuit parameters which do not change at the instant of the state change)

3. Solve for the circuit in its new steady-state.

7.3 RC Circuits

- Consider the circuit shown below.



  - The voltage source was disconnected for some time $t < 0$ (which we assume was long enough for the circuit to be in steady-state)
  - The only energy storage element in the circuit is the capacitor so the state variable will be the voltage across the capacitor (i.e. the only circuit parameter which will not change when the switch is closed).
  - The capacitor may have an initial charge $v(0) = V_0$.
  - For simplicity of the example will assume this initial charge is $V_0 = 0V0$.

- We can choose to study what happens when we close the switch by looking at any of the circuit parameters, but if we choose to focus on the state variable (the capacitor voltage) then the analysis is considerably easier (since it is continuous at the state change)

- If for example we had chosen to study the capacitor current, then we would need to find the initial value for this current which is quite difficult since it changes discontinuously.

- So solving for the above circuit (in terms of the capacitor voltage state variable):

  - Current flows through the circuit due to the initial voltage on the capacitor $v(0) = V_0$ and the energy supplied by the voltage source V_s

  - We can write the I-V relationship for the capacitor (which gives us the current through the circuit)

$$i = C\frac{dv}{dt}$$

  - But we want an equation which is just in terms of the state variable v so we need to eliminate i from the above equation

  - Applying KVL around the loop gives

$$V_s - iR - v = 0$$

- Solving for i

$$i = \frac{V_s - v}{R}$$

- Substituting this into our previous equation we get an DE equation which is just in terms of the state variable v

$$\frac{V_s - v}{R} = C\frac{dv}{dt}$$

- This is a first-order differential equation (DE) in one unknown v.

- In order to solve this DE we would first like to put it into **standard form** (i.e. having the coefficients of the highest order differential operator equal to 1, and placing the variables on the left hand side, in order of the differential operators, and the constants on the right hand side)

- $\frac{dv}{dt}$ is the highest order differential operator, so we can make its coefficient 1 by dividing the equation through by C

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}$$

(DE in standard form)

- This is a first order DE and it has a solution of the form

$$v = v_f + v_n$$

- where v_f is the **forced response** (due to the power source) and v_n is the **natural response** (due to the initial charge on the capacitor)

- Hence, v_f and v_n must satisfy the following equations

$$\frac{dv_f}{dt} + \frac{v_f}{RC} = \frac{V_s}{RC} \text{ (nonhomogeneous DE)}$$

$$\frac{dv_n}{dt} + \frac{v_n}{RC} = 0 \text{ (homogenous DE)}$$

- We can observe that the forced response v_f satisfies the original DE while the natural response v_n only satisfies the left hand side of the original DE (i.e. its solution does not depend on the power source V_s).

7.3.1 The Forced Response

- To solve for the forced response, we first notice that the forced response will have the same time dependence as the power supply
  - So for a dc power supply we can try solution of $v_f = A$ (i.e. assume that v_f is a constant)
  - Substituting this trial value into

$$\frac{dv_f}{dt} + \frac{v_f}{RC} = \frac{V_s}{RC}$$

  - we get

$$(0) + \frac{A}{RC} = \frac{V_s}{RC}$$

$$A = V_s$$

  - Hence, the forced response is simply

$$v_f = V_s$$

7.3.2 The Natural Response

- For the natural response we need to find a v_n which will solve the DE

$$\frac{dv_n}{dt} + \frac{v_n}{RC} = 0$$

  - This DE requires that the value of v_n have the same form as its differential $\frac{dv_n}{dt}$

  - A natural choice for the solution is a v_n of the form $v_n = ke^{st}$ (where k is a constant of dimension volts, and s is a constant with dimensions of $\mathrm{sec}onds^{-1}$)

  - Substituting this trial solution into the DE. we get that

$$ske^{st} + \frac{ke^{st}}{RC} = 0$$

  - Collecting terms

$$ke^{st}\left(s + \frac{1}{RC}\right) = 0$$

- We reject the trivial solution $k = 0$ and since $e^{st} \neq 0$ we can divide both sides through by ke^{st}

- The only nontrivial solution therefore to this equation is when the bracketed term is zero

$$s + \frac{1}{RC} = 0$$

- This equation is known as the **characteristic equation** for the circuit.

- This characteristic equation has a single root which occurs when

$$s = -\frac{1}{RC}$$

- This root tells us how fast the natural response decays to zero. The is termed at the **time constant** of the circuit.

- Traditionally, this decay constant is denoted by τ and the solution for v_n is written in the form

$$v_n = ke^{-\frac{t}{\tau}}$$

- where τ has units of seconds and is given by $\tau = RC$.

- We still need to solve for k, but we can **only** do this once we have an equation for the **total response** of the circuit.

- Significance of the time constant:

  - The expression for the natural response will always include a term of the form $e^{-\frac{t}{\tau}}$.

  - For large values of t the natural response will decay to zero

$$\lim_{t \to \infty}\left[e^{-\frac{t}{\tau}}\right] = 0$$

  - The time constant τ determines *how fast* the circuit settles down to its equilibrium value (steady state value)

  - For example,

    - if $\tau = 1s$ then the natural response decays to $e^{-1} = 36.788\%$ of its starting value every second since we have that

$$v_n(1) = V_0 e^{-\frac{1}{1}} = V_0 \times 36.788\%$$

- On the other hand, if $\tau = 1ms$ then the natural response decay by 36.788% each millisecond.

7.3.3 The Total Response

- We can now write the total response in terms of our solutions for the natural and forced responses as,

$$v(t) = v_n(t) + v_f(t) = ke^{-\frac{t}{\tau}} + V_s$$

  - This is our solution to the differential equation

$$\frac{dv(t)}{dt} + \frac{v(t)}{RC} = \frac{V_s}{RC}$$

  - This solution is valid for all times t
  - Thus we can use this total response to solve for k
    - We know the initial conditions for the circuit
    - At $t = 0$, we are given that the initial charge on the capacitor is V_0

- Substituting $t = 0$ into the above equation gives

$$V_0 = ke^{-\frac{(0)}{\tau}} + V_s = k + V_s$$

- Solving for k we have that

$$k = V_0 - V_s$$

- We can now substitute this value for k back into our equation for the total response to get

$$v(t) = (V_0 - V_s)e^{-\frac{t}{\tau}} + V_s = V_s\left(1 - e^{-\frac{t}{\tau}}\right) + V_0 e^{-\frac{t}{\tau}}$$

- Note that the voltage starts at its initial value and changes exponentially towards its final value with the time constant τ which is given by the values of R and C only.

- If we evaluate $v(t)$ around $t = 0$, then for some small value ε we can observe that

$$v(t) = \begin{cases} V_0 \text{ when } t=0+\varepsilon \\ V_0 \text{ when } t=0-\varepsilon \end{cases}$$

- Proving that sudden changes in the circuit will **not** result in sudden changes in the capacitor voltage

- After we have found the total response for the circuits state variable $v(t)$, we can now solve for other circuit variables of interest.

  - For example, the resistor voltage can be easily found as

$$v_R(t) = i(t)R = C\frac{dv}{dt} \times R = RC(V_s - V_0)e^{-\frac{t}{\tau}}$$

  - This equation though is only valid for $t \geq 0^+$ since it assumes that the switch has been closed and that current can flow in the circuit.

  - To find $v_R(t)$ for $t \leq 0^-$ we need to look at the steady state circuit which existed before the switch was closed

  - In this case, $i(t) = 0$ for $t \leq 0^-$, hence $v_R(t) = 0$ for $t \leq 0^-$

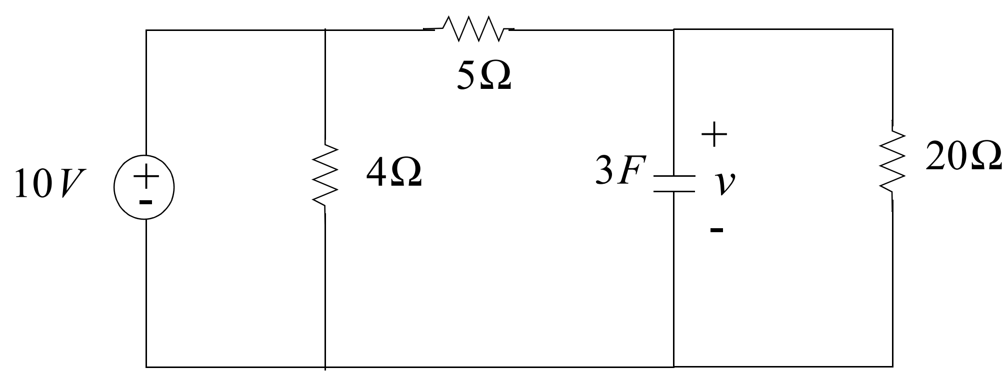
- Hence, the complete expression for $v_R(t)$ is

$$v_R(t) = \begin{cases} 0 & \text{for } t \le 0^- \\ RC(V_s - V_0)e^{-\frac{t}{\tau}} & \text{for } t \ge 0^+ \end{cases}$$

- There is a discontinuous change in $v_R(t)$

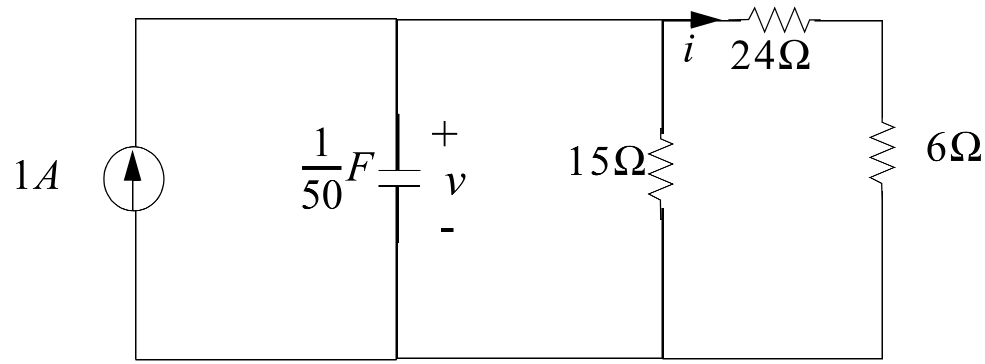
- The fact that the state variables have no such discontinuities makes it much easier to solve for the circuit in terms of them first, then use those solutions to get the other circuit parameters of interest.

Ex. 7.1 Find the total response for the following circuit (assuming that $v(0) = -5V$)



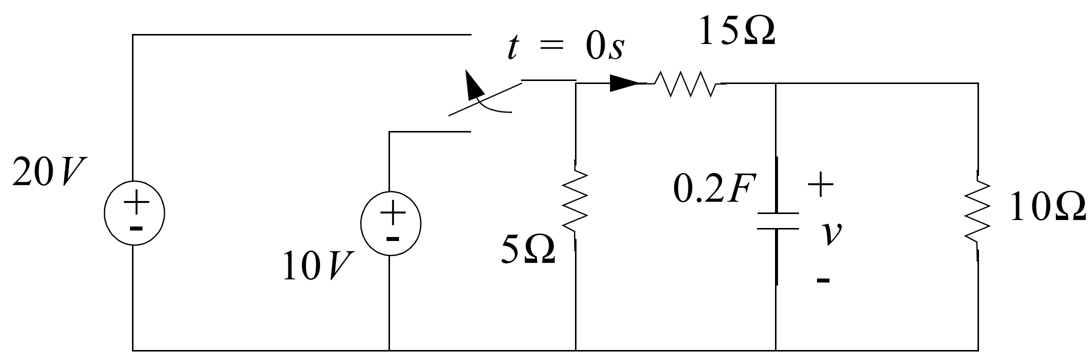
Solution:

Ex. 7.2 Find the total response for the voltage $v(t)$ for the circuit shown below given that $v(0) = 0V$. Find $i(t)$.



Solution:

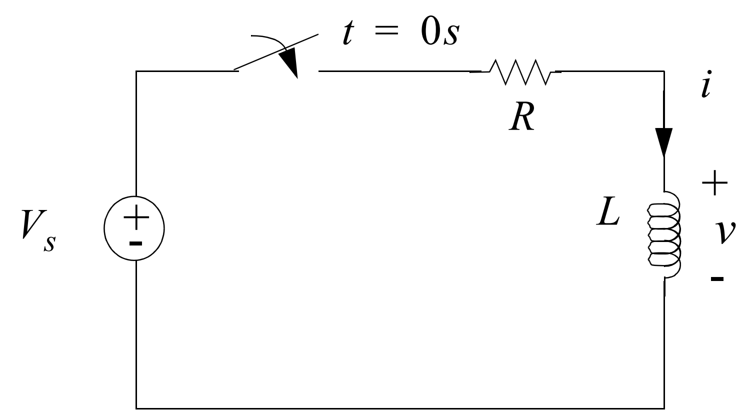
Ex. 7.3 The circuit shown below is in steady-state when $t < 0$. The switch flips as shown at $t = 0$. find how i and v change with time for $t \geq 0$.



Solution:

7.4 RL Circuits

- The above work deals only with energy storage elements which are in the form of capacitors.
- We can also store energy in terms of the magnetic flux across an inductor.
  - We know that the current through an inductor cannot change instantaneously
  - Hence for RL circuit the state variable of interest is the inductor current
- Given the following circuit



  - Current flows in this circuit due to the initial energy stored on the inductor, given by $i(0) = I_0$, and due to the energy supplied by the voltage source V_s.

- Since we know that the inductor current is the state variable we can begin by writing the I-V relationship for the inductor

$$v = L\frac{di}{dt}$$

- We require a DE just in terms of the state variable, so we need an another expression for v in terms of i

- We can write the KVL equation around the loop as

$$V_s - iR - v = 0$$

- Note that this equation only “exists” once the switch has been closed.

- We can use this equation to substitute for v in the inductor’s I-V relationship

$$V_s - iR - L\frac{di}{dt} = 0$$

- This gives us a DE just in terms of the state variable i, but we need to express it in standard form

$$\frac{di}{dt} + i\frac{R}{L} = \frac{V_s}{L}$$

- This is a first-order DE and the solution to this equation will be of the form

$$i = i_f + i_n$$

  - where i_f is the forced response due to the voltage source and i_n is the natural response due to the initial current through the inductor.

- As in the previous section, the forced and natural response need to solve the homogenous and nonhomogeneous DEs given by

$$\frac{di_f}{dt} + i_f\frac{R}{L} = \frac{V_s}{L}$$

$$\frac{di_n}{dt} + i_n\frac{R}{L} = 0$$

7.4.1 The Forced Response

- The forced response has the same time dependency as the power supply, so for a dc power supply, we try a constant value for $i_f = A$

- Substituting this value into the nonhomogeneous DE gives

$$(0) + A\frac{R}{L} = \frac{V_s}{L}$$

$$A = \frac{V_s}{R}$$

- which gives the forced response simply as

$$i_f = \frac{V_s}{R}$$

7.4.2 The Natural Response

- For the natural response we attempt a trial solution of the form

$$i_n = ke^{st}$$

- Since it needs to solve the homogenous DE of $\frac{di_n}{dt} + i_n\frac{R}{L} = 0$

- Substituting this trial solution in the DE gives

$$ske^{st} + ke^{st}\left(\frac{R}{L}\right) = 0$$

- Simplifying by collecting like terms

$$ke^{st}\left(s + \frac{R}{L}\right) = 0$$

- The only non-trivial solution to the equation is when

$$s + \frac{R}{L} = 0$$

- This equation is the characteristic equation for the RL circuit

- The solution to this equation is given by

$$s = -\frac{R}{L}$$

- Thus the solution for the natural response is

$$i_n = ke^{-\frac{R}{L}t} = ke^{-\frac{t}{\tau}}$$

- where $\tau = \frac{L}{R}$ is the **time constant** of the RL circuit.

- As with the RC circuit we must use the total response of the circuit to solve for k.

7.4.3 The Total Response

- Combining the above result we have a solution of the form

$$i(t) = \frac{V_s}{R} + ke^{-\frac{t}{\tau}}$$

  - This solution applies at all times, hence we can solve for k by evaluating this equation at $t = 0$ (where we know the initial conditions)

$$i(0) = \frac{V_s}{R} + ke^{-\frac{(0)}{\tau}} = I_0$$

  - Hence, we can solve for k as

$$k = I_0 - \frac{V_s}{R}$$

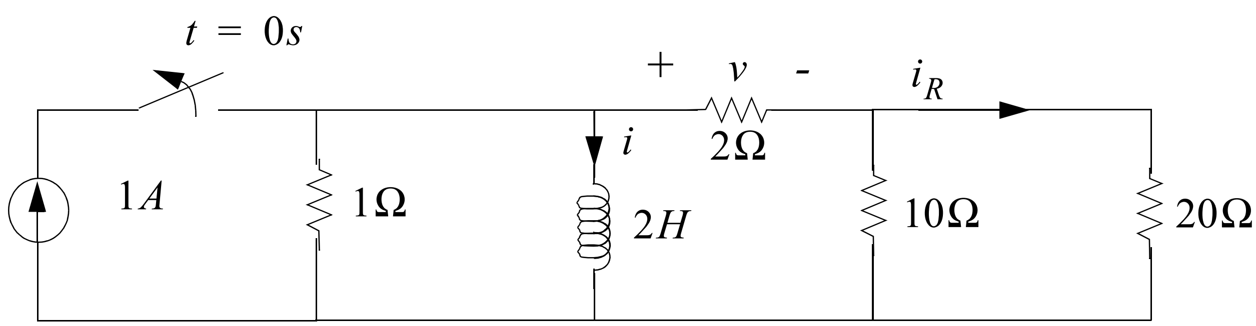
  - Thus our complete solution for $i(t)$ is

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-\frac{t}{\tau}}$$

$$i(t) = \frac{V_s}{R}\left(1 - e^{-\frac{t}{\tau}}\right) + I_0 e^{-\frac{t}{\tau}}$$

- Note that the current starts at its initial value and changes exponentially towards its final value with a time constant τ that depends only on the values of R and L.

Ex. 7.4 The circuit below was in steady-state prior to the switch being opened at $t = 0$. Find the values of i, i_R, and v for $t \geq 0$.



Solution:

7.5 Short-Cut method for First-Order Circuits

- First-order RC and RL circuits lend themselves to a simple technique for finding the time-behaviour of any of their circuit parameters according to the following steps

1. Make sure that the circuit does not have any dependent current or voltage sources

2. Find the Thevenin (or Norton) equivalent circuit seen by the capacitor or inductor.

3. Find the forced, or steady-state, response by studying the circuit under dc or ac conditions, depending on the type of the power supply connected to the circuit. For a dc source, the capacitor is replaced by an open circuit while an inductor is replaced by a short-circuit. The relevant state variable is then evaluated from this simplified circuit.

4. Compute the circuits time constant as

$$\tau = R_T C \text{ for RC circuits}$$

$$\tau = \frac{L}{R_T} \text{ for RL circuits}$$

5. Write the forced response as

$$x_f = V_T \text{ for RC circuits}$$

$$x_f = \frac{V_T}{R_T} = I_N \text{ for RL circuits}$$

6. Write the natural response as

$$x_n = ke^{-\frac{t}{\tau}}$$

7. Write the total response as

$$x = x_f + x_n$$

8. Solve for the constant k from the initial condition as

$$x(0) = x_f + k$$

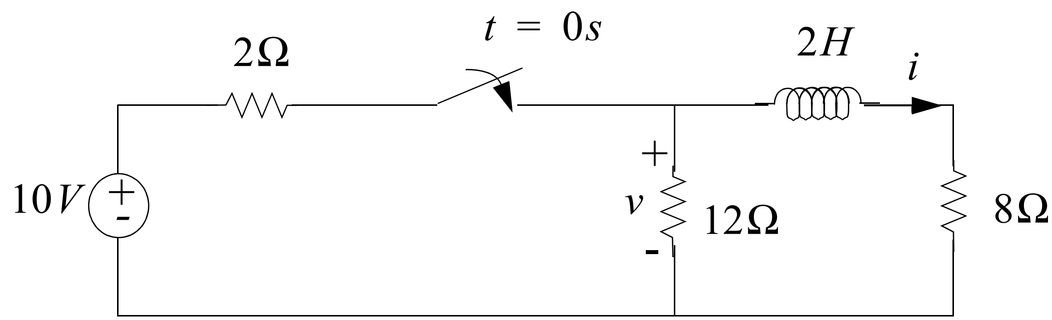
$$k = x(0) - x_f$$

9. Write the total response as

$$x(t) = x_f + [x(0) - x_f]e^{-\frac{t}{\tau}} = x_f\left(1 - e^{-\frac{t}{\tau}}\right) + x(0)e^{-\frac{t}{\tau}}$$

10. Use the solution obtained for the state variable to solve for any of the remaining circuit parameters of interest.

Ex. 7.5 For the circuit below find the values for v and i for $t \geq 0$, given that the current in the inductor was $i(0^-) = -3A$ when $t < 0$.

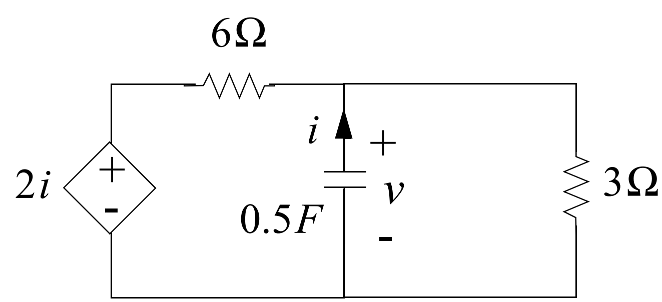


Solution:

7.6 Case when Dependent Sources are Present

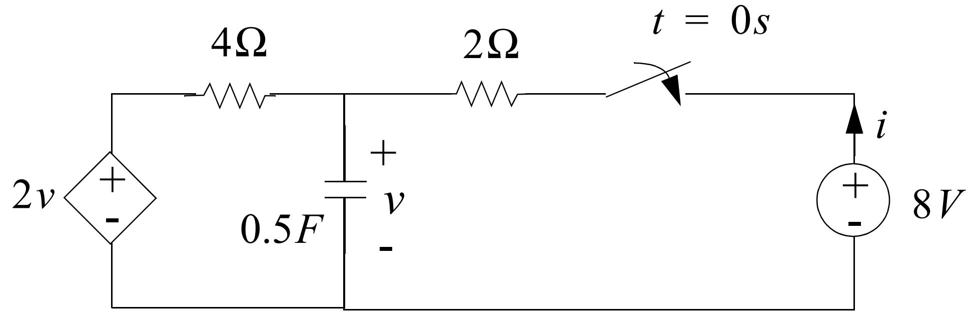
- When dependent sources are present we are unable to use the short-cut method to quickly arrive at the total response.
  - We have to use our circuit theorems (KVL, KCL, Norton, or Thevenin equivalent circuits, etc.) to solve such problems as the following examples illustrate.

Ex. 7.6 Find v for $t \geq 0$ if $v(0) = 3V$.



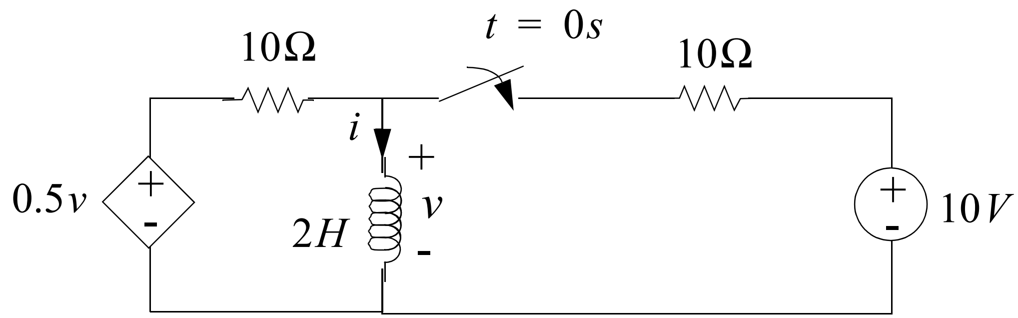
Solution:

Ex. 7.7 Find v for $t \geq 0$ if $v(0) = 3V$ assuming the circuit is in steady-state for $t < 0$



Solution:

Ex. 7.8 Find v for $t \geq 0$ assuming the circuit is in steady-state for $t < 0$.



Solution:

Assignment #7

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

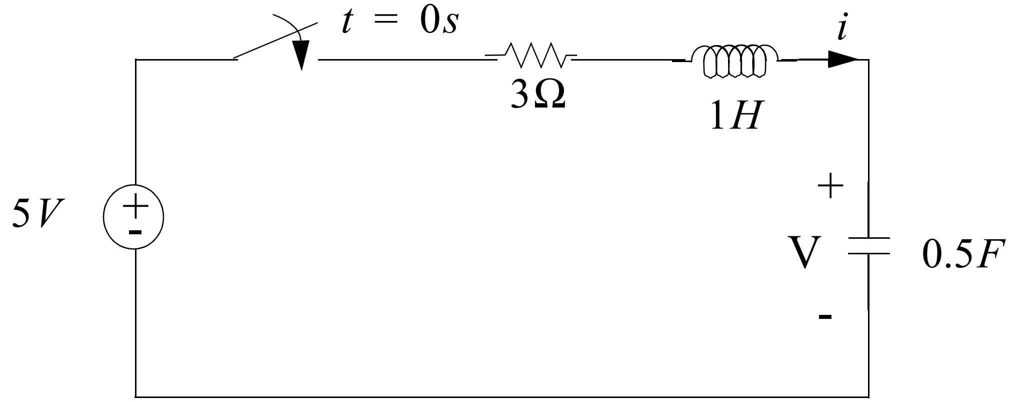
Chapter 8
Second-Order RC and RL Circuits

8.1 Introduction

- In the previous Chapter, we considered circuits which have only one energy storage element.
    - These circuits produce first-order differential equations which we learned to solve.
- In this Chapter, we consider circuits containing two energy storage elements (i.e. an inductor and a capacitor, two inductors, or two capacitors).
    - These circuits will produce second-order differential equations which we will need to solve.
- As before, we will analyze these circuits by looking at the circuit parameters which do not change instantly when the state of the circuit changes
    - These are called the state variables
    - We know from before that for inductors their current is a state variable, and for capacitors their voltage is a state variable.

8.2 Series RLC Circuit

- Consider the RLC circuit below



- For time $t < 0$ the voltage source was not connected (since the switch was open)
- But at time $t = 0$ the switch closed, and the voltage source became connected to the circuit
- At some arbitrarily small time before the switch closed (i.e. at $t = 0 - \varepsilon$ for some small ε) we are told that the inductor had an initial current $i(0) = 1A$ and the capacitor had an initial voltage across it of $v(0) = 2V$.
- The exact values for these initial inductor current and capacitor voltage are for illustration. The state variables will be in some initial state though (either at zero, if the circuit has been in an un-powered steady-state for a long time, or at some non-zero constant value if the circuit has some energy stored in it.)

- When the switch is closed the state of the circuit is disturbed and we would like to know how it transitions to its new equilibrium (steady) state and what that steady-state will be.

  - To determine this we focus our analysis on the state variables i and v.

  - Since both i and v are state variables we can develop a differential equation describing the circuit's transient behaviour from either of them

- Assume we begin by looking at the state variable v

  - Writing KVL around the loop gives

$$5 - 3i - \frac{di}{dt} - v = 0$$

  - This is an equation in terms of both state variable i and v

  - But we want a DE in terms of only one of the state variables (and we have stated above that we will choose to write the DE equation in terms of v)

  - So we need another equation which relates i and v,

- The I-V relationship for the capacitor gives

$$i = C\frac{dv}{dt} = 0.5\frac{dv}{dt}$$

- Substituting this equation in for i in the DE above we obtain

$$5 - 3\left(0.5\frac{dv}{dt}\right) - \frac{d}{dt}\left(0.5\frac{dv}{dt}\right) - v = 0$$

$$5 - 1.5\frac{dv}{dt} - 0.5\frac{d^2v}{dt^2} - v = 0$$

- Placing in standard form (left-hand side ordered according to highest differential operator, then dividing through such that the highest differential operator has a coefficient of 1, and the right-hand side being just the constant term).

$$\frac{d^2v}{dt^2} + 3\frac{dv}{dt} + 2v = 10$$

- This is a second-order nonhomogeneous DE in v and (as in the first-order DE case of Chapter 13) its solution can be written as the sum of two components

$$v = v_f + v_n$$

- where v_f is the **forced response**, due to the power source(s) in the circuit, and v_n is the **natural response**, due to the initial energy stored within the circuit's energy storage elements (i.e. the initial voltage on the capacitor, and the initial current through the inductor)

8.2.1 The Forced Response

- As in Chapter 13, we solve for the forced response by solving the nonhomogeneous DE given (for the above circuit) by

$$\frac{d^2 v_f}{dt^2} + 3\frac{dv_f}{dt} + 2v_f = 10$$

  - We can observe that the forced response v_f must satisfy the original DE.
  - We need to solve this equation for v_f
    - We known that the forced response will have the same time dependence as the power supply
    - For a dc power supply we can try a solution for v_f of

$$v_f = A$$

- i.e. the trial solution is where v_f equals a constant.

- Substituting this value in to the above DE gives

$$(0)^2 + 3(0) + 2A = 10$$

- Simplifying

$$A = 5V$$

- Hence the forced response is given by

$$v_f = A = 5V$$

8.2.2 The Natural Response

- The natural response if described by the homogenous DE where the forcing term is zero

$$\frac{d^2 v_n}{dt^2} + 3\frac{dv_n}{dt} + 2v_n = 0$$

  - For a natural response we attempt a solution in the form of

$$v_n = ke^{st}$$

- where k is a constant with dimensions of volts and s is a constant with dimensions of $\sec onds^{-1}$.

- Substituting this trial equation into the homogenous DE

$$ks^2e^{st} + 3kse^{st} + 2ke^{st} = 0$$

- Since we exclude the trivial solution of $k = 0$, and since $e^{st} \neq 0$, we can divide both sides through by ke^{st}

$$s^2 + 3s + 2 = 0$$

- This equation is called the **characteristic equation** for the circuit.

- This is a quadratic equation in s and it has two roots

$$s = -1 \text{ and } s = -2$$

- We assumed a solution of the form $v_n = ke^{st}$ but since we have two roots we known that the actual form of the solution is

$$v_n = k_1e^{-t} + k_2e^{-2t}$$

- We still need to find the values for k_1 and k_2 but we can only find those values once we have the total response.

8.2.3 The Total Response

- We now have the total response for the example circuit of the form

$$v = v_f + v_n$$

$$v = 5 + k_1 e^{-t} + k_2 e^{-2t}$$

- This solution applies at all times. So we can use this equation in solving for k_1 and k_2 by substituting the initial conditions at $t = 0$ into the above equation.

$$5 + k_1 + k_2 = 2$$

- or simplifying

$$k_1 + k_2 = -3$$

- This is one equation in two unknowns so we need to find another independent equation relating k_1 and k_2

  - We began our analysis with the two equations

$$5 - 3i - \frac{di}{dt} - v = 0$$

$$i = C\frac{dv}{dt} = 0.5\frac{dv}{dt}$$

- So, since we now have a solution for v in the form of $v = 5 + k_1 e^{-t} + k_2 e^{-2t}$ we can use either of the above equations to generate another equation in terms of k_1 and k_2.(note: these equations are valid for $t \geq 0$)

- The second equation is simpler so substituting in our solution for v

$$i(t) = 0.5\frac{d}{dt}(k_1 e^{-t} + k_2 e^{-2t})$$

$$i(t) = 0.5(-k_1 e^{-t} - 2k_2 e^{-2t})$$

- Substituting in $t = 0$

$$i(0) = 0.5(-k_1 - 2k_2)$$

- We know from the initial conditions that $i(0) = 1$ therefore

$$1 = 0.5(-k_1 - 2k_2)$$

- Simplifying

$$k_1 + 2k_2 = -2$$

- Now we have two equations in terms of just the two unknowns k_1 and k_2

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

- Solving we get that $k_1 = -4$ and $k_2 = 1$

- Substituting these values into our equation for the response we get that the total response for the circuit is given by

$$v(t) = -4e^{-t} + e^{-2t} + 5 \text{ V}$$

  - We can check that this value for $v(t)$ satisfies the original DE

$$\frac{d^2v}{dt^2} + 3\frac{dv}{dt} + 2v = 10$$

  - From the equations above we have that $\frac{dv}{dt} = 4e^{-t} - 2e^{-2t}$ and $\frac{d^2v}{dt^2} = -4e^{-t} + 4e^{-2t}$ therefore substituting these values into the DE gives

$$[-4e^{-t} + 4e^{-2t}] + 3[4e^{-t} - 2e^{-2t}] + 2[-4e^{-t} + e^{-2t} + 5 \text{ V}] = 10$$

- Simplifying and collecting exponential terms

$$(-8e^{-t} + 12e^{-t} - 4e^{-t}) + (4e^{-2t} - 2e^{-2t} + 2e^{-2t}) + 10 = 10$$

$$10 = 10 \blacksquare$$

- Therefore $v(t)$ satisfies the DE describing the circuits operation.

8.3 The Roots of the Characteristic Equation

- We saw in the previous section that the second-order circuits DE produced a characteristic equation of the form

$$s^2 + bs + c = 0$$

  - This is a standard second order quadratic equation in s with $a = 1$
  - We know from algebra that this equation has two roots which are given by the quadratic formula as

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  - There are three possibilities for the two roots (note that there are **always** 2 roots)

    1. The two roots are real and distinct (Overdamped circuit)

2. The two roots are real and equal (Critically damped circuit)

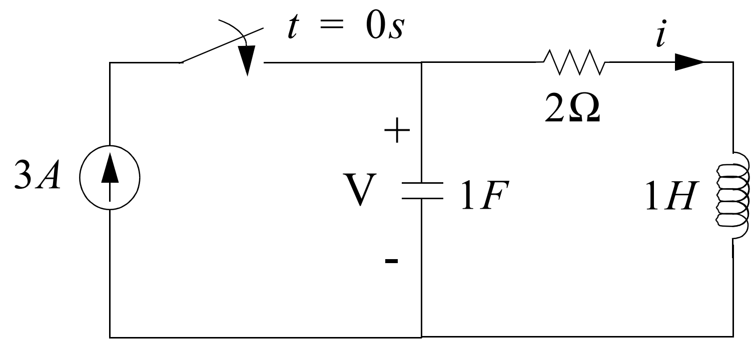
3. The two roots are complex conjugates of each other (Underdamped circuit)

- (Note: complex roots to polynomial equations must always come in complex conjugate pairs.)

- The previous section covered the first case, the next two sections cover the latter two cases.

8.4 The Roots are Real and Equal

• Consider the following circuit



- At time $t = 0$ the switch closes and connects the current source to the circuit.

- We need to find an equation for $i(t)$ at $t \geq 0$ given the initial conditions $v(0) = 1V$ and $i(0) = 2A$

  - Applying KCL at the top left node we get

$$3 = \frac{dv}{dt} + i$$

  - We are interested in the state variable i so we need to eliminate v from the above equation. To do this we need another equation relating i and v.

  - We can use the I-V relationship of the 1H inductor, given by $v = L\frac{di}{dt} = \frac{di}{dt}$, in the the KVL equation around the right loop to get

$$v = 2i + \frac{di}{dt}$$

  - Combining this equation with the one above and eliminating v gives

$$\frac{d^2 i}{dt^2} + 2\frac{di}{dt} + i = 3$$

  - This is a second-order DE and its solution will be in the form

$$i = i_f + i_n$$

- As before we solve this equation by solving for the forced and natural responses in turn.

8.4.1 The Forced Response

- The forced response must satisfy the nonhomogeneous DE

$$\frac{d^2 i_f}{dt^2} + 2\frac{di_f}{dt} + i_f = 3$$

  - The forced response must have the same time dependency as the power source.
  - For dc circuits we try a solution of the form

$$i_f = A$$

  - Substituting in to the DE we get

$$0^2 + 2(0) + A = 3$$

  - Therefore the forced response is given by

$$i_f = 3A$$

8.4.2 The Natural Response

- The natural response must satisfy the homogeneous DE

$$\frac{d^2 i_n}{dt^2} + 2\frac{di_n}{dt} + i_n = 0$$

  - As before we try the trial solution of $i_n = ke^{st}$
  - Substituting this trial solution into the DE gives

$$ks^2 e^{st} + 2kse^{st} + ke^{st} = 0$$

  - Since we reject the trivial solution of $k = 0$ and $e^{st} \neq 0$ we can divide through by ke^{st}

$$s^2 + 2s + 1 = 0$$

  - This is the characteristic equation for the circuit. It has two real equal root both given by

$$s = -1$$

  - The fact that the characteristic equation has two equal root means that the form of the solution cannot be

$$i_n = k_1 e^{-t} + k_2 e^{-t} = (k_1 + k_2)e^{-t}$$

- since this is a solution to a first-order differential equation

- Therefore we choose for the natural response a solution of the form

$$i_n = k_1 e^{-t} + k_2 t e^{-t}$$

- We can check this solution works by substituting it into the DE

$$\frac{d^2 i_n}{dt^2} + 2\frac{di_n}{dt} + i_n = 0$$

- where $\frac{di_n}{dt} = -k_1 e^{-t} + k_2 e^{-t} - k_2 t e^{-t}$ and $\frac{d2i_n}{dt^2} = k_1 e^{-t} - k_2 e^{-t} - k_2 e^{-t} + k_2 t e^{-t}$

$$[k_1 e^{-t} - k_2 e^{-t} - k_2 e^{-t} + k_2 t e^{-t}] + 2[-k_1 e^{-t} + k_2 e^{-t} - k_2 t e^{-t}] + k_1 e^{-t} + k_2 t e^{-t} = 0$$

$$[k_1 e^{-t} - 2(k_1 e^{-t}) + k_1 e^{-t}] + [-k_2 e^{-t} - k_2 e^{-t} + 2(k_2 e^{-t})] + [k_2 t e^{-t} + 2(-k_2 t e^{-t}) + k_2 t e^{-t}] = 0$$

$$[0] + [0] + [0] = 0$$

- As before, we cannot solve for the values of k_1 and k_2 until we have the total response.

8.4.3 The Total Response

- The total response can now be written as

$$i(t) = i_f(t) + i_n(t)$$

$$i(t) = 3 + (k_1 + k_2 t)e^{-t}$$

- To find the values for k_1 and k_2 we need to known the values for $i(t)$ and $\frac{di(t)}{dt}$ at $t = 0$
- From the initial conditions we known that $i(0) = 2$
  - Substituting this value into our total response (at $t = 0$) gives

$$2 = 3 + k_1$$

  - Therefore

$$k_1 = -1$$

- The value of $\frac{di(t)}{dt}$ at $t = 0$ can be found by evaluating

$$v = 2i + \frac{di}{dt}$$

- at $t = 0$ and using the initial condition for the capacitor that $v(0) = 1V$

- This gives

$$v(0) = 2i(0) + \left.\frac{di}{dt}\right|_{t=0}$$

$$1 = 2(2) + [-(k_1 + k_2 t)e^{-t} + k_2 e^{-t}]\Big|_{t=0}$$

$$1 = 4 - k_1 + k_2$$

- Simplifying

$$k_1 - k_2 = 3$$

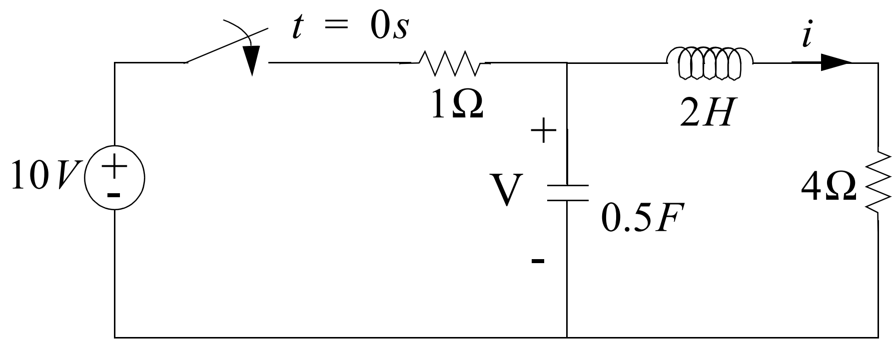
- Substituting $k_1 = -1$ then given $k_2 = -4$

- Therefore the total response for the circuit (which had two real equal roots for its characteristic equation) is

$$i(t) = 3 - (1 + 4t)e^{-t}$$

8.5 The Roots are Complex Conjugates

- Consider the circuit shown below



- To analyze this circuit we need a DE in terms of one of the state variables (i.e. i or v)
- Applying KVL around the right loop gives

$$v = 2\frac{di}{dt} + 4i$$

- This is a DE in terms of both state variables but we want a single equation in terms of just one of the state variable.
- So we need to find another equation that relates v and i

- Applying KCL to the top middle node gives

$$\frac{v-10}{1} + 0.5\frac{dv}{dt} + i = 0$$

- Substituting in for v from the previous equation we have that

$$\frac{\left[2\frac{di}{dt} + 4i\right] - 10}{1} + 0.5\frac{d}{dt}\left[2\frac{di}{dt} + 4i\right] + i = 0$$

- Simplifying

$$2\frac{di}{dt} + 4i - 10 + \frac{d^2 i}{dt^2} + 2\frac{di}{dt} + i = 0$$

- Placing in standard form

$$\frac{d^2 i}{dt^2} + 4\frac{di}{dt} + 5i = 10$$

- We know this equation will have a solution of the form

$$i(t) = i_f(t) + i_n(t)$$

8.5.1 The Forced Response

- The forced response must satisfy the nonhomogeneous DE

$$\frac{d^2 i_f}{dt^2} + 4\frac{di_f}{dt} + 5i_f = 10$$

  - Since the power supply is DC, we seek a solution of the form

$$i_f = A$$

  - Substituting this into the nonhomogeneous DE we get

$$(0)^2 + 4(0) + 5A = 10$$

$$A = 2$$

  - Thus we have that

$$i_f(t) = 2A$$

8.5.2 The Natural Response

- The natural response must satisfy the homogenous DE

$$\frac{d^2 i_n}{dt^2} + 4\frac{di_n}{dt} + 5i_n = 0$$

- The characteristic equation for this homogeneous DE is given by

$$s^2 + 4s + 5 = 0$$

- The roots of this equation are

$$s = -2 \pm j$$

- Therefore we can write i_n as

$$i_n(t) = k_1 e^{(-2+j)t} + k_2 e^{(-2-j)t}$$

$$i_n(t) = (k_1 e^{jt} + k_2 e^{-jt})e^{-2t}$$

$$i_n(t) = Re(k_1[\cos t + j\sin t] + k_2[\cos(t) - j\sin(t)])e^{-2t}$$

  - Note: k_1 and k_2 will themselves be complex when we have complex conjugate roots.

  - $i_n(t)$ can be simplified to:

$$i_n(t) = (k_1 \cos t + k_2 \sin t)e^{-2t}$$

- The constants k_1 and k_2 can then be solved for by using the total response.

- **Clarification:**
- Where does the "$k_2 \sin t$"term given above come from?
  - We have defined (in Chapter 7) that the time domain signals are the real part our complex phasor notation (i.e., $v(t) = Re[\cos(\omega t + \phi) + j\sin(\omega t + \phi)]$)
  - So to get $i_n(t)$ we want to determine $i_n(t) = Re[(k_1 e^{jt} + k_2 e^{-jt})]e^{-2t}$
  - Because we have complex conjugate roots (i.e., $s_2 = s_1^*$), k_1 and k_2 will be complex numbers themselves and, in fact, $k_2 = k_1^*$ (i.e., they will also be complex conjugates)
  - Assume $k_1 = a + bj$ then we can expand $Re[(k_1 e^{jt} + k_1^* e^{-jt})]$ as

$$Re[(k_1 e^{jt} + k_1^* e^{-jt})] = Re\{(a + bj)[\cos(t) + j\sin(t)] + (a - bj)[\cos(t) - j\sin(t)]\}$$

  - Simplifying and taking only the real terms,

$$Re[(k_1 e^{jt} + k_1^* e^{-jt})] = [a\cos(t) - b\sin(t)] + [a\cos(t) - b\sin(t)] = 2[a\cos(t) - b\sin(t)]$$

- We can replace the unknown constants a and b with two arbitrary constants, which we will name k_1 and k_2. (These are different than k_1 and k_2 used above - this is allowed since we do not know the actual values for the original k_1 and k_2)

- We can also subsume the -'ve sign and the "2" into these new constants to simplify the resulting equation.

- This allows us to write

$$i_n(t) = Re[(k_1 e^{jt} + k_2 e^{-jt})]e^{-2t} = [k_1 \cos(t) + k_2 \sin(t)]e^{-2t}$$

- This is the form that is given above.

- This approach obviously works for all cases when we get complex conjugate roots from our second order differential equation.

8.5.3 The Total Response

- The total response has the form

$$i(t) = i_f(t) + i_n(t)$$

$$i(t) = 2 + (k_1 \cos t + k_2 \sin t)e^{-2t}$$

- To find k_1 and k_2 we need to find the values of $i(t)$ and $\frac{di(t)}{dt}$ when $t = 0$ (Since this this the time when the circuit changed states.)

- From the initial conditions we know that $i(0) = 2A$. Therefore substituting $t = 0$ into the above equation gives

$$i(0) = 2 + (k_1 \cos(0) + k_2 \sin(0))e^{-2(0)} = 4$$

$$k_1 = 2A$$

- We also derived in our analysis the expression $v = 2\frac{di}{dt} + 4i$ which we can evaluate at $t = 0$ since we have the initial conditions for $i(0) = 4$ and $v(0) = 6$, and we have and expression for $i(t)$ which we can differentiate and evaluate at $t = 0$

$$v(0) = 2\frac{di}{dt}\bigg|_{t=0} + 4i(0)$$

$$6 = 2\left(\frac{d}{dt}[2 + (k_1 \cos t + k_2 \sin t)e^{-2t}]\right)\bigg|_{t=0} + 4(4)$$

$$-5 = [-2e^{-2t}(k_1 \cos t + k_2 \sin t) + e^{-2t}(-k_1 \sin t + k_2 \cos t)]\big|_{t=0}$$

$$-5 = -2k_1 + k_2$$

- Substituting in the solution for k_1 we get that

$$k_2 = -1$$

- Therefore the total response is given by

$$i(t) = 2 + (2\cos t - \sin t)e^{-2t} A$$

- We can now use this derived total response for the state variable i to solve for any other circuit parameter of interest.

  - For example if we now need to solve for $v(t)$ then we know from the circuit that

$$v(t) = 2\frac{di(t)}{dt} + 4i(t)$$

  - Hence,

$$v(t) = 2\frac{d}{dt}(2 + (2\cos t - \sin t)e^{-2t}) + 4(2 + (2\cos t - \sin t)e^{-2t})$$

$$v(t) = 2[-2e^{-2t}(2\cos t - \sin t) + e^{-2t}(-2\sin t - \cos t)] + 8 + (8\cos t - 4\sin t)e^{-2t}$$

$$v(t) = [-8\cos t + 4\sin t + 8\cos t - 4\sin t - 2\sin t - \cos t]e^{-2t} + 8$$

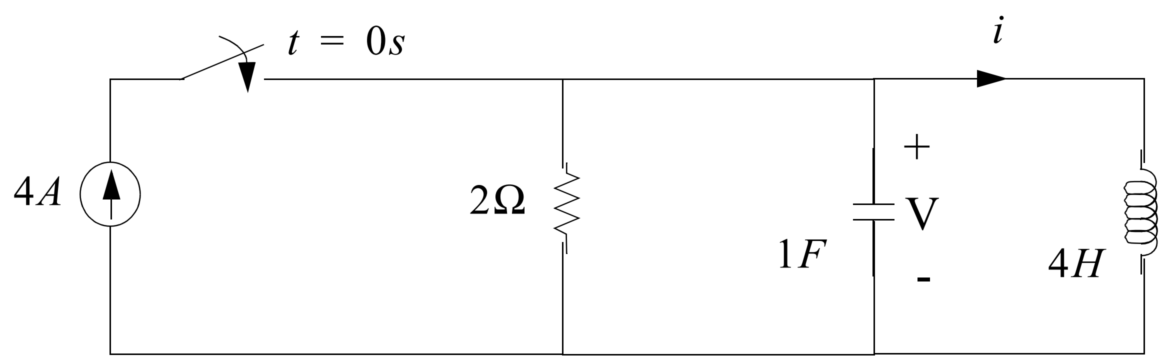
$$v(t) = (-2\sin t - \cos t)e^{-2t} + 8V$$

- Note that this solution is valid for all t.

8.6 Parallel RLC Circuits

- Obviously, if we can construct series RLC circuits, then we can also construct parallel RLC circuits and like their series counterparts they will also give rise to second-order DE of the state variables.

  - Consider the circuit shown below



  - The circuit contains two energy storage elements and two state variables (the current through the inductor, and the voltage across the capacitor)

  - The circuit also changes state at $t = 0$ when the switch is closed which causes the 4A current source to be connected to the circuit.

- Prior to the switch being close the inductor has an initial current $i(0)$ and the capacitor has an initial voltage $v(0)$. If the circuit is such that we cannot derive these initial values by assuming the circuit is in steady state for a long time before the switch is closed then they must be given.

- Assume for this circuits that $v(0) = 0V$ and $i(0) = 0A$.

- As before we analyze the circuit by first analyzing the state variables (since they are the only circuit parameters which do not change when the state of the circuit changes)

- If we apply KCL at the node which i leaves for $t \geq 0$ then we obtain

$$4 = \frac{v}{2} + i + \frac{dv}{dt}$$

- This equation though is in terms of both the state variables i and v. We want to find an DE in terms of just one of the state variable so we need to find another independent equation relating v and i.

- Applying KVL to the right most loop we obtain

$$v = 4\frac{di}{dt}$$

- Substituting this equation into the one above we get that

$$4 = 2\frac{di}{dt} + i + 4\frac{d^2 i}{dt^2}$$

- Putting this equation into standard form

$$\frac{d^2 i}{dt^2} + \frac{1}{2}\frac{di}{dt} + \frac{1}{4}i = 1$$

- This DE has a solution of the form

$$i(t) = i_f(t) + i_n(t)$$

8.6.1 The Forced Response

- The forced response must satisfy the nonhomogeneous DE

$$\frac{d^2 i_f}{dt^2} + \frac{1}{2}\frac{di_f}{dt} + \frac{1}{4}i_f = 1$$

- Since we have a dc power source we try the solution

$$i_f = A$$

- Substituting this solution into the DE above

$$\frac{1}{4}A = 1$$

- Hence

$$i_f = 4A$$

8.6.2 The Natural Response

- The natural response must satisfy the homogeneous DE

$$\frac{d^2 i_n}{dt^2} + \frac{1}{2}\frac{di_n}{dt} + \frac{1}{4}i_n = 0$$

  - We attempt a trial solution of the form

$$i_n = ke^{st}$$

  - Substituting this trial solution into the homogeneous DE gives

$$ks^2e^{st} + 0.5kse^{st} + 0.25ke^{st} = 0$$

  - Dividing through by ke^{st} (since this term can never be zero)

$$s^2 + 0.5s + 0.25 = 0$$

- This is the characteristic equation for the circuit and it has two complex conjugate roots located at

$$s = -0.25 \pm 0.5j$$

- Substituting these roots into our trial solution we get that

$$i_n = k_1 e^{(-0.25 + 0.5j)t} + k_2 e^{(-0.25 - 0.5j)t}$$

- Simplifying and expressing an in terms of sinusiods

$$i_n = [k_1 \cos(0.5t) + k_2 \sin(0.5t)]e^{-0.25t}$$

- To solve for the constants k_1 and k_2 we need to compute the total response.

8.6.3 The Total Response

- We now have equations for the forced and natural responses for this circuit so we can write the total response as

$$i(t) = i_f(t) + i_n(t)$$

$$i(t) = 4 + (k_1 \cos(0.5t) + k_2 \sin(0.5t))e^{-0.25t}$$

  - This equation is valid for all t.

- But we still need to solve for the constant k_1 and k_2

- We can do this by using the initial conditions to evaluate $i(t)$ and $\frac{di(t)}{dt}$ for $t = 0$, which is the time when the state change occurs. This will give us two independent equations in the two unknown k_1 and k_2.

  - At $t = 0$ the initial current is given as $i(0) = 0A$ substituting this value into the above equation for $t = 0$ gives

$$i(0) = 4 + (k_1 \cos(0.5(0)) + k_2 \sin(0.5(0)))e^{-0.25(0)} = 0A$$

$$4 + k_1 = 0$$

$$k_1 = -4$$

- We need to find another equation for which we can evaluated $\frac{di(t)}{dt}$ at $t = 0$ and which is in terms of the other initial condition $v(0) = 0V$.

  - We began with the equation $v = 4\frac{di}{dt}$ evaluating this at $t = 0$ gives

$$v(0) = 4\left.\frac{di}{dt}\right|_{t = 0} = 0V$$

- Substituting in our expression for $i(t)$ and differentiating gives

$$4\frac{d}{dt}[(4 + (k_1\cos(0.5t) + k_2\sin(0.5t))e^{-0.25t})]\Big|_{t=0} = 0V$$

$$[-0.25e^{-0.25t}(k_1\cos(0.5t) + k_2\sin(0.5t)) + e^{-0.25t}(-0.5k_1\sin(0.5t) + 0.5k_2\cos(0.5t))]\Big|_{t=0} = 0$$

$$-0.25k_1 + 0.5k_2 = 0$$

- Substituting in the value for k_1 from above

- $-0.25(-4) + 0.5k_2 = 0$

- $k_2 = -2$

- Therefore the total response is given by

$$i(t) = 4 + (-4\cos(0.5t) - 2\sin(0.5t))e^{-0.25t}A$$

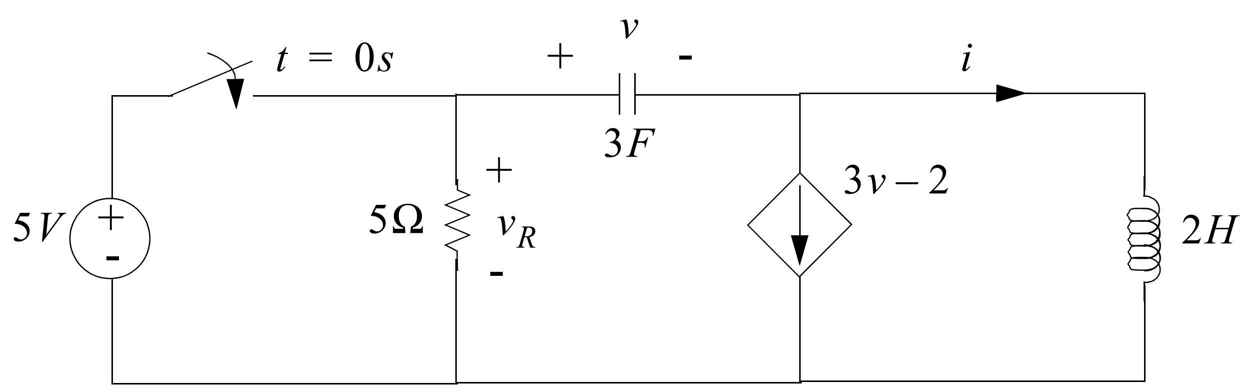
8.7 Case when Dependent Sources are Present

- Transient analysis when dependent sources are present is not much different from the case when no dependent sources are present.

- The only difference is that we need to remember to take the functional form of the dependency into account when we write our equations describing the circuit

- Other than that the analysis procedure proceeds as before.

Ex. 8.1 Consider the following circuit. Assume that at $t = 0$ the capacitor voltage was $v(0) = 1V$ and the inductor current was $i(0) = 2A$. Find the response of the current through the 5-Ω resistor for $t \geq 0$.



Solution:

- We begin by identifying the two state variables i and v

- We can apply KCL at the node joining the capacitor, inductor, and the dependent source which gives

$$3\frac{dv}{dt} = 3v - 2 + i$$

  - Simplifying

$$i = 3\frac{dv}{dt} - 3v + 2$$

- This gives us one of our equations but it is in terms of both the state variables. We want an equation in terms of just one of the state variable.

- We know that the forced response will be do to the independent source so we want this second equation to also include the independent source.

  - At $t \geq 0$ we can apply KVL around the outside loop (remember that we need an equation which relates i and v) which gives

$$5 - v - 2\frac{di}{dt} = 0$$

- Differentiating our previous expression for i we have that

$$\frac{di}{dt} = \frac{d}{dt}\left[3\frac{dv}{dt} - 3v + 2\right] = 3\frac{d^2v}{dt^2} - 3\frac{dv}{dt}$$

- Substituting this into our KVL equation gives

$$5 - v - 2\left[3\frac{d^2v}{dt^2} - 3\frac{dv}{dt}\right] = 0$$

$$-6\frac{d^2v}{dt^2} + 6\frac{dv}{dt} - v = 5$$

- Placing in standard form

$$\frac{d^2v}{dt^2} - \frac{dv}{dt} + \frac{v}{6} = \frac{5}{6}$$

- We know that this equation has a solution of the form $v = v_f + v_n$

    - v_f must satisfy the nonhomogeneous DE therefore

$$v_f = 5$$

    - v_n must satisfy the homogeneous DE which has the characteristic equation

$$s^2 - s + \frac{1}{6} = 0$$

- Remember that we obtain this characteristic equation by assuming a solution of the form $v_n = ke^{st}$ and substituting it into the homogenous DE.

- The roots of this characteristic equation are

$$s = \frac{-1 \pm \sqrt{1 - \frac{4}{6}}}{2} = -\frac{1}{2} \pm \frac{1}{2\sqrt{3}}$$

- The characteristic equation therefore has two complex roots and we known that v_n has the form

$$v_n = k_1 e^{\left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)t} + k_2 e^{\left(-\frac{1}{2} - \frac{1}{2\sqrt{3}}\right)t}$$

- Now we can write the total response as $v = v_f + v_n$ which gives

$$v = 5 + k_1 e^{\left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)t} + k_2 e^{\left(-\frac{1}{2} - \frac{1}{2\sqrt{3}}\right)t}$$

- We can now find the values for k_1 and k_2 by applying the initial conditions

- We know that $v(0) = 1$ hence

$$v(0) = 5 + k_1 e^{\left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)(0)} + k_2 e^{\left(-\frac{1}{2} - \frac{1}{2\sqrt{3}}\right)(0)} = 1$$

- Therefore we obtain the equation

$$5 + k_1 + k_2 = 1$$

- Simplifying

$$k_1 + k_2 = -4$$

- This gives us one equation in the two unknowns k_1 and k_2

- But we also know that $i(0) = 2A$ and that $i(t) = 3\frac{dv(t)}{dt} - 3v(t) + 2$ hence

$$i(0) = \left\{ 3\frac{d}{dt}\left[5 + k_1 e^{\left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)t} + k_2 e^{\left(-\frac{1}{2} - \frac{1}{2\sqrt{3}}\right)t}\right] - 3\left[5 + k_1 e^{\left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)t} + k_2 e^{\left(-\frac{1}{2} - \frac{1}{2\sqrt{3}}\right)t}\right]\right\}\Bigg|_{t=0} + 2 = 2$$

- Simplifying

$$3\left(-\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)k_1+3\left(-\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)k_2-15-3k_1-3k_2 = 0$$

$$\left(-\frac{1}{2}+\frac{1}{2\sqrt{3}}-1\right)k_1+\left(-\frac{1}{2}-\frac{1}{2\sqrt{3}}-1\right)k_2 = 5$$

$$\left(-\frac{3}{2}+\frac{1}{2\sqrt{3}}\right)k_1+\left(-\frac{3}{2}-\frac{1}{2\sqrt{3}}\right)k_2 = 5$$

- We now have two equations in two unknowns and we can solve for k_1 and k_2 as

$$\begin{bmatrix} 1 & 1 \\ \left(-\frac{3}{2}+\frac{1}{2\sqrt{3}}\right) & \left(-\frac{3}{2}-\frac{1}{2\sqrt{3}}\right) \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

- The solution to which gives

$$k_1 = -3.7321$$

$$k_2 = -0.2679$$

- Therefore the total response is given by

$$v(t) = 5-3.7321e^{\left(-\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)t}-0.2679e^{\left(-\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)t}\ V$$

- Since we now have the solution for the state variable $v(t)$ it is straight forward to find the solution for the voltage across the 5-Ω resistor for $t \geq 0$ using our linear circuit theorems

  - Applying KVL around the loop including the resistor, capacitor and inductor gives

$$v_R = v + 2\frac{di}{dt}$$

  - From the work above we know

$$v(t) = 5 - 3.7321e^{\left(-\frac{1}{2}+\frac{1}{2\sqrt{3}}\right)t} - 0.2679e^{\left(-\frac{1}{2}-\frac{1}{2\sqrt{3}}\right)t} V$$

  - and that

$$\frac{di}{dt} = 3\frac{d^2v}{dt^2} - 3\frac{dv}{dt}$$

  - Hence, it is a simple matter of substitution (and a bit of math) to get $v_R(t)$ as

$$v_R(t) = 5 - 9.4642e^{-0.2113t} - 2.5354e^{-0.7887t} \text{ V}$$

8.8 General Solution Steps (for first and second-order circuits)

- The overview, the following are the steps which are used to solve for circuits in which the state of the circuit changes during the circuits operation. Hence, we need to analyze the circuit's transient behavior and determine the new state which the circuits settles to.

  - Analysis Steps:

    1. Reduce the circuit as much as possible (i.e. combine series and parallel circuit element as per the circuit theorems)

    2. If the initial conditions are given Goto step 4.

    3. Solve for the initial conditions of the state variables assuming the circuit is in steady-state prior to the state change.

    4. Identify the state variables.

    5. Develop a DE equation in terms of the state variables using KCL, KVL, etc.

    6. If there is only one state variable then it is a first-order circuit. Use Thevenin or Norton's theorems to simplify the circuit if possible. Use circuit theorems to construct a DE, in terms of the circuit's state variable, which describes the circuit's operation. Goto step 9.

7. If there are two state variables then it is a second-order circuits and you need to develop a second independent DE equation in terms of the two state variable (using KCL, KVL, etc.) (Note: choose equations which relate the two state variables and include the independent sources, since these give rise to the forced response).

8. In one of the differential equations solve for one of the state variables, substitute this solution into the other DE to get a second order DE which is just in terms of the chosen state variable.

9. Place the DE in standard form.

10. Determine the forced response through a trial solution of the same time dependence as the power source (i.e. for a dc power source try a trial solution of $v_f = A$ or $i_f = A$). Substitute this trial solution into the nonhomogeneous DE to determine v_f or i_f.

11. For the natural response. Write the characteristic equation of the homogeneous DE. (obtained by assuming a natural response of the form ke^{st} and substituting this trial solution into the homogeneous DE)

12. Solve for the roots of the characteristic equation

13. If it is a first order DE there is only one root and the natural response has the form $v_n = ke^{st}$ or $i_n = ke^{st}$. Goto Step 18.

14. If it is a second-order DE then there are two roots and three possible cases for the roots.(where the second-order DE is in terms of a capacitor voltage or inductor current).

15. If the roots are real and unequal the natural response is of the form $v_n = k_1e^{s_1t} + k_2e^{s_2t}$ or $i_n = k_1e^{s_1t} + k_2e^{s_2t}$ where s_1 and s_2 are the roots of the characteristic equation.

16. If the roots are real and equal the natural response is of the form $v_n = k_1e^{st} + k_2te^{st}$ or $i_n = k_1e^{st} + k_2te^{st}$.

17. If the roots are complex conjugates the natural response is of the form $v_n = (k_1\cos t + k_2\sin t)e^{Re[s]t}$ or $i_n = (k_1\cos t + k_2\sin t)e^{Re[s]t}$ where $s = a \pm bj$ are the complex conjugate roots of the characteristic equation.

18. Write the total response as $v(t) = v_f(t) + v_n(t)$ or $i(t) = i_f(t) + i_n(t)$ using the solutions to the forced and natural responses found above.

19. This total response is valid for all t for the state variable, so use the initial conditions to determine the values of k_1 and k_2 (or just k in the case of a first-order circuit). For second-order circuits this will require generating two equations in the two unknowns k_1 and k_2. This is straight forward though since the initial conditions for both state variable are known. One equation will come from the solution to the DE by substituting in one of the initial conditions. The other equation will come from substituting the other initial condition into one of the previously derived circuit equations (note: in doing this one may need to the value of the derivative of the state variable evaluated at the time of the state change (i.e. $\left.\frac{dv(t)}{dt}\right|_{t=0}$ or $\left.\frac{di(t)}{dt}\right|_{t=0}$))

20. Solve for the other circuit parameters of interest in terms of the solution which was obtained for the state variable through application of standard linear circuit theorems.

Assignment #8

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 9
Phasors

9.1 Introduction

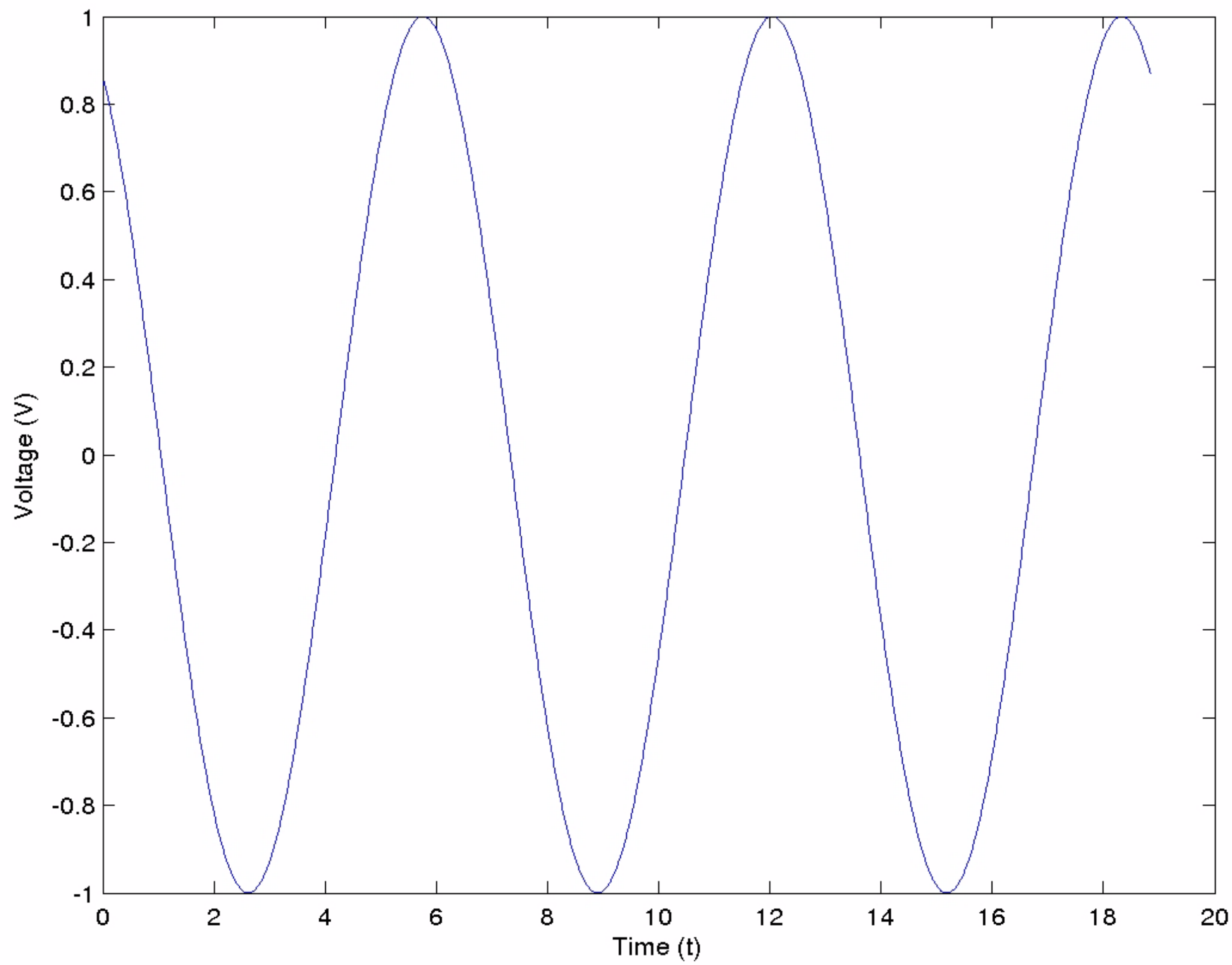
- To this point we have only dealt with circuits which are in dc steady state.
- We would like to extend our results and theorems so that they can be used for circuits which are excited by sinusoidal power sources.
  - We can use complex analysis (**phasors**) to simplify this process when the circuit's power sources have a single frequency ω.
  - **Frequency domain analysis** is the study of circuits with power source frequencies other than dc (i.e when $\omega > 0$).

9.2 Sinusoidal Waveforms

- Assume that we have a voltage source which is given by

$$v(t) = V\cos(\omega t + \phi)$$

- A plot of this waveform is shown below



- We say that the voltage waveform is expressed in **standard form** when it is written in the form shown above.

- To use phasors we must always put the power sources in **standard form**.

- This waveform has 3 parameters

    1. V: the **amplitude** (volts)

    2. ω: the **angular frequency** (radians/sec)

    3. ϕ: the **phase** (radians or degrees)

Ex. 9.1 Find the parameters of the given current waveform

$$i(t) = 10\cos(100t + 30°)\ \text{A}$$

Solution:

Ex. 9.2 Find the current value in the above example at $t = 3s$

Solution:

Ex. 9.3 Express the waveform $v(t) = 4\sin(100t + 90°)$ in standard form.

Solution:

9.3 Sinusoidal Waveform Period and Frequency

- The waveform $v(t) = V\cos(\omega t + \phi)$ repeats itself at regular time intervals T, which we call the **period** of the waveform.

  - T is given from the equation

$$\omega T = 2\pi$$

$$T = \frac{2\pi}{\omega} \text{ (sec)}$$

  - The frequency of this waveform is the inverse of its period

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ (Hz)}$$

  - Alternatively we can relate the frequency and the angular frequency by

$$\omega = 2\pi f \text{ rad/s}$$

9.4 Complex Numbers

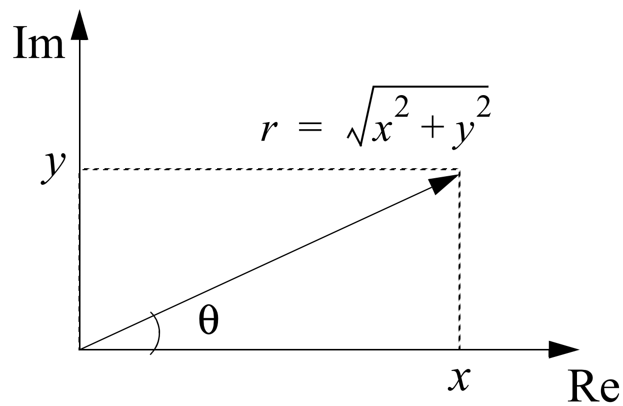
- A complex number z can be represented in **rectangular form** (also known as **Cartesian form**) as

$$z = x + jy$$

- where $j = \sqrt{-1}$ and both x and y are real numbers denoting the real and imaginary parts of z respectively.

- It follows that $j^2 = -1$ (as electrical engineers we use j instead of i, which is use in mathematics, because i is already used to denote current)

- z can be represented in the complex plane as follows



- The real and imaginary parts can be extracted from z using the real (Re) and imaginary (Im) functions

$$y = Im(z)$$

$$x = Re(z)$$

- We can represent z in **polar form** as

$$z = r\angle\theta$$

- where the **magnitude** r and the **angle** θ are given by

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

- Alternately give a complex number in its polar form we can express it in its rectangular form as

$$r\angle\theta = r\cos\theta + jr\sin\theta$$

- The **conjugate** of a complex number z is denoted by z^* and is obtained by replacing j with $-j$ (this is equivalent to mirroring the complex number about the real axis).

$$z = a + jb$$

$$z^* = a - jb$$

9.5 Euler's Formula

- There is one more way which we can represent a complex number which is **Euler's form**

$$re^{j\theta} = r\cos\theta + jr\sin\theta$$

- This formula relates the **polar form** to the exponential function.

Ex. 9.4 Illustration of the validity of Euler's Formula

- Square both sides of the above equation

$$(e^{j\theta})^2 = (r\cos\theta + jr\sin\theta)^2$$

- Expanding the right hand side

$$e^{j2\theta} = \cos^2\theta - \sin^2\theta + j2\sin\theta\cos\theta$$

- Simplifying through trigonometric identities

$$e^{j2\theta} = \cos 2\theta + j\sin 2\theta$$

- This results confirms that Euler's formula produces the correct result

Ex. 9.5 Express the complex number $z = 3 - j4$ in polar and Euler's form

Solution:

- Euler's formula also allows us to easily prove the counter-intuitive identity

$$\frac{1}{j} = -j$$

  - We can prove this result of follows

$$j = \cos 90° + j\sin 90° = (0) + j(1)$$

  - By writing the complex number in Euler's form we have that

$$j = e^{j90°}$$

  - This allows us to write,

$$\frac{1}{j} = \frac{1}{e^{j90°}} = e^{-j90°}$$

  - Placing $e^{-j90°}$ back into rectangular form we have that

$$\frac{1}{j} = e^{-j90°} = \cos 90° + j\sin(-90°) = (0) + j(-1) = -j$$

  - which proves the identity ■

9.6 Phasors

- What is a phasor? and Why do we need phasors?
- From Chapter 6 we learned that
  - The current through a capacitor is given by $i = C\frac{dv}{dt}$
  - The voltage across a capacitor is given by $v = \frac{1}{C}\int_{-\infty}^{t} i d\tau$
  - The current through an inductor is given by $i = L\int_{-\infty}^{t} v d\tau$
  - The voltage across an inductor is given by $v = L\frac{di}{dt}$
- We would like a notation which will allow us to avoid having to deal with complicated equations involving differentials and integrals. We would like simple equations like we had when we were just dealing with resistive circuits.

- We would also like to deal with sinusoidal power sources. Integration and differentiation across these functions is easy (sine to cosine and vice versa) but it is messy
  - we need to make sure we track the necessary sign changes
  - and we would need to do integration and differentiation by parts to track the effects of the sine and cosine functions' arguments.
- Phasors can help us solve these problems (and make our solutions easier).
  - Phasors are based on the following observation

Representing a sinusoidal waveform as an exponential function allows us to express I-V relationships for inductors and capacitors as if they are simple resistors with *complex* resistance.

  - Using Euler's form we can represent sinusiods as exponentials.
  - This is nice since exponential functions are the ***only functions*** which do not change form with respect to integration and differentiation operators.
- What is a Phasor?
  - Starting with a sinusoidal voltage expressed in standard form

$$v(t) = V\cos(\omega t + \phi)$$

- The above equation can be viewed as the real part of a complex sinusiod

$$v(t) = VRe[\cos(\omega t + \phi) + j\sin(\omega t + \phi)]$$

- where $Re[.]$ is a function which returns the real part of a complex number (i.e. $Re[a - jb] = a$)

- Now we have the voltage source described in a form where we can use Euler's formula

$$v(t) = VRe[e^{j(\omega t + \phi)}]$$

- Since our signals are always actually real we can drop the $Re[.]$ part

$$v(t) = Ve^{j(\omega t + \phi)}$$

- Under the assumption that when we write this we know that we are only referring to the real part of the complex number which $v(t)$ now describes.

- Now we can use the rules of exponentiation to separate out the part due to the sinusiods frequency ω and the part due to its phase ϕ

$$v(t) = Ve^{j\omega t} \times e^{j\phi}$$

- This allows us to group the two terms which do not involve time together

$$v(t) = \boldsymbol{V}e^{j\omega t}$$

$$\boldsymbol{V} = Ve^{j\phi}$$

- $\boldsymbol{V}$ is a complex number which is called the **voltage phasor**.

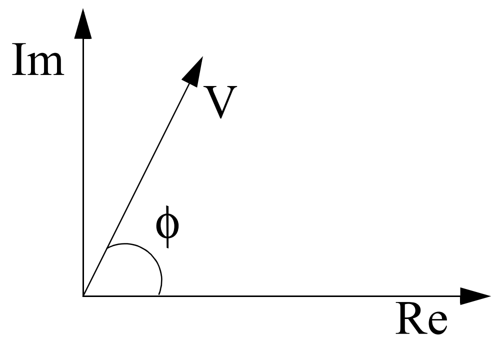
- Since the voltage phasor is a complex number we can represent it in any of or three equivalent forms

  - Euler's Form: $\boldsymbol{V} = Ve^{j\phi}$

  - Rectangular Form: $\boldsymbol{V} = V(\cos\phi + j\sin\phi)$

  - Polar Form: $\boldsymbol{V} = V\angle\phi$

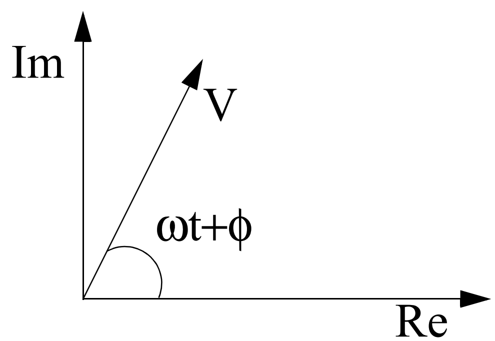
- Graphically the phasor $\boldsymbol{V}$ can be represented as follows



- Note that the phasor is only the representation of the non-time dependent portion of the sinusiod.

- Relative to the complete waveform we have that

$v(t) = V\sin(\omega t + \phi) = [Ve^{j\phi}]e^{j\omega t} = \boldsymbol{V}e^{j\omega t}$ or graphically



- This shows that the magnitude of the vector remains constant but the angle the vector V makes with the x-axis changes with time.

- Vector V rotates in a counter clockwise direction as time progresses.

- The angular frequency of rotation equals the angular frequency ω of the voltage waveform

- The x-component of the vector equals the amplitude of $v(t)$ at as a function of t and is given by

$$v(t) = V\cos(\omega t + \phi)$$

- This corresponds to the projection of the complex vector V onto the x-axis.

- This is the value which we would actually measure on a real circuit using a multimeter.

- The complex part of the vector allows the sinusoidal power source to be expressed in Euler's formula which allows us to only need to do integration and differentiation across exponential functions when we solve for our circuit parameters.

Ex. 9.6 Suppose we have a sinusoidal current source expressed in standard form as

$$i(t) = 140\cos(377t + 40°)$$

- What are the parameters of that source and what is the phasor associated with it?

Solution:

Ex. 9.7 Suppose we have a sinusoidal voltage source expressed as

$$v(t) = 10\sin(50t - 30°)$$

- What are the parameters of that source and what is the phasor associated with it?

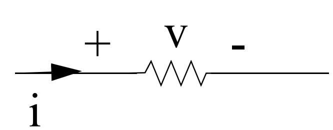
Solution:

Ex. 9.8 A sinusoidal voltage waveform has $f = 60Hz$ and a voltage phasor of $\mathbf{V} = 4\angle 30° V$. Find the time-dependent expression for this waveform.

Solution:

9.7 Phasor Current-Voltage Law for Resistors

- Assume a sinusoidal current is passing through a resistor R.



- The current is assumed to be expressed as

$$i(t) = I\cos(\omega t + \phi)$$

- The current phasor is

$$\boldsymbol{I} = Ie^{j\phi}$$

- The voltage across the resistor is determined by Ohm's Law

$$v(t) = i(t)R$$

- Substituting our expression in for $i(t)$ from above we get that

$$v(t) = IR\cos(\omega t + \phi)$$

- The phasor for this sinusoidal voltage waveform is

$$\boldsymbol{V} = IRe^{j\phi}$$

- But we can notice that $Ie^{j\phi}$ is just the current phasor given above. Therefore we can write the voltage phasor in terms of the resistance and current phasor as

$$\boldsymbol{V} = \boldsymbol{I}R$$

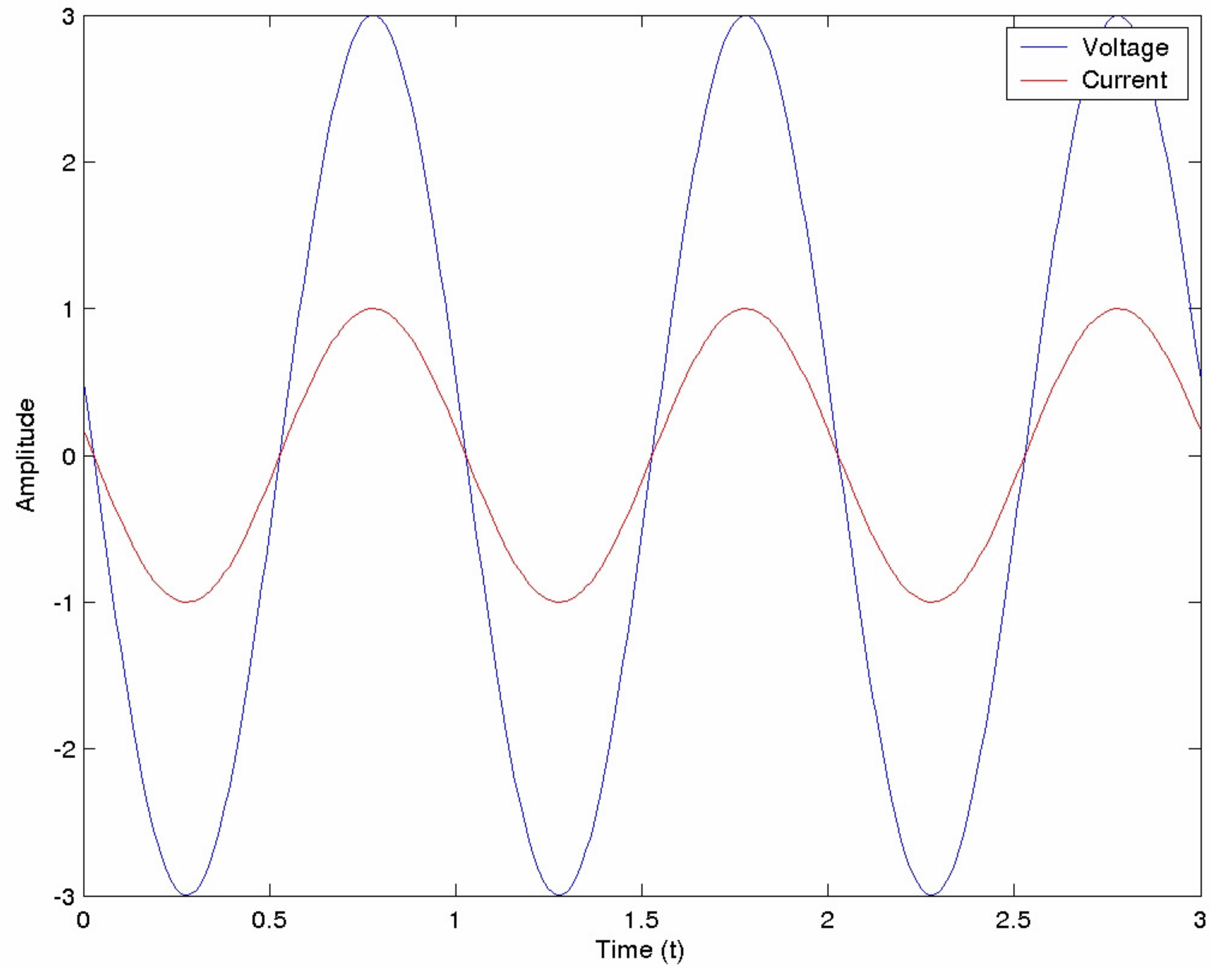
- But this is nothing more than applying Ohm's law using phasor notation.

- Alternatively we can also write

$$\boldsymbol{I} = G\boldsymbol{V}$$

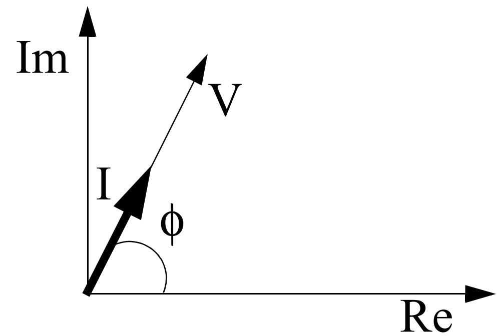
- So Ohm's law holds under phasor notation.

- From this result we can plot the voltage waveform which results when a sinusoidal current is passed through a resistor



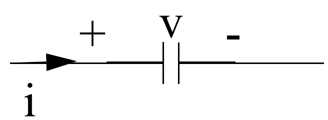
- Notice that both the current and voltage waveforms are in phase (i.e. there is no phase shift between the waveforms caused by passing through the resistor.)

- In terms of the voltage and current phasors this can be graphically shown as the fact that the two vectors are at the same phase angle



9.8 Phasor Current-Voltage Law for Capacitors

- Assume a sinusoidal voltage across a capacitor C.



- The voltage is assumed to be expressed as

$$v(t) = V\cos(\omega t + \phi)$$

- The current phasor is

$$\boldsymbol{V} = Ve^{j\phi}$$

- The voltage across the capacitor is determined by the expression

$$i(t) = C\frac{dv(t)}{dt}$$

- Substituting our expression in for $v(t)$ from above we get that

$$i(t) = -\omega CV\sin(\omega t + \phi)$$

- To express this waveform in standard form we must first take the negative sign into the expression

$$i(t) = \omega CV\sin(\omega t + \phi + 180°)$$

- Now we must convert the sine to a cosine using $\sin\theta = \cos(\theta - 90°)$

$$i(t) = \omega CV\cos(\omega t + \phi + 180° - 90°)$$

- Simplifying we are now in the standard form

$$i(t) = \omega CV\cos(\omega t + \phi + 90°)$$

- The phasor for this sinusoidal current waveform is

$$\boldsymbol{I} = \omega CVe^{j(\phi + 90°)} = j\omega CVe^{j\phi}$$

- Since

$$e^{j90°} = \cos(90° + j\sin 90°) = j$$

- But we can notice that $Ve^{j\phi}$ is just the voltage phasor given above. Therefore we can write the current phasor in terms of the capacitance, angular frequency, and voltage phasor as

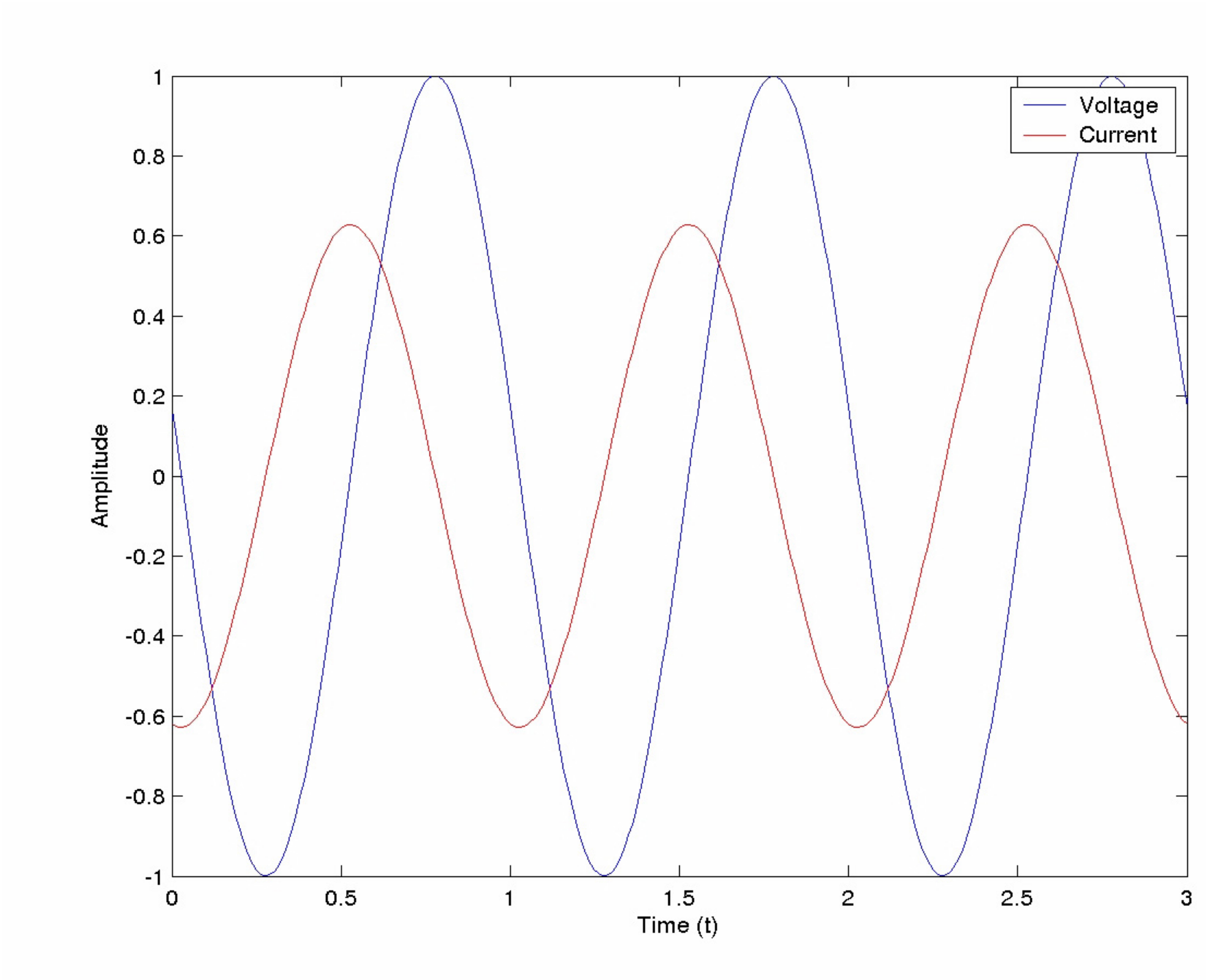
$$\boldsymbol{I} = j\omega C\boldsymbol{V}$$

- Alternatively we can also write

$$\boldsymbol{V} = \frac{\boldsymbol{I}}{j\omega C} = -j\frac{\boldsymbol{I}}{\omega C}$$

- which is the current voltage law for a capacitor using phasor notation.
- Notice that the I-V law for the capacitor, when written in phasor notation, does not involve integration or differentiation.
- Notice also that the I-V relationship for the capacitor now look similar to Ohm's law for a resistor except now we have a complex proportionality constant.

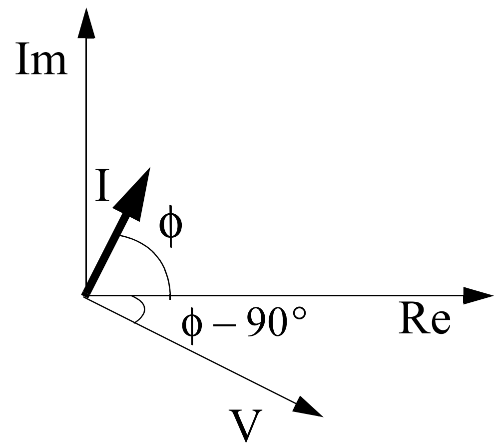
- From this result we can plot the current waveform which results when a sinusoidal voltage is passed through a capacitor



- Notice that the current and voltage waveforms are now out of phase (i.e. the is a phase shift between the waveforms caused by the capacitor.)

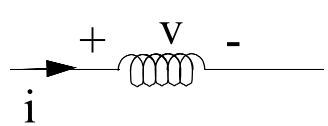
- We say that the current phasor ***leads*** the voltage phasor by $90°$

- In terms of the voltage and current phasors this can be graphically shown as the fact that the current phasor is at a phase angle $\phi + 90°$ to the current phasor.



9.9 Phasor Current-Voltage Law for Inductors

- Assume a sinusoidal current across an inductor L.



- The current is assumed to be expressed as

$$i(t) = I\cos(\omega t + \phi)$$

- The current phasor is

$$\boldsymbol{I} = Ie^{j\phi}$$

- The voltage across the inductor is determined by the expression

$$v(t) = L\frac{di(t)}{dt}$$

- Substituting our expression in for $i(t)$ from above we get that

$$v(t) = -\omega LI\sin(\omega t + \phi)$$

- To express this waveform in standard form we must first take the negative sign into the expression

$$v(t) = \omega LI\sin(\omega t + \phi + 180°)$$

- Now we must convert the sine to a cosine using $\sin\theta = \cos(\theta - 90°)$

$$v(t) = \omega LI\cos(\omega t + \phi + 180° - 90°)$$

- Simplifying we are now in the standard form

$$v(t) = \omega LI\cos(\omega t + \phi + 90°)$$

- The phasor for this sinusoidal current waveform is

$$\boldsymbol{V} = \omega L I e^{j(\phi + 90^\circ)} = j\omega L I e^{j\phi}$$

- Since

$$e^{j90^\circ} = \cos(90^\circ + j\sin 90^\circ) = j$$

- But we can notice that $Ie^{j\phi}$ is just the voltage phasor given above. Therefore we can write the current phasor in terms of the capacitance, angular frequency, and voltage phasor as

$$\boldsymbol{V} = j\omega L\boldsymbol{I}$$

- Alternatively we can also write

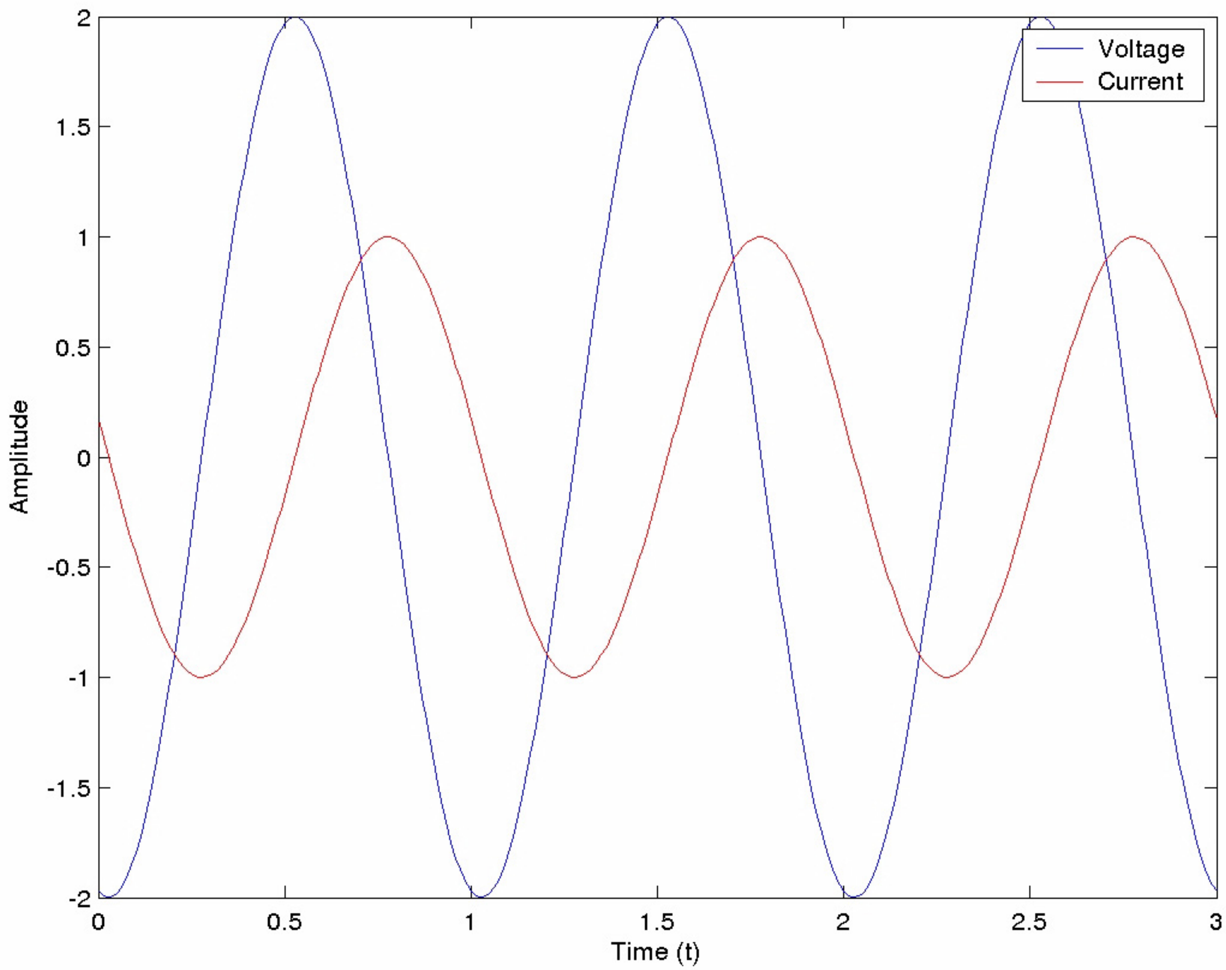
$$\boldsymbol{I} = \frac{\boldsymbol{V}}{j\omega L} = -j\frac{\boldsymbol{V}}{\omega L}$$

- which is the current voltage law for a inductor using phasor notation.

- Notice that the I-V law for the inductor, when written in phasor notation, does not involve integration or differentiation.

- Notice also that the I-V relationship for the inductor now look similar to Ohm's law for a resistor except now we have a complex proportionality constant.

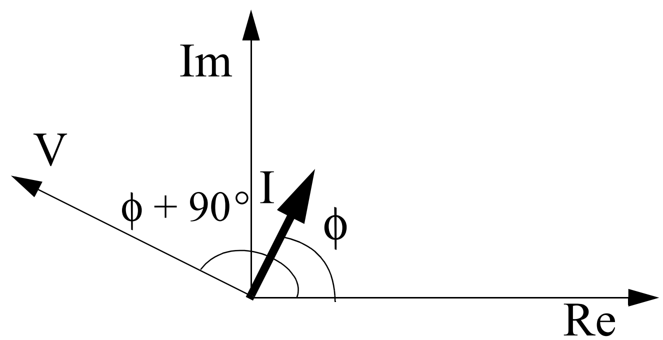
- From this result we can plot the voltage waveform which results when a sinusoidal current is passed through a inductor



- Notice that the current and voltage waveforms are now out of phase (i.e. the is a phase shift between the waveforms caused by the inductor.)

- We say that the current phasor ***lags*** the voltage phasor by $90°$ (since $-j = e^{-90°} = \cos(-90°) + j\sin(-90°)$)

- In terms of the voltage and current phasors this can be graphically shown as the fact that the current phasor is at a phase angle $\phi - 90°$ to the current phasor.



9.10 Impedance

- When a sinusoidal voltage or current source is connected to two or more elements in a circuit, the current and voltage at the terminals of the source can be described in general by

$$v(t) = V\cos(\omega t + \phi_v)$$

$$i(t) = I\cos(\omega t + \phi_i)$$

- All the circuit variable will have the same frequency ω but each circuit variable will have its own amplitude and phase angle

- The phasors for an element's current and voltage will be given by

$$\boldsymbol{V} = V\angle\phi_v$$

$$\boldsymbol{I} = I\angle\phi_i$$

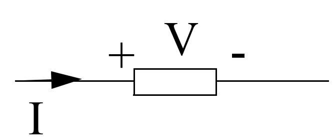
- The ***impedance*** seen by the source is defined as the ration of the voltage phasor to the current phasor

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}$$

  - Using the phasor expressions above for the voltage and current we can write

$$\boldsymbol{Z} = \frac{V\angle\phi_v}{I\angle\phi_i} = \frac{V}{I}\angle(\phi_v - \phi_i)$$

- The circuit symbol for impedance is



- The current always enters the impedance at the positive end, following our convention for passive elements.

- **<u>IMPORTANT NOTE</u>**

Impedance is a complex number.

It is not a phasor.

It does not correspond to a time-domain waveform.

It is just the ratio of two complex numbers.

- Since impedance is a complex number we can express it in polar as well as in rectangular forms

$$\boldsymbol{Z} = Z\angle\phi$$

$$\boldsymbol{Z} = R + jX$$

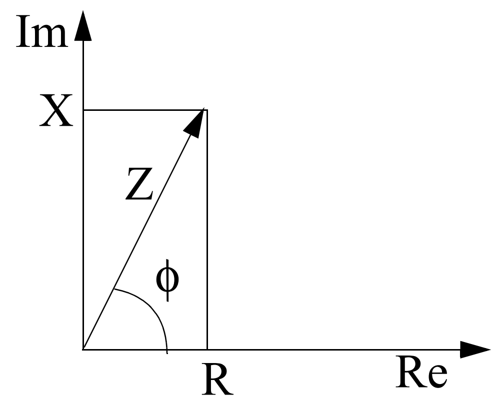
- where R is the **resistance** and X is the **reactance** and are given by

$$R = Z\cos\phi$$

$$X = Z\sin\phi$$

- Both R and X are measured in ohms, as is the impedance Z

- Since Z is a complex number we can represent it in the complex plane



- The components of Z are related to each other as

$$Z = \sqrt{R^2 + X^2}$$

$$\phi = \tan^{-1}\frac{X}{R}$$

$$R = Z\cos\phi$$

$$X = Z\sin\phi$$

Ex. 9.9 What are the impedances of a resistor, an inductor, and a capacitor?

Solution:

Ex. 9.10 Given a voltage source $\boldsymbol{V} = 10\angle 30°$ supplies a current $\boldsymbol{I} = 5\angle 60°$. What is the impedance seen by the source and is the element capacitive or inductive?

Solution:

9.11 Admittance

- The reciprocal of impedance is ***admittance***

$$\boldsymbol{Y} = \frac{1}{\boldsymbol{Z}}$$

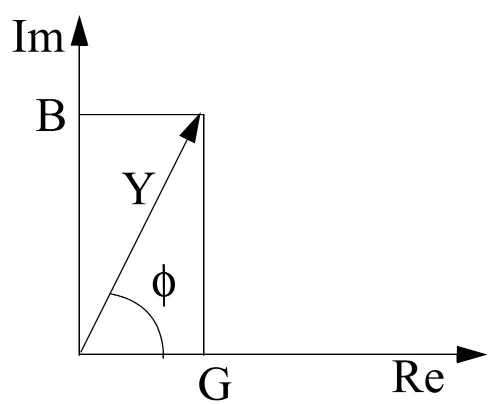
- Admittance, like impedance, is a complex number and can be expressed in polar, rectangular, and Euler's forms

$$\boldsymbol{Y} = Y\angle\phi$$

$$\boldsymbol{Y} = Ye^{j\phi}$$

$$\boldsymbol{Y} = G + jB$$

  - where G is the **conductance** and B is the **susceptance**.
  - Both G and B are measured in Siemens, as is the admittance $\boldsymbol{Y}$
  - Admittance, like impedance, is a complex number and ***NOT*** a phasor and it does ***NOT*** correspond to a time-dependent waveform.
  - We can represent $\boldsymbol{Y}$ is the complex plane



- where each of the components of ***Y*** are related to each other as

$$Y = \sqrt{G^2 + B^2}$$

$$\phi = \tan^{-1}\frac{B}{G}$$

$$G = Z\cos\phi$$

$$B = Z\sin\phi$$

Ex. 9.11 What are the admittance of a resistor, an inductor, and a capacitor?

Solution:

Ex. 9.12 Given a voltage source $\boldsymbol{V} = 5\angle 20°$ supplies a current $\boldsymbol{I} = 10\angle -40°$. What is the admittance seen by the source and is the element capacitive or inductive?

Solution:

- The summary of the current voltage laws in the time and frequency domains for our three type of elements is

Element	**Resistor**	**Capacitor**	**Inductor**
Impedance	R	$-\frac{j}{\omega C}$	$j\omega L$
Admittance	G	$j\omega C$	$-\frac{j}{\omega L}$

9.12 Kirchhoff's' Laws for Phasors

- Now that we have the phasor notation for sinusoidal source waveforms and have Ohm's law for complex numbers, we would like to know if our circuit theorem developed in the previous chapters still hold and can be applied to solve for circuit parameters in the complex domain.

- Kirchhoff's Voltage Law still holds when phasors are used.

  - KVL states that the sum of voltage drops around a closed loop is zero

$$v_1 + v_2 + \ldots + v_n = \sum_{i=1}^{n} v_i = 0$$

  - When all the voltages are sinusoidal and have the same frequency ω, the above equation becomes

$$V_1 e^{j(\omega t + \phi_1)} + V_2 e^{j(\omega t + \phi_2)} + \ldots + V_n e^{j(\omega t + \phi_n)} = \sum_{i=1}^{n} V_i e^{j(\omega t + \phi_i)} = 0$$

  - By dividing by the common term $e^{j\omega t}$ we get

$$\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_n = \sum_{i=1}^{n} \boldsymbol{V}_i = 0$$

  - where $\boldsymbol{V}_i = V_i \angle \phi_i ; i = 1, 2, \ldots, n$ are the phasors of the voltages around the loops.

- Thus KVL holds for phasors ■

- A similar argument holds for KCL for currents entering a node.
  - When n currents enter a node we have

$$i_1 + i_2 + \ldots + i_n = \sum_{i=1}^{n} i_i = 0$$

  - When all the currents are sinusoidal and have the same frequency ω, the above equation becomes

$$I_1 e^{j(\omega t + \phi_1)} + I_2 e^{j(\omega t + \phi_2)} + \ldots + I_n e^{j(\omega t + \phi_n)} = \sum_{i=1}^{n} I_i e^{j(\omega t + \phi_i)} = 0$$

  - By dividing by the common term $e^{j\omega t}$ we get

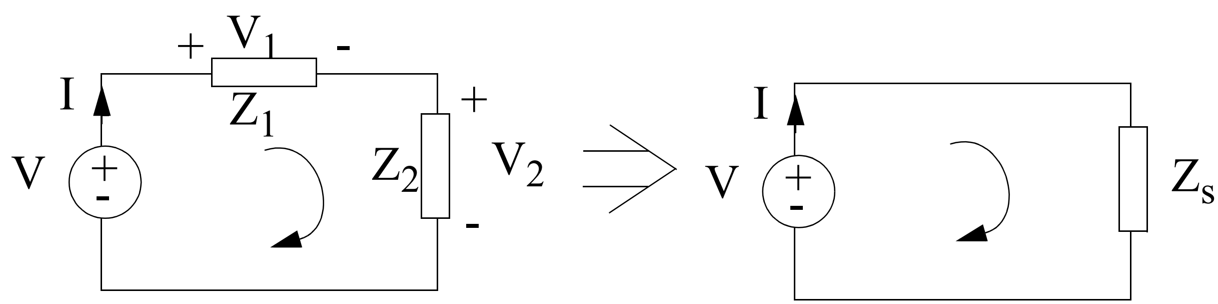
$$\boldsymbol{I}_1 + \boldsymbol{I}_2 + \ldots + \boldsymbol{I}_n = \sum_{i=1}^{n} \boldsymbol{I}_i = 0$$

  - where $\boldsymbol{I}_i = I_i \angle \phi_i ; i = 1, 2, \ldots, n$ are the phasors of the voltages around the loops.
  - Thus KCL also holds for phasors ■

9.13 Series Impedance

- Given two impedances in series we can ask the question: What is their equivalent resistance?

  - We can use KVL to answer this question



(a) (b)

  - Applying KVL to circuit (a) we have

$$-\boldsymbol{V} + \boldsymbol{V}_1 + \boldsymbol{V}_2 = 0$$

  - Applying the complex version of Ohm's law to each impedance

$$V = IZ_1 + IZ_2$$

$$I = \frac{V}{Z_1 + Z_2}$$

- Applying the complex version of Ohm's law to the circuit (b) we have

$$V = IZ_s$$

$$I = \frac{V}{Z_s}$$

- If the current and voltage in both circuits are identical then for the circuits to be equivalent we must have

$$Z_s = Z_1 + Z_2$$

Impedances sum when they are connected in series.

- The above case can be generalized to the case of *n* series-connected impedances. In that case the equivalent series impedance is given by

$$Z_s = \sum_{i=1}^{n} Z_i$$

Ex. 9.13 A current source of $i(t) = 5\cos(250t - 30°)$ is connected to a series connection of $R = 10\Omega$ and $C = 2mF$. Find the voltage across each circuit element.

Solution:

9.13.1 Voltage Division

- Given the series-connected impedances we can also ask what is the voltage across each of the impedances. From circuit (a) above and the discussion which followed we have that

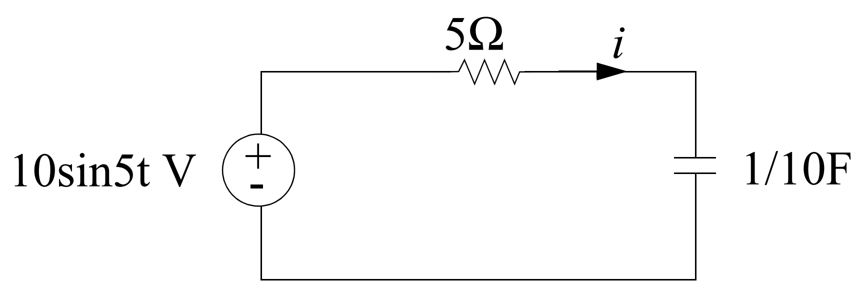
$$V_1 = IZ_1 = \frac{Z_1}{Z_1 + Z_2}V$$

$$V_2 = IZ_2 = \frac{Z_2}{Z_1 + Z_2}V$$

- where the current is given by $I = \dfrac{V}{Z_1 + Z_2}$ as solved for above.

- Series impedances therefore perform *voltage division*.

Ex. 9.14 Find i in the in the circuit below?



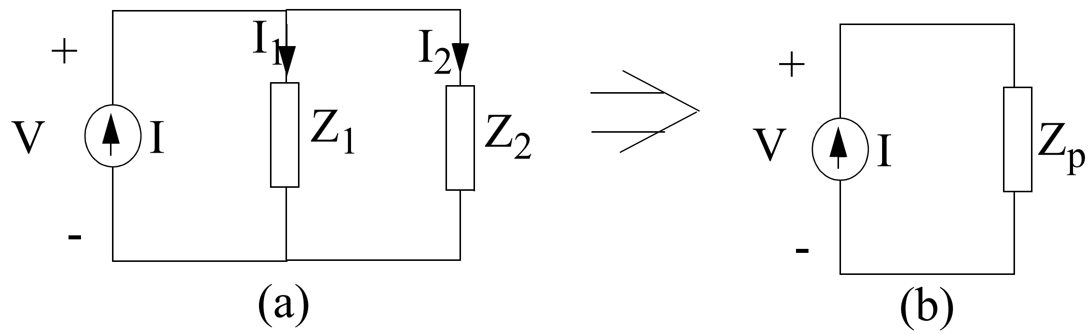
Solution:

Ex. 9.15 A voltage source is connected to three series impedences.Given that $\boldsymbol{V} = 100\angle 30°$, $\boldsymbol{Z}_1 = 3\angle 30°$, $\boldsymbol{Z}_2 = 5\angle -30°$, and $\boldsymbol{Z}_3 = 6\angle 15°$. Find the current which flows through the circuit and that the sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_3$ equals the applied voltage $\boldsymbol{V}$.

Solution:

9.14 Parallel Impedances

- We can ask, what is the equivalent circuit of two impedances in parallel?



- Applying KCL and Ohm's law to circuit (a) we have

$$I = I_1 + I_2$$

$$I = \frac{V}{Z_1} + \frac{V}{Z_2} = Y_1 V + Y_2 V$$

$$V = \frac{I}{Y_1 + Y_2}$$

- Applying Ohm's law to circuit (b) we have

$$V = IZ_p = \frac{I}{Y_p}$$

- Circuits (a) and (b) are equivalent when

$$Y_P = Y_1 + Y_2$$

- or in terms of impedances

$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

- For *n* impedances in parallel these results can be generalized to give

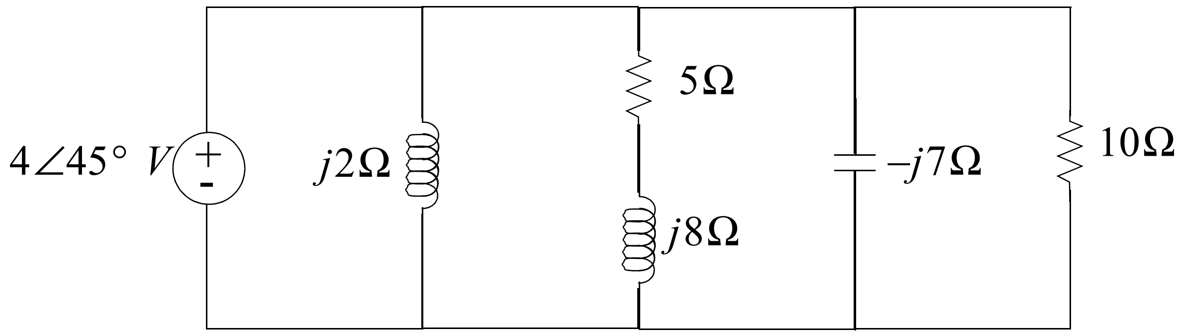
$$Y_p = \sum_{i=1}^{n} Y_i$$

$$Z_p = \frac{1}{\left[\sum_{i=1}^{n} Y_i\right]} = \frac{1}{Y_1 + Y_2 + \ldots + Y_n} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \ldots + \frac{1}{Z_n}}$$

Ex. 9.16 A voltage source of $v(t) = 5\cos(250t - 30°)$ is connected to a parallel connection of $R = 10\Omega$ and $C = 2000\mu F$. Find the current flowing in each circuit element.

Solution:

Ex. 9.17 For the circuit shown below find the equivalent impedance and the total current which flows through in the voltage source



Solution:

9.14.2 Current Division

- The currents through each of the resistors in circuit (a) above are given by

$$I_1 = VY_1 = \frac{Y_1}{Y_1 + Y_2}I$$

$$I_2 = VY_2 = \frac{Y_2}{Y_1 + Y_2}I$$

  - In terms of impedances these equations become

$$I_1 = \frac{Z_2}{Z_1 + Z_2}I$$

$$I_2 = \frac{Z_1}{Z_1 + Z_2}I$$

- The above result can be generalized to *n* parallel-connected impedances

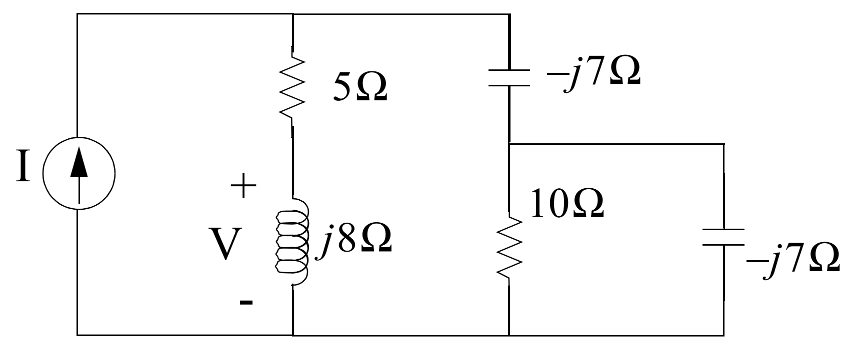
$$I_i = \frac{Y_i}{Y_p}I, \qquad 1 \le i \le n$$

where Y_P is given by

$$Y_p = \sum_{i=1}^{n} Y_i$$

- Parallel impenitences therefore perform *current division.*

Ex. 9.18 In the circuit shown below the voltage drop across the inductor is $\boldsymbol{V} = 30V$. What is the value of the current source?

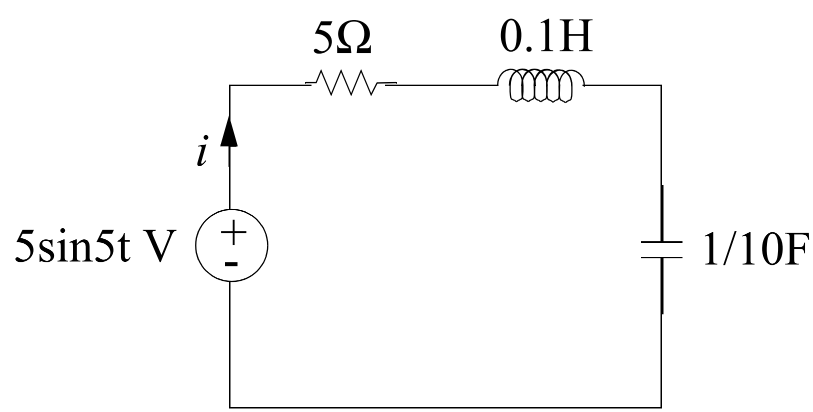


Solution:

9.15 Phasor Circuits

- Using phasors we can analyze circuits containing resistors, capacitors, and inductors, with sinusoidal power sources.
  - Phasors allow us to treat each of these elements as just complex impedances
  - This greatly simplifies our analysis
- To do this though we must first convert all of the circuit element to their equivalent complex impedances and convert all of the power sources to their phasor notation.

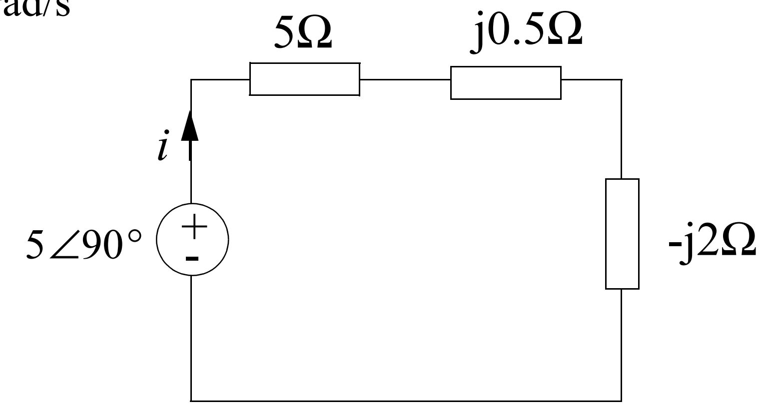
Ex. 9.19 Solve for the current i in the following circuit



Solution:

- Step 1: Convert the circuit to its phasor equivalent form

$\omega = 5 \text{ rad/s}$



- where the phasor for the voltage source is found by first placing the source in standard form

$$v(t) = 5\sin 5t = 5\cos(5t - 90°) = [5\angle -90°]e^{j5t}$$

$$\omega = 5 \text{ rad/s}$$

$$\phi = -90°$$

- were the equivalent impedances are found by

$$Z_L = j\omega L = j(0.1H)(5 \text{ rad/s}) = j0.5\Omega$$

$$Z_C = -\frac{j}{\omega C} = -\frac{j}{(5 \text{ rad/s})\left(\frac{1}{10}F\right)} = -j2\Omega$$

- Once, we have all the complex impedances we can simplify the circuit by using our theorems for series and parallel impedance equivalents

$$Z_s = Z_R + Z_L + Z_C = 5 + j0.5 - j2 = 5 - j1.5\ \Omega$$

- We can now solve for i using the complex form of Ohm's Law

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{Z_s} = \frac{5\angle -90°}{5 - j1.5\ \Omega}$$

- Note that to solve this equation we need to put the numerator's and denominator's complex numbers in the same form.

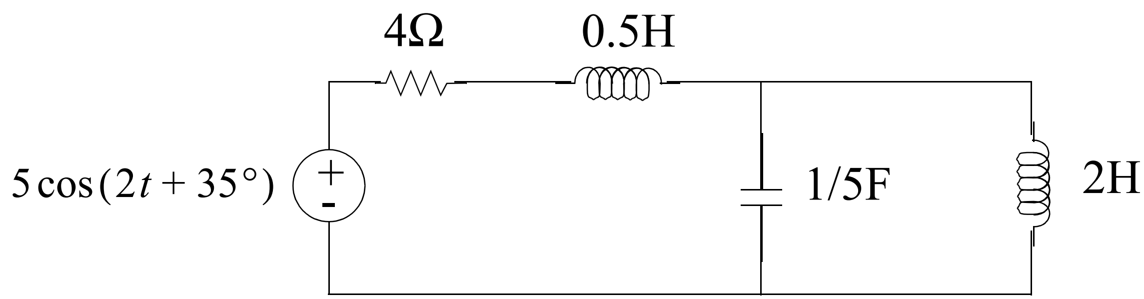
- Given it is division it is easiest if we put the denominator into polar form

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{Z_s} = \frac{5\angle -90°}{5 - j1.5\ \Omega} = \frac{5\angle -90°}{5.22\angle -16.70°} = 0.9578\angle(-90° - (-16.70°)) = 0.9578\angle -73.3008°$$

- Now we have a phasor value for $\boldsymbol{I}$, but we were asked to get the time domain function for i so we need to convert from the phasor back to the time domain function

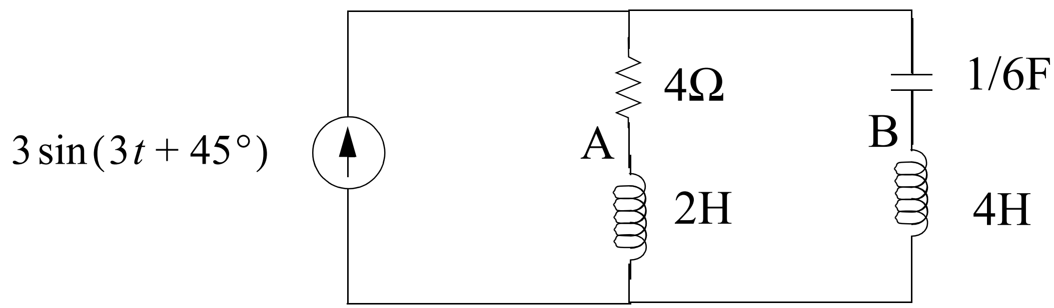
$$i(t) = 0.9578\cos(5t - 73.3008°)\ \text{A}$$

Ex. 9.20 Find the voltage across each element in the circuit shown below



Solution:

Ex. 9.21 Determine the voltage v_{AB} in the circuit below



Assignment #9

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 10
AC Analysis Using Phasors

10.1 Introduction

- We would like to use our linear circuit theorems (Nodal analysis, Mesh analysis, Thevenin and Norton equivalent circuits, Superposition, etc.) to be able to solve for circuit parameters in phasor circuits.

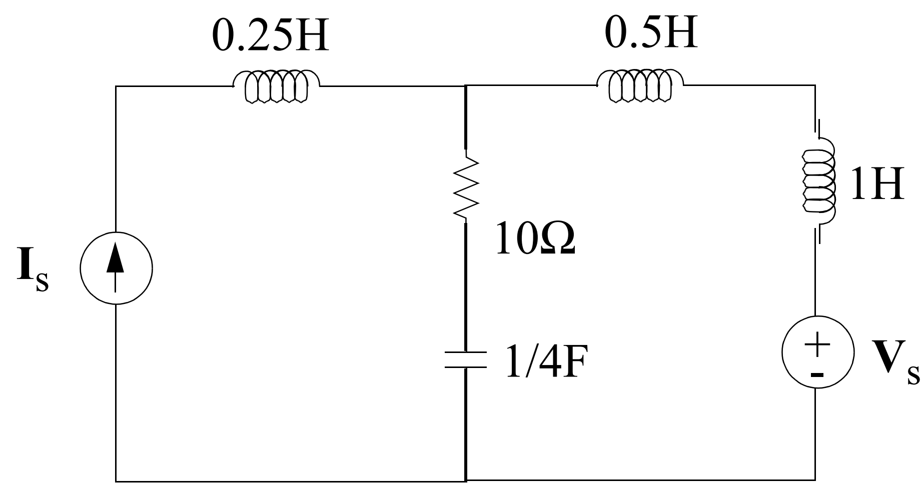
- In this chapter we will show that all of these theorems hold, we just need to convert the circuit we are given to its phasor equivalent, and then use the complex domain versions of these theorems to solve for the circuit parameters we are interested in.
  - We need to remember though, that if we are asked for a time domain circuit parameter, then once we have the phasor solution we must convert the phasor back into its time domain equivalent.

10.2 Superposition

- We know from Chapter 3 that superposition is a property of **ALL** linear circuits.
  - Changing the circuit notation from time domain to phasor domain does not change the linearity of the circuit.

- Therefore, superposition continues to hold for phasor circuits.

Ex. 10.1 Find the current through the voltage source assuming $\boldsymbol{I_s} = 5\angle{-25°}A$, $\boldsymbol{V_s} = 3\angle{45°}V$ and $\omega = 4$ rad/s



Solution:

10.3 Thevenin's Theorem

- From Chapter 3 we had that Thevenin's theorem states

Any linear circuit can be represented at a given pair of nodes by an equivalent circuit consisting of a single voltage source and in series with a single resistor.

- Phasor equivalents circuits of linear circuits are still linear circuits so Thevenin's theorem holds.
  - But we need to write Thevenin's theorem in its complex form

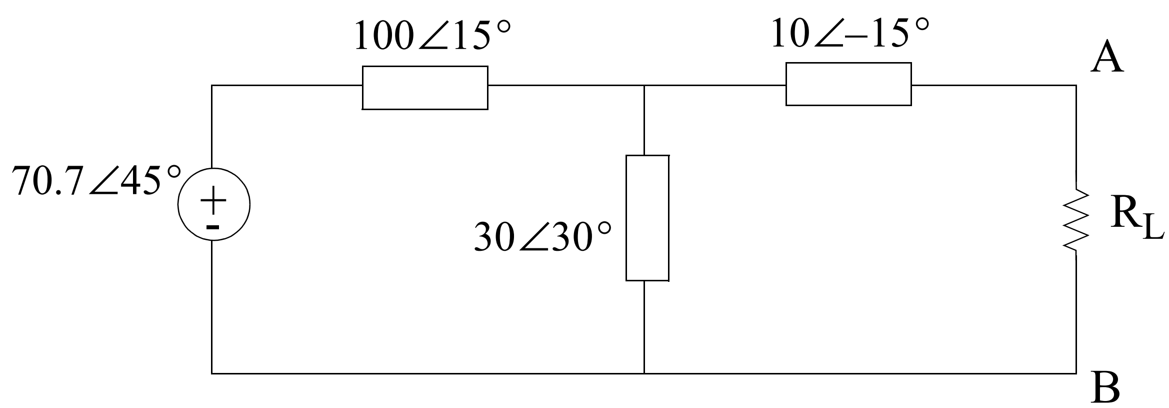
Any linear circuit can be represented at a given pair of nodes by an equivalent circuit consisting of a single voltage source in series with a single *impedance*.

- Where
  - $\boldsymbol{V}_T = \boldsymbol{V}_{oc}$ is the Thevenin voltage
  - $\boldsymbol{I}_{sc}$ is the short-circuit current
  - and $Z_T = \dfrac{\boldsymbol{V}_{oc}}{\boldsymbol{I}_{sc}}$ is the Thevenin impedance.

- So everything is the same as before except that we are expressing the voltages and currents in terms of phasors and we now have complex impedances (instead of just the resistances we had in Chapter 3).

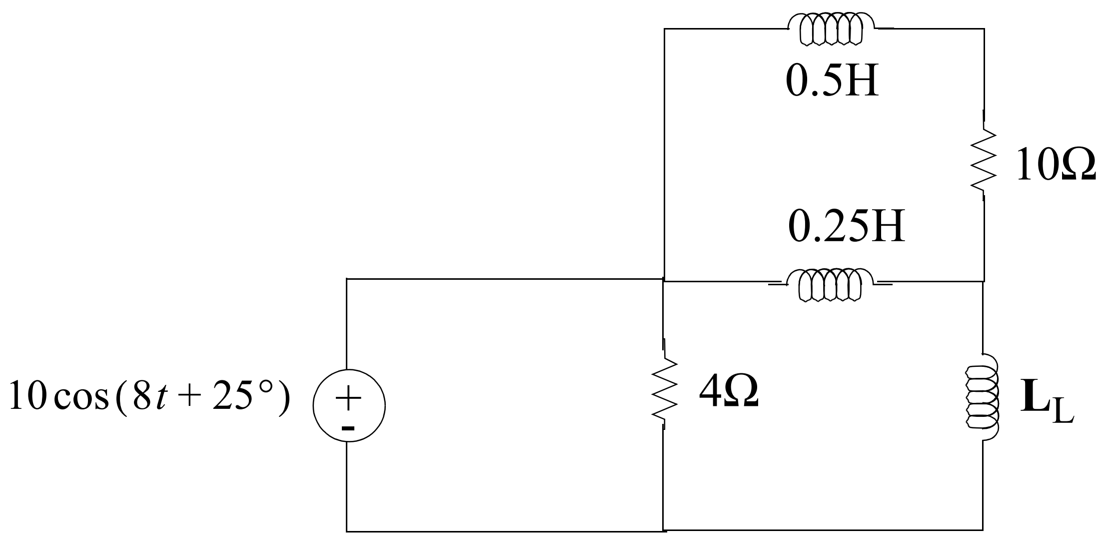
- The application of Thevenin's theorem has not changed.

Ex. 10.2 Given the circuit below find its Thevenin equivalent with respect to nodes A and B.



Solution:

Ex. 10.3 Given the following circuit find it Thevenin equivalent at the load inductor



Solution:

10.4 Norton's Theorem

- From Chapter 3 we had that Norton's theorem states

Any linear circuit can be represented at a given pair of nodes by an equivalent circuit consisting of a single current source in parallel with a single resistor.

- Phasor equivalents circuits of linear circuits are still linear circuits so Norton's theorem holds.
  - But we do need to write Norton's theorem in its complex form
  - The complex form of Norton's theorem states that

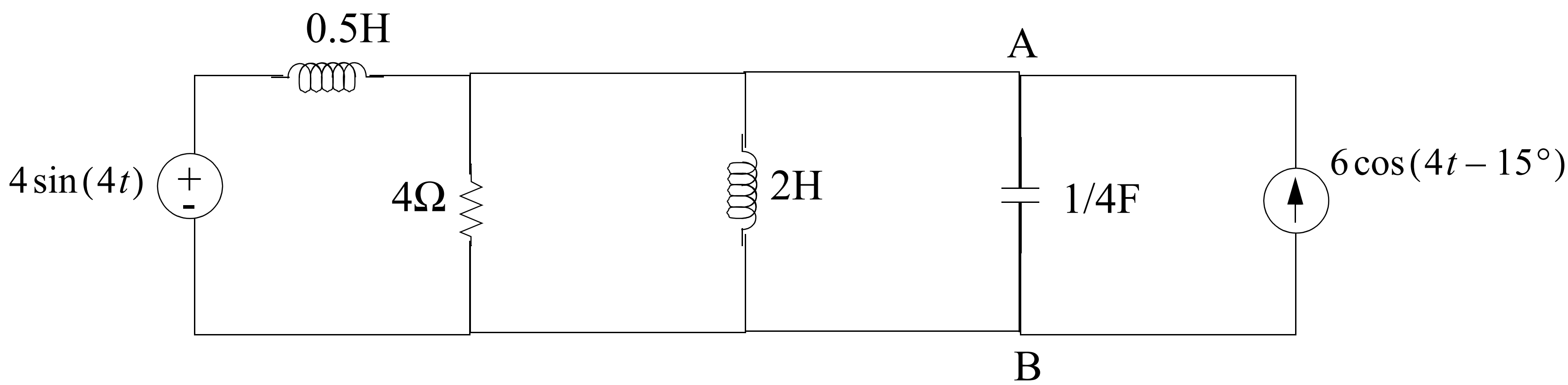
Any linear circuit can be represented at a given pair of nodes by an equivalent circuit consisting of a single current source in parallel with a single *impedance*.

- Where
  - $\boldsymbol{I}_N = \boldsymbol{I}_{sc}$ is the Norton current
  - $\boldsymbol{V}_{oc} = \boldsymbol{V}_T$ is the open-circuit voltage
  - and $Z_N = \frac{\boldsymbol{V}_{oc}}{\boldsymbol{I}_{sc}} = Z_T$ is the Norton impedance.

- So everything is the same as before except that we are expressing the voltages and currents in terms of phasors and we now have complex impedances (instead of just the resistances we had in Chapter 3).

- The application of Norton's theorem has not changed.

Ex. 10.4 Given the circuit below find its Norton equivalent with respect to nodes A and B.



Solution:

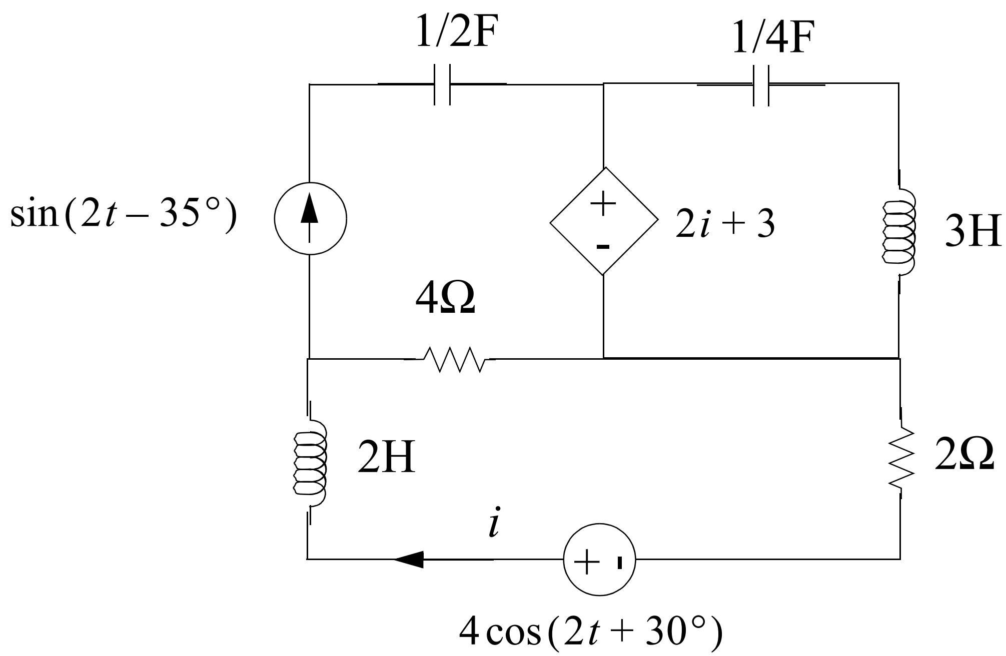
10.5 Nodal Analysis

- Nodal analysis is just a systematic way of applying KCL to a circuit, based on the assumption that the law of conservation of charge holds (i.e. charge cannot accumulate at a node, hence sum of currents entering a node must equal the sum of currents leaving the node)

- Converting a circuit to complex impedances and phasor notation for the power sources does not change this assumption that the law of conservation of charge holds.

- Therefore, nodal analysis applies to phasor circuits

Ex. 10.5 Find the node voltages for the following circuit



Solution:

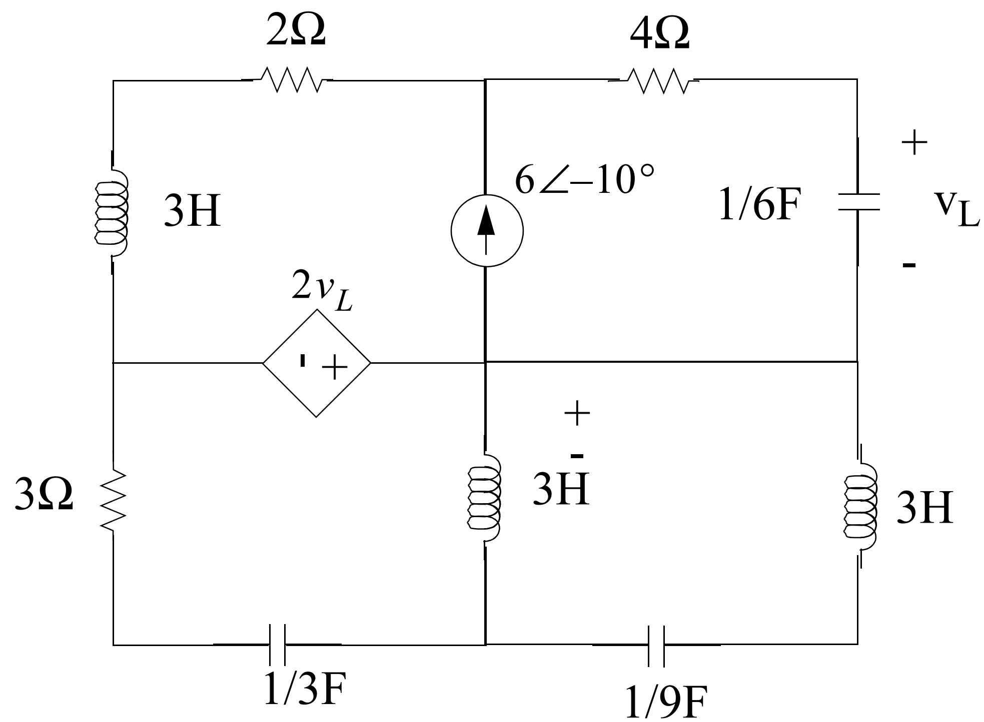
10.6 Mesh Analysis

- Mesh analysis is just a systematic way of applying KVL to a circuit, based on the assumption that the law of conservation of energy holds (i.e. the energy gained around a closed loop must be zero hence the energy gained around a loop must equal the energy lost around the same loop)

- Converting a circuit to complex impedances and phasor notation for the power sources does not change this assumption that the law of conservation of energy holds.

- Therefore, mesh analysis applies to phasor circuits

Ex. 10.6 Solve for the mesh currents in the following circuit and give the time domain voltage across the load capacitor assuming $\omega = 3$ rad/s.

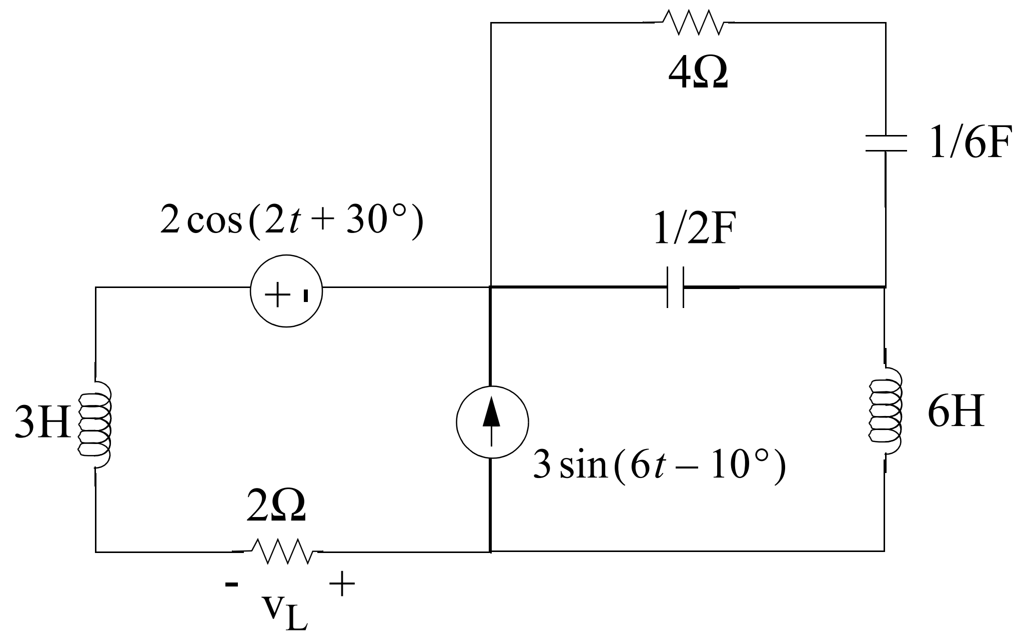


Solution:

10.7 Sources with Different Frequencies

- How do we solve when circuits contain power sources at different frequencies?
  - Remember that phasor notation assumes that the frequency ω is constant throughout the circuit
  - But we also have linear circuits so we can use superposition to separate out the power sources at different frequencies.
  - We can then use phasor analysis to solve each of these new circuits
  - Once we have the phasor solutions we can convert them back to time domain
  - We can now combine the time domain solution to get the overall solution for the original circuit.
  - If the power sources are at different frequencies then we **MUST** solve for their effects independently (via superposition) and then combine the time domain results to get the complete circuit solution.
  - This is because phasors are defined for only singular frequencies. Hence, phasor analysis can only deal with one of the frequencies (sources) at a time.

Ex. 10.7 Solve for the voltage across the load resistor in the following circuit.



Solution:

Assignment #10

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 11
AC Power
Sections 10.1 - 10.7 (Text)

11.1 Introduction

- In Chapter 1 we studied power consumption in the DC case.
- This chapter studies power consumption in the case of ac conditions (i.e. where the circuits are excited by sinusoidal power sources)
- The quantity of interest is the **average power** and how it can be obtained through knowing phasor voltages and currents in the circuit.

11.2 RMS Value

- Assume a periodic current passes through a resistor

$$i(t) = I_m \cos(\omega t + \phi)$$

  - We want to know the equivalent dc current (I_{rms}) that delivers the same amount of average power in the resistor
  - This equivalent dc current is called the **effective value** of the periodic current

- From calculus we know that the average value of a function over an interval $[a, b]$ is given by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

- Mathematically, we can write can therefore write the **average power** consumed by the resistor over **one period** of the sinusiod as

$$P = RI_{rms}^{\ 2} = \frac{1}{T}\int_0^T Ri^2(t)dt$$

- Thus the effective current I_{rms} is found from the equation

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}$$

- rms in the subscript is an abbreviation for “root mean square”

- An easy way to remember the formula for finding an effective value of a periodic function is from the *rms* abbreviation which implies we must do the following steps

    1. Find the square of the given function or waveform ("square")

    2. Find the average value of the squared waveform ("mean")

    3. Find the square root of the average value ("root")

- Hence,

$$RMS \equiv \text{root of the mean of the square}$$

- RMS values can be obtained for any periodic waveform

- In this course we mainly deal with sinusoidal waveforms so in this chapter we will focus on how to compute *rms* values for sinusoidal functions,

Ex. 11.1 Find the I_{rms} if the current waveform is

$$i = I_m \cos(\omega t + \phi)$$

- By substituting our function for i into the equation for I_{rms} given above we have that and performing the integration over one period of the sinusiod,

$$I_{rms} = \sqrt{\frac{1}{T}\int_{\phi}^{(T+\phi)} [I_m\cos(\omega t+\phi)]^2 dt}$$

- Since we are concerned with one complete period we can shift the current by ϕ and drop ϕ from the above equation,

$$I_{rms} = \sqrt{\frac{\omega I_m^2}{2\pi}\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t)dt} = \frac{I_m}{\sqrt{2}}$$

- Therefore the average power consumed by a resistor when a current $i = I_m\cos(\omega t+\phi)$ flows through it is given by

$$P = \frac{I_m^2}{2}R = I_{rms}^2 R$$

- Keep in mind that for ac circuits we are interested in the average power, P, not the instantaneous power, p.

- *rms* values give us the equivalent dc current which would produce the same power consumption (i.e. they give us the average power consumption over one period of the sinusoidal waveform).

Ex. 11.2 Find the rms value of the current waveform

$$i(t) = 10|\sin(\omega t)|$$

Solution:

- Note that the period is $T = \frac{\pi}{\omega}$ for the above signal

- apply the formula from above

$$I^2_{rms} = \frac{1}{T}\int_0^T (10|\sin(\omega t)|)^2 dt$$

$$I^2_{rms} = \frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} 100\sin^2(\omega t)dt = \frac{100\omega}{\pi}\left[-\frac{1}{2}\sin(\omega t)\cos(\omega t) + \frac{1}{2}\omega t\right]\frac{1}{\omega}\Bigg|_0^{\frac{\pi}{\omega}}$$

$$I^2_{rms} = \frac{100\omega}{\pi}\left(\frac{\pi}{2}\right)\left(\frac{1}{\omega}\right) = 50\text{ A}^2$$

- If our power source is expressed as a phasor then we can easily express its rms value in phasor notation as follows,

$$\boldsymbol{I} = I_m \angle \phi_i$$

$$\boldsymbol{I_{rms}} = \frac{I_m}{\sqrt{2}} \angle \phi_i$$

$$\boldsymbol{V} = V_m \angle \phi_v$$

$$\boldsymbol{V_{rms}} = \frac{V_m}{\sqrt{2}} \angle \phi_v$$

Ex. 11.3 What is the rms value of the sinusoidal voltage waveform

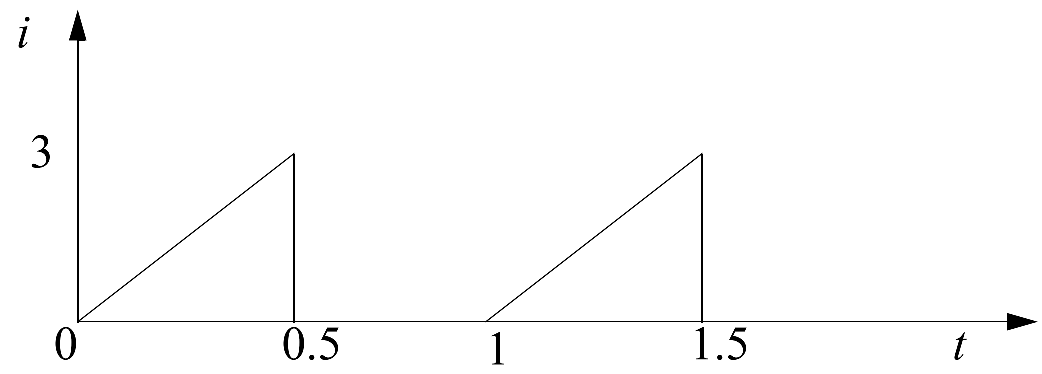
$$v(t) = 10\cos(100t + 60°)$$

Solution:

Ex. 11.4 Given an rms phasor voltage $\boldsymbol{V}_{rms} = 100e^{j30°}\,\mathrm{V}$, write its corresponding time-domain waveform if the angular frequency is $\omega = 100\ \mathrm{rad/s}$.

Solution:

Ex. 11.5 What is the rms value of the sawtooth waveform given below



Solution:

11.3 Average Power

- Consider an impedance in which the current through it is i and the voltage across it is v.
  - The instantaneous power consumed by the impedance is given by

$$p = vi$$

  - We are interested in the power consumed by the impedance when the voltage and current are sinusoidal.
  - Assume we have

$$v = V_m \cos(\omega t + \phi_v)$$

$$i = I_m \cos(\omega t + \phi_i)$$

  - From Chapter 7 we know that these correspond to the two phasors

$$\boldsymbol{V} = V_m e^{j\phi_v}$$

$$\boldsymbol{I} = I_m e^{j\phi_i}$$

  - The instantaneous power p in this case is given by

$$p = [V_m \cos(\omega t + \phi_v)][I_m \cos(\omega t + \phi_i)] = V_m I_m \cos(\omega t + \phi_v) \cos(\omega t + \phi_i)$$

- But we have the trigonometric identity that

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

- Thus we have

$$p = \frac{1}{2}V_m I_m[\cos([\omega t + \phi_v] + [\omega t + \phi_i]) + \cos([\omega t + \phi_v] - [\omega t + \phi_i])]$$

$$p = \frac{1}{2}V_m I_m[\cos(2\omega t + \phi_v + \phi_i) + \cos(\phi_v - \phi_i)].$$

- Therefore, the instantaneous power consists of two terms

  1. a DC term: $\frac{1}{2}V_m I_m \cos(\phi_v - \phi_i)$

  2. an AC term: $\frac{1}{2}V_m I_m \cos(2\omega t + \phi_v + \phi_i)$

- Since average power is defined over one period of the waveform then the AC term will be zero,

$$\frac{1}{T}\int_0^T \left[\frac{1}{2}V_m I_m \cos(2\omega t + \phi_v + \phi_i)\right]dt = 0$$

- So the average power is only a function of the DC component of the instantaneous power.

- The average power consumed by the element (over one period of the sinusiod) is therefore given by,

$$P = \frac{1}{2}V_m I_m \cos(\phi_v - \phi_i) = V_{rms} I_{rms} \cos(\phi)$$

- where

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

- are the rms values of the voltage and current magnitudes respectively. (Note that these are **NOT** the rms phasors, which would be denoted $\boldsymbol{V}_{rms}$ and $\boldsymbol{I}_{rms}$, but only the magnitudes of the respective phasors)

- and the phase angle ϕ is given by

$$\phi = \phi_v - \phi_i$$

- ϕ is the phase difference between the voltage and current sources.

- The rms voltage and current **phasors** can therefore be written as

$$\boldsymbol{V}_{rms} = V_{rms}e^{j\phi_v}$$

$$\boldsymbol{I}_{rms} = I_{rms}e^{j\phi_i}$$

Ex. 11.6 The current and voltage time-domain waveforms for an elements are $i(t) = 15\sin(100t + 80°)$ mA and $v(t) = 5\cos(100t + 60°)$ V. Calculate the average power consumption in this element.

Solution:

11.4 The Power Factor

- In the section above we found that the average power consumed by the impedance is given by

$$P = V_{rms}I_{rms}\cos(\phi_v - \phi_i) = V_{rms}I_{rms}\cos(\phi)$$

  - The term $\cos\phi = \cos(\phi_v - \phi_i)$ is called the **power factor** and is abbreviated **pf**.
  - For passive loads the phase angle difference $\phi_v - \phi_i$ is always $-90° \le (\phi_v - \phi_i) \le 90°$.
  - This assures that the real part of the power consumed is positive (i.e. that the element consuming the power is a passive element)
  - Hence, the power factor is **always positive** (for passive elements) and will be in the range of

$$0 \le pf \le 1$$

  - In **capacitive circuits** the current leads the voltage and we have a **leading power factor**, $\phi_v - \phi_i < 0$.
  - In **inductive circuits** the current lags the voltage and we have a **lagging power factor**, $\phi_v - \phi_i > 0$

Ex. 11.7 Assume three elements which have currents and voltages as follows.

Element	**RMS Phasor Voltage (V)**	**RMS Phasor Current (A)**
Element A	$15e^{15°}$	$5e^{15°}$
Element B	$10e^{45°}$	$2e^{105°}$
Element C	$2e^{20°}$	$15e^{-70°}$

Find the power factor in each element and the power consumed.

Solution:

11.5 Complex (Apparent) Power

- In Section 11.3 we derived an expression for the power consumed in an element as

$$P = V_{rms}I_{rms}\cos(\phi_v - \phi_i) = V_{rms}I_{rms}\cos(\phi)$$

  - where $\phi = \phi_v - \phi_i$
  - Note that the average power is a real number.

- We can write the average power as the real part of a complex number

$$P = V_{rms}I_{rms}Re[\cos\phi + j\sin\phi] = Re[\boldsymbol{S}]$$

  - where $\boldsymbol{S}$ is interpreted as the **complex power**, or **apparent power**.

$$S = V_{rms}I_{rms}\cos(\phi) + jV_{rms}I_{rms}\sin(\phi)$$

$$S = P + jQ$$

  - where P is the **real** or **average power** defined as

$$P = V_{rms}I_{rms}\cos\phi \text{ Watts}$$

  - and Q is the **imaginary** or **reactive power** defined as

$$Q = V_{rms}I_{rms}\sin\phi \text{ VAR}$$

- where Q is measured **volt-ampere reactive (VAR)**.

- The complex or apparent power S is a complex number which may be expressed in Euler's form

$$S = V_{rms} I_{rms} e^{j\phi}$$

  - where S is measured in volt-amperes (VA)

- For lagging power factor (inductive circuits), ϕ is positive and Q will be positive.

- For leading power factor (capacitive circuits), ϕ is negative and Q will be negative.

- Based on the above,

  - When we are given the rms voltage and current phasors

$$\boldsymbol{V}_{rms} = V_{rms} e^{j\phi_v}$$

$$\boldsymbol{I}_{rms} = I_{rms} e^{j\phi_i}$$

  - the apparent power is given by

$$\boldsymbol{S} = V_{rms} I_{rms} e^{j(\phi_v - \phi_i)}$$

- Note that this equation is an equation of the rms voltage and current magnitudes and their respective phase angles.

- It is NOT written in terms of the rms voltage and current phasors.

- To write the apparent power in terms of the rms voltage and current phasors we must write

$$S = \boldsymbol{V}_{rms}\boldsymbol{I}^*_{rms} = V_{rms}I_{rms}e^{j(\phi_v - \phi_i)}$$

- were $\boldsymbol{I}^*_{rms}$ is the **conjugate** of the rms current phasor $\boldsymbol{I}^*_{rms} = I_{rms}e^{-j\phi_i}$.

- The conjugation is required in order to get the difference in phase angles $\phi = \phi_v - \phi_i$.

- The above equation gives us a means of calculating the average power used by an element in phasor notation.

Ex. 11.8 Determine the power consumed by a resistor R, and inductor L, and a capacitor C when an rms voltage $V_{rms} = V\cos\omega t$ is applied across each element.

Solution:

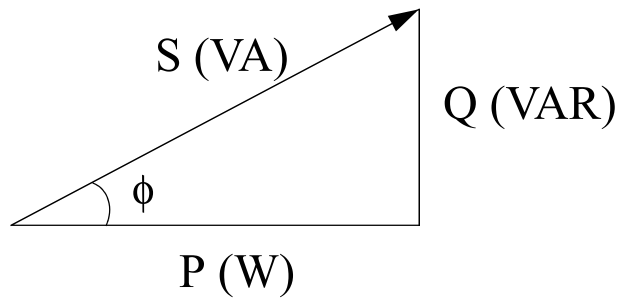
Ex. 11.9 Given a circuit with impedance $Z = 10 + j10\ \Omega$ and an applied rms voltage $\boldsymbol{V} = 20\angle 30°$, find the complex power consumed by the circuit.

Solution:

11.6 The Power Triangle

- The different components of the complex power $\boldsymbol{S}$ form what is known as the **power triangle**, which is the representation of the apparent power in the complex plane.

- The figure below shows the power triangle for an inductive circuit ($Q > 0$ so the circuit must be inductive)



Ex. 11.10 Assume an impedance of $Z = 3 + 4j\ \Omega$ and an applied rms voltage $V = 120\angle 30°$. Determine the power consumed by this impedance.

Solution:

Ex. 11.11 An impedance consumes $500\ \mathrm{W}$ at a power factor of 0.707 leading. The applied voltage $v = 170\sin(314t + 15°)$ V. Determine the component values of the power and the impedance.

Solution:

11.7 Impedance and Power

- Assume an impedance Z where a rms voltage $\boldsymbol{V}$ is applied and gives rise to a rms current $\boldsymbol{I}$ such that

$$\boldsymbol{V}_{rms} = Ve^{j\phi_v}$$

$$\boldsymbol{I}_{rms} = Ie^{j\phi_i}$$

- We can then write

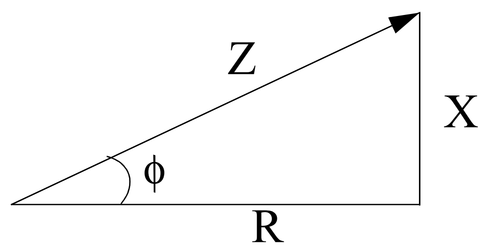
$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}}$$

$$\boldsymbol{Z} = \frac{\sqrt{2}\boldsymbol{V}_{rms}}{\sqrt{2}\boldsymbol{I}_{rms}} = \frac{\boldsymbol{V}_{rms}}{\boldsymbol{I}_{rms}} = \frac{V}{I}e^{j(\phi_v - \phi_i)} = \frac{V}{I}e^{j\phi}$$

- The figure below shows the representation of $\boldsymbol{Z}$ in the complex plane where

$$R = |\boldsymbol{Z}|\cos\phi = Z\cos\phi$$

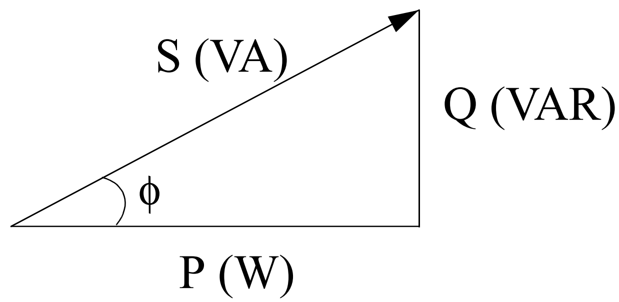
$$X = |\boldsymbol{Z}|\sin\phi = Z\sin\phi$$



- The power consumed by this impedance is given by

$$\boldsymbol{S} = \boldsymbol{V}\boldsymbol{I}^* = V_{rms}I_{rms}e^{j\phi}$$

- The power triangle for this impedance can be drawn in the complex plane as



- where

$$P = |\boldsymbol{S}|\cos\phi = S\cos\phi$$

$$Q = |\boldsymbol{S}|\sin\phi = S\sin\phi$$

- Comparing the impedance triangle and the power triangle we can see that they are very similar.

- For an inductive impedance X is positive and Q is positive.

- For a capacitive impedance X is negative and Q is negative.

- Using this realization, we can now express the power consumed by an impedance using the following relations

$$\boldsymbol{S} = \boldsymbol{V}_{rms}\boldsymbol{I}_{rms}^{\ *} = \frac{|\boldsymbol{V}_{rms}|^2}{\boldsymbol{Z}^*} = |\boldsymbol{I}_{rms}|^2\boldsymbol{Z}$$

$$P = Re[\boldsymbol{S}] = |\boldsymbol{V}_{rms}||\boldsymbol{I}_{rms}^{\ *}|\cos\phi = \frac{|\boldsymbol{V}_{rms}|^2}{|\boldsymbol{Z}|^2}R = |\boldsymbol{I}_{rms}|^2R$$

$$Q = Im[\boldsymbol{S}] = |\boldsymbol{V}_{rms}||\boldsymbol{I}_{rms}^{\ *}|\sin\phi = \frac{|\boldsymbol{V}_{rms}|^2}{|\boldsymbol{Z}|^2}X = |\boldsymbol{I}_{rms}|^2X$$

  - The above equations need not be memorized since they can always be quickly derived from

$$\boldsymbol{V} = \boldsymbol{I}\boldsymbol{Z}$$

$$\boldsymbol{S} = \boldsymbol{V}_{rms}\boldsymbol{I}_{rms}^{\ *}$$

Ex. 11.12 A circuit has an applied voltage $v(t) = 100\sin(20t + 15°)$ and a current $i(t) = 2\sin(20t - 20°)$. Determine the impedance and power triangle.

Solution:

Ex. 11.13 A voltage of $v(t) = 99\cos(6000t + 30°)$ is applied across an impedance $\boldsymbol{Z}$ such that the power consumed in $P = 940\ \text{W}$ with a power factor of $pf = 0.707$ leading. Find the value of $\boldsymbol{Z}$ and the value of it elements.

Solution:

11.8 Power Factor Correction

- It is desirable in industrial application to supply power to a load with a unity power factor.
  - In this case, the load appears to the power utility as a simple resistor.
  - Therefore, the current supplied to the load will be reduced and the ohmic losses in the transmission lines will also be reduced.

- Assume an rms voltage $\boldsymbol{V}_{rms}$ is applied to a pure restive load (unity pf) and produces some rms current I_1 in that load.
  - The power supplied to the load is given by

$$P = V_{rms}I_1\cos(\phi_v - \phi_i) = V_{rms}I_1(1) = V_{rms}I_1$$

- Assume the same rms voltage $\boldsymbol{V}_{rms}$ is applied to a load without a unity pf, which produces some current I_2 in the load.
  - The power supplied to the load in this case is given by

$$P = V_{rms}I_2\cos(\phi_v - \phi_i)$$

- Comparing the two cases when the power consumed is equal we have that

$$V_{rms}I_1 = V_{rms}I_2\cos(\phi_v - \phi_i)$$

$$I_2 = \frac{I_1}{\cos(\phi_v - \phi_i)} > I_1$$

  - More current is drawn by the non-unity pf load for equal power consumption.
  - This is the reason why power utilities penalize industrial customers that have loads with non-unity power factors.

- How do we correct for non-unity power factors?
  - Industry usually has inductive loads which produce lagging power factors
  - A lagging power factor can be made a unity power factor by placing a capacitor in parallel with the load, where the value of the capacities is such that the load becomes purely resistive (i.e. $Im[Z]$ is made equal to 0).
  - A leading power factor can be made purely resistive by placing an inductor in parallel with the load.

Ex. 11.14 An industrial load $\boldsymbol{Z} = 5\angle 10°\ \Omega$ is supplied by an rms voltage $\boldsymbol{V} = 100\angle 30°$ at 60 Hz. Find the capacitor value required to change the power factor to a unity pf and find the current supplied before and after the power factor correction.

Solution:

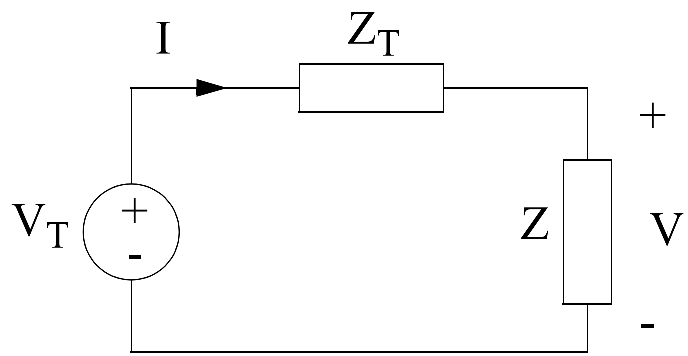
Ex. 11.15 The power consumed in an impedance is 100 W at a power factor of 0.8 lagging. If the applied voltage is $v(t) = 25\sin(300t + 45°)$ V, determine the current flowing in the impedance and the value of the impedance.

Solution:

11.9 Maximum Power Transfer

- In Chapter 3, we studied maximum power transfer for purely resistive circuits operating in DC conditions, we would like to perform a similar analysis for impedance circuits operating in AC conditions.

- Consider the following Thevenin equivalent circuit (remember we can replace any linear circuit about its load with it Thevenin equivalent circuit about the load).



- Assume that we can vary the load impedance Z to get the maximum power out of the circuit.

- Applying KVL to the circuit we get

$$V_T = I(Z_T + Z)$$

- Thus,

$$I = \frac{V_T}{(Z_T + Z)}$$

- Applying voltage division across the load

$$V = \frac{Z_T}{Z_T + Z} V_T$$

- From the above two equations the power delivered to the load is

$$S = VI^* = \frac{|V_T|^2 Z}{|Z_T + Z|^2}$$

- The real power consumed by the load is

$$P = \frac{|V_T|^2 R}{|Z_T + Z|^2}$$

- But the source and load impedances can be written as

$$Z_T = R_T + jX_T$$

$$Z = R + jX$$

- Therefore the real power can be written as

$$P = |V_T^2| \frac{R}{(R_T + R)^2 + (X + X_T)^2}$$

- The value of X which maximizes P is

$$X = -X_T$$

- since this causes the $(X + X_T)^2$ term in the denominator to become 0.

- and the real power becomes

$$P = |V_T|^2 \frac{R}{(R_T + R)^2}$$

- The maximum value for P is found by differentiating the above equation with respect to R

$$\frac{dP}{dR} = |V_T|^2 \frac{R_T - R}{(R_T + R)^3}$$

- The maximum occurs when $\frac{dP}{dR} = 0$ which occurs when

$$R = R_T$$

- We now have the real and imaginary parts of Z required for the maximum power transfer.

- The maximum power transfer to the load Z in a Thevenin equivalent circuit is that the load impedance must be the complex conjugate of the Thevenin impedance.

$$Z = Z_T^* = R_T - jX_T$$

- At this value the output voltage and current will be

$$V' = V_T \frac{Z_T^*}{2R_T}$$

$$I' = \frac{V_T}{2R_T}$$

- The maximum complex power delivered to the load will be

$$S' = V'I'^* = \frac{|V_T|^2}{4R_T^2} Z_T^*$$

- And the maximum power available from the circuit is

$$P' = \frac{|V_T|^2}{4R_T}$$

- In terms of the Norton equivalent circuit I_N and Z_T, the load current, voltage, and maximum power are given by

$$V' = \frac{I_N |Z_T|^2}{2R_T}$$

$$I' = \frac{I_N Z_T}{2R_T}$$

$$P' = \frac{|I_N|^2 |Z_T|^2}{4R_T}$$

Ex. 11.16 A voltage source of $V = 10\text{ V rms}$ has an internal impedance of $Z = 3\angle 30°\ \Omega$ and drives a serially connected load impedance Z. What is the value of Z which will maximize the power transfer and what is the value of that power?

Solution:

Assignment #11

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

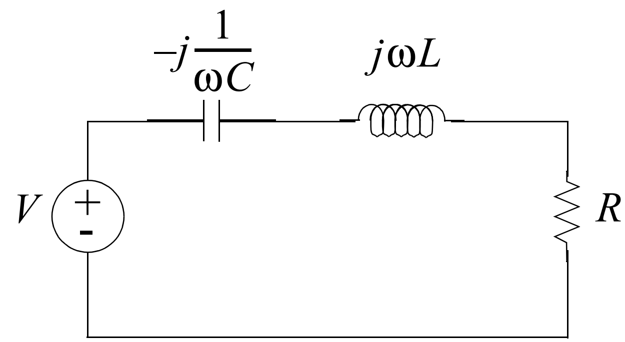
Chapter 12
Series and Parallel Resonance

12.1 Introduction

- Resonance circuits are circuits which behave much differently for a narrow range of frequencies than they do across the broad range of all frequencies.
  - The text derives resonance theorem via Laplace transform analysis and Bode plots.
  - We will not cover those methods in class (but you will cover them in 3rd year)
- Resonance circuits are useful in many applications such as the extraction of signals from background noise.
  - Of importance in resonance circuits is how selective the circuit is at filtering out noise which is very close to the circuits resonant frequency.
  - A circuit is in resonance when the applied voltage $\boldsymbol{V}$ and the resulting current $\boldsymbol{I}$ are in phase.
  - Thus, at **resonance** the equivalent impedance is **purely resistive**.

12.2 Series Resonance

- The following shows a series RLC circuit.



- Assume that the input phasor voltage is

$$\boldsymbol{V} = V\angle 0°$$

- The impedance seen by the source is described by

$$\boldsymbol{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

- which consists of a constant real part R and an imaginary part $\left(\omega L - \frac{1}{\omega C}\right)$ which varies with the frequency ω

- At a particular frequency, denoted ω_0, the imaginary part vanishes completely

$$\omega_0 L - \frac{1}{\omega_0 C} = 0$$

- Thus,

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ rad/s}$$

- This frequency ω_0 is called the **resonance frequency** of the series RLC circuit.

- The resonance frequency in Hertz is given by

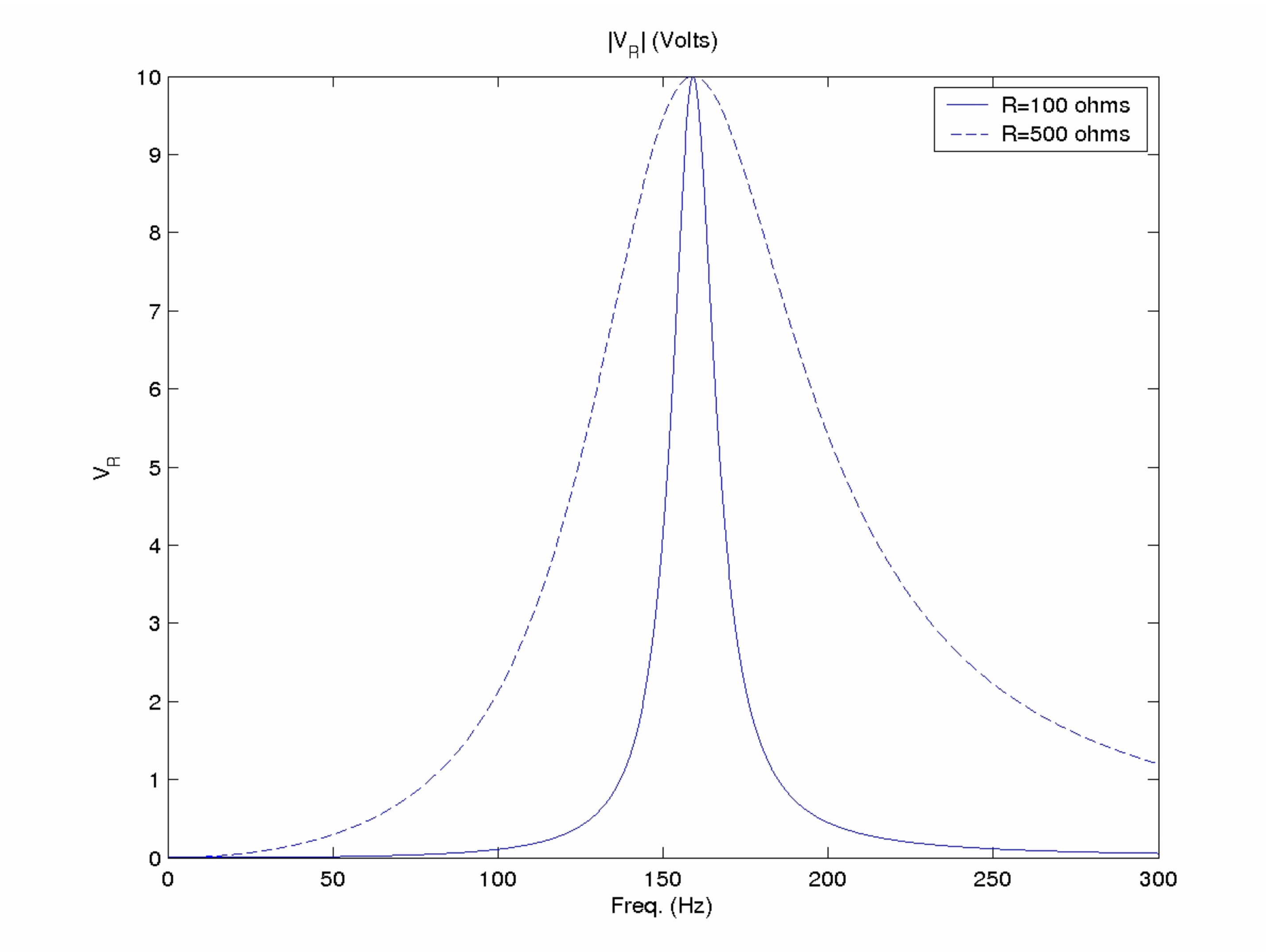
$$\omega = 2\pi f$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \text{ Hz}$$

- In a series resonance circuit, the voltage that develops across the load resistor is usually the desired output voltage, and it is given by

$$\boldsymbol{V}_R = \boldsymbol{V} \times \frac{R}{\boldsymbol{Z}} = \frac{\boldsymbol{V}R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

- The figure below show the dependency of the magnitude of this output voltage on the operating frequency for the parameters $V = 10\angle 0°$, $L = 1$ H, and $C = 1\mu F$, and two values of R: 100 and 500 Ω.



- The resonance frequency is given by the equation $f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$ Hz and does not depend on R. For this parameters used above we have that

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\ \text{Hz} = \frac{1}{2\pi}\sqrt{\frac{1}{(1)(1e^{-6})}} = 159\ \text{Hz}$$

- Notice from the figure that the output voltage is maximum at the resonance frequency.

- At resonance ($f_o = \text{Hz}159$) we can write,

$$\boldsymbol{Z} = R + j(0)$$

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{R}$$

$$\boldsymbol{V}_R = \boldsymbol{V}$$

- At resonance, the resistor current is in phase with the input voltage since the circuit appears to the source as a purely resistive circuit.

- The resistor voltage will exactly equal the source voltage since at resonance the effects of the capacitor and inductor cancel each other out.

12.2.1 Series Resonance Bandwidth and Quality Factor

- We saw in the figure above that as the resistance is changed in the series RLC circuit the sharpness of the output voltage spike changes.
  - The sharpness of the resonant peak of the output voltage versus frequency is a useful measure of the frequency selectivity of the series RLC circuit.
  - We define the **bandwidth** B of the resonant circuit by the equation

$$B = \omega_2 - \omega_1$$

  - where ω_1 and ω_2 are defined at the angular frequencies where the magnitude of the output voltage (voltage across the resistor) has the value

$$|\boldsymbol{V}_{1,2}| = \frac{|\boldsymbol{V}|}{\sqrt{2}}$$

  - In filtering this is termed the point were the signal is “3dB down” since

$$20\log_{10}(|\boldsymbol{V}_{1,2}|) = 20\log_{10}\left(\frac{|\boldsymbol{V}|}{\sqrt{2}}\right) = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) + 20\log_{10}(|\boldsymbol{V}|) = -\boldsymbol{3dB} + 20\log_{10}(|\boldsymbol{V}|)$$

- Given the above definitions for ω_1 and ω_2 we can derive their respective equations as follows,

  - Using voltage division we can write

$$\frac{|\boldsymbol{V}|}{\sqrt{2}} = \frac{R}{|\boldsymbol{Z}_{1,2}|} \times |\boldsymbol{V}|$$

  - Solving for the complex impedance $\boldsymbol{Z}_{1,2}$

$$|\boldsymbol{Z}_{1,2}| = \sqrt{R^2 + X^2} = \sqrt{2}R$$

  - Which means that $|X| = R$ when $\omega = \omega_1$ or ω_2

  - At the lower frequency ω_1, we have that $X_C > X_L$ and we can write

$$\frac{1}{\omega_1 C} - \omega_1 L = R$$

  - At the higher frequency ω_2, we have that $X_L > X_C$ and we can write

$$\omega_2 L - \frac{1}{\omega_2 C} = R$$

- Solving the above two equations for ω_1 and ω_2 respectively we have that

$$\omega_1 = \frac{1}{2L}\left(\sqrt{R^2 + 4\frac{L}{C}} - R\right)$$

$$\omega_2 = \frac{1}{2L}\left(\sqrt{R^2 + 4\frac{L}{C}} + R\right)$$

- From before we had that the bandwidth is defined as $B = \omega_2 - \omega_1$ therefore for a series RLC circuit the bandwidth is given by

$$B = \frac{1}{2L}\left(\sqrt{R^2 + 4\frac{L}{C}} + R\right) - \frac{1}{2L}\left(\sqrt{R^2 + 4\frac{L}{C}} - R\right) = \frac{R}{L} \text{ rad/s}$$

Ex. 12.1 Design a series RLC circuit such that its resonance frequency is 60 Hz, the resonant bandwidth is 10 Hz, and R = 25 Ω.

Solution:

Ex. 12.2 Design a series RLC resonant circuit such that its resonance frequency is 5kHz and the resonant bandwidth is 100 Hz.

Solution:

12.2.2 The Quality Factor

- An important parameter related to the resonance phenomena the quality factor Q which is defined as the ration of the resonant frequency ω_0 to the bandwidth B:

$$Q = \frac{\omega_0}{B}$$

- For series RLC circuits we know from above that $B = \frac{R}{L}$ therefore the quality factor can be written as

$$Q = \frac{\omega_0}{B} = \frac{\omega_0 L}{R}$$

- We also know from above that $\omega_0 = \sqrt{\frac{1}{LC}}$ rad/s therefore

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

- The quality factor is a measure of the sharpness of the resonance curve.

- Highly selective filters have a very sharp resonance curve peak, showing that they only select frequencies which are very close to the resonance frequency.

- We can see from the equation above that the smaller R is the higher the quality factor.

- Higher values for Q, than are available through the passive circuit elements we study in this class, can be obtained by using active circuits which contain operational amplifiers or through using digital signal processing techniques.

- The highest quality factor yet achieved was by Mossbauer during his Ph.D. work in Germany in the mid-sixties through processes related to the emission of gamma radiation by radioactive nuclei.

- In general, the quality factor is defined by the ratio

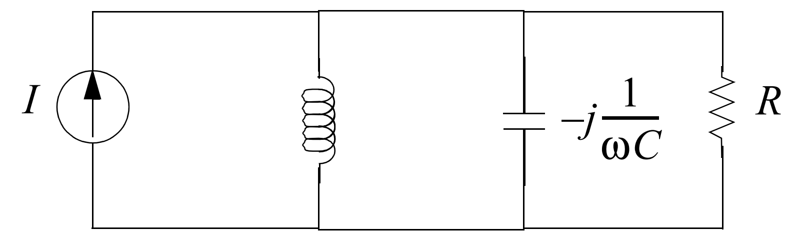
$$Q = 2\pi\frac{\text{Energy stored in the system}}{\text{Energy dissipated per cycle}}$$

Ex. 12.3 Design a series RLC resonant circuit such that its resonant frequency is 500kHz and its quality factor is 100,

Solution:

12.3 Parallel Resonance

- The following shows a parallel RLC circuit.



- Assume that the input phasor current is

$$\boldsymbol{I} = I\angle 0°$$

- The admittance seen by the source is described by

$$Y = G + j\left(\omega C - \frac{1}{\omega L}\right)$$

- which consists of a fixed real part G and an imaginary part $\left(\omega C - \frac{1}{\omega L}\right)$ which varies with the frequency ω

- At a particular frequency, denoted ω_0, the imaginary part vanishes completely

$$\omega_0 C - \frac{1}{\omega_0 L} = 0$$

- Thus,

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ rad/s}$$

- This frequency ω_0 is called the **resonance frequency** of the parallel RLC circuit.

- The resonance frequency in Hertz is given by

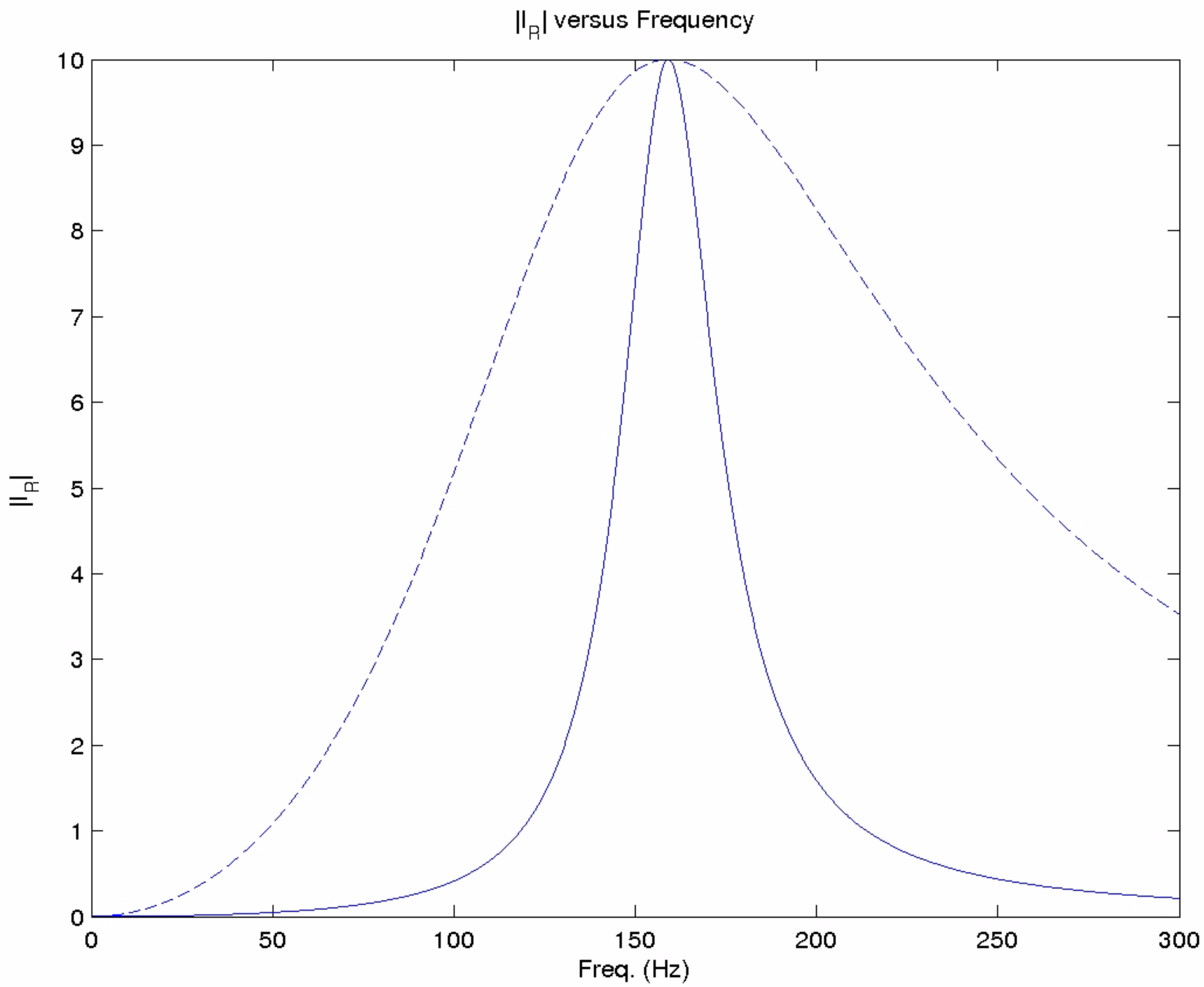
$$\omega = 2\pi f$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \text{ Hz}$$

- In a series resonance circuit, the voltage that develops across the load resistor is usually the desired output voltage, and it is given by

$$\boldsymbol{V}_R = \frac{\boldsymbol{I}}{Y} = \frac{\boldsymbol{I}}{G + j\left(\omega C - \frac{1}{\omega L}\right)}$$

- The figure below show the dependency of the magnitude of the output current on the operating frequency for the parameters $I = 10\angle 0°$, $L = 1$ H, and $C = 1\mu F$, and two values of R: 1kΩ and 5kΩ.



- The resonance frequency is given by the equation $f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$ Hz and does not depend on R.

- For the parameters used above we have that

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\text{ Hz} = \frac{1}{2\pi}\sqrt{\frac{1}{(1)(1e^{-6})}} = 159\text{ Hz}$$

- Notice from the figure that the output current is maximum at the resonance frequency.

- At resonance ($f_o = \text{Hz}159$) we can write,

$$Y = G + j(0)$$

$$\boldsymbol{V} = \boldsymbol{I}R$$

$$\boldsymbol{I}_R = \boldsymbol{I}$$

- At resonance, the voltage across the circuit is in phase with the input current since the circuit appears to the source as a purely resistive circuit.

- The resistor current will exactly equal the source current since at resonance the effects of the capacitor and inductor cancel each other out.

12.3.3 Parallel Resonance Bandwidth and Quality Factor

- We saw in the figure above that as the resistance is changed in the parallel RLC circuit the sharpness of the output voltage spike changes.

  - The sharpness of the resonant peak of the output voltage versus frequency is a useful measure of the frequency selectivity of the series RLC circuit.

  - As in the series case the **bandwidth** B of the resonant circuit by the equation

$$B = \omega_2 - \omega_1$$

  - where ω_1 and ω_2 are defined at the angular frequencies where the magnitude of the output voltage (voltage across the resistor) has the value

$$|\boldsymbol{I}_{1,2}| = \frac{|\boldsymbol{I}|}{\sqrt{2}}$$

  - For the parallel RLC circuit ω_1 and ω_2 we can be derived as follows,

    - Using current division we can write

$$\frac{|\boldsymbol{I}|}{\sqrt{2}} = \frac{G}{|\boldsymbol{Y}_{1,2}|} \times |\boldsymbol{I}|$$

- Solving for the complex admittance $Y_{1,2}$

$$|Y_{1,2}| = \sqrt{G^2 + B^2} = \sqrt{2}G$$

- Which means that $|B| = G$ when $\omega = \omega_1$ or ω_2

- At the lower frequency ω_1, we have that $B_L > B_C$ and we can write

$$\frac{1}{\omega_1 L} - \omega_1 C = G$$

- At the higher frequency ω_2, we have that $B_C > B_L$ and we can write

$$\omega_2 C - \frac{1}{\omega_2 L} = G$$

- Solving the above two equations for ω_1 and ω_2 respectively we have that

$$\omega_1 = \frac{1}{2C}\left(\sqrt{G^2 + 4\frac{C}{L}} - G\right)$$

$$\omega_1 = \frac{1}{2C}\left(\sqrt{G^2 + 4\frac{C}{L}} + G\right)$$

- From before we had that the bandwidth is defined as $B = \omega_2 - \omega_1$ therefore for a parallel RLC circuit the bandwidth is given by

$$B = \frac{G}{C} \text{ rad/s}$$

Ex. 12.4 Design a parallel RLC circuit such that its resonance frequency is 60 Hz, the resonant bandwidth is 10 Hz, and R = 25 Ω.

Solution:

Ex. 12.5 Design a parallel RLC resonant circuit such that its resonance frequency is 500kHz and the resonant bandwidth is 100 Hz.

Solution:

12.3.4 The Quality Factor

- As in the series case we have that the quality factor Q is defined as the ration of the resonant frequency ω_0 to the bandwidth B:

$$Q = \frac{\omega_0}{B}$$

- For parallel RLC circuits we know from above that $B = \frac{G}{C}$ therefore the quality factor can be written as

$$Q = \frac{\omega_0}{B} = \frac{\omega_0 C}{G} = \omega_0 CR$$

- We also know from above that $\omega_0 = \sqrt{\frac{1}{LC}}$ rad/s therefore

$$Q = R\sqrt{\frac{C}{L}}$$

Ex. 12.6 Design a parallel RLC resonant circuit such that its resonant frequency is 500kHz and its quality factor is 100,

Solution:

Table 16: Table summarizing the formulae for series and parallel RLC resonant circuits

Parameter	Series RLC	Parallel RLC
ω_0 (rad/s)	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
B (rad/s)	$\frac{R}{L}$	$\frac{1}{RC}$
Q	$\frac{1}{R}\sqrt{\frac{L}{C}}$	$R\sqrt{\frac{C}{L}}$

Assignment #12

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

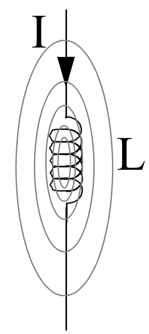
Chapter 13
Mutual Inductance

13.1 Introduction

- We are all familiar with the ac-to-dc converters that power out printers, scanners, computers, stereo's, etc.
  - These devices require a transformer for their operation.
  - Transformers are also used in the transmission of electrical power across the power grid.
  - Electromagnetic coupling is the basic phenomena responsible for transformer operation.
- In this chapter, we study the phenomena of magnetic coupling through mutual inductance, and then we discuss the principle of operation of an ideal transformer.
  - Mesh (loop) analysis is the primary circuit theorem which we will use to solve for circuits containing mutual inductance.
  - The nature of such circuits is not well suited to nodal analysis.

13.2 Self Inductance

- Consider the inductor shown below, when the current I passes through the inductor magnetic lines of force are produced.



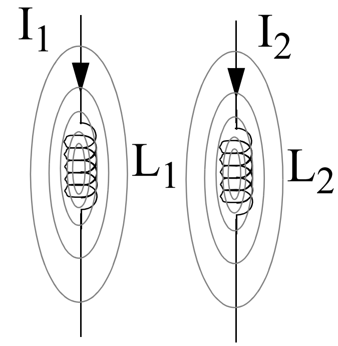
  - This is what gives the self inductance L and we get the usual phasor equation

$$\boldsymbol{V} = j\omega L\boldsymbol{I}$$

  - where ω is the operating angular frequency of the current phasor.

13.3 Mutual Inductance

- Assume that we have two inductors in close proximity as shown below



- When a sinusiods current I_1 passes through the first inductor, a magnetic field is created around it, and part of this magnetic field cuts the coils of the second inductor.
- The voltage on inductor L_1 is given by the same phasor equation as before

$$\boldsymbol{V}_1 = j\omega L_1 \boldsymbol{I}_1$$

- But, the same current I_1 gives rise to a voltage on the second inductor, denoted $\boldsymbol{V}_{21}$, given by

$$\boldsymbol{V}_{21} = \pm j\omega M \boldsymbol{I}_1$$

- Where M is the mutual inductance between the two inductors.

- The $\pm$ sign arises because the direction of the induced voltage might be opposite to the assumed reference voltage.

- M is measured in Henries just like the self inductance L.

- In general, when currents flow in two inductors at the same time, the total induced voltages in the two inductors are given by

$$\boldsymbol{V}_1 = j\omega L_1\boldsymbol{I}_1 \pm j\omega M\boldsymbol{I}_2$$

$$\boldsymbol{V}_2 = j\omega L_2\boldsymbol{I}_2 \pm j\omega M\boldsymbol{I}_1$$

- Each inductor will have two voltage components:

    1. A component due to the self inductance.

    2. A component due to the mutual inductance

- Note that the mutual inductance coefficient $j\omega M$ is the same for both voltage equations.

Ex. 13.1 Assume two coupled inductors $L_1 = 1H$ and $L_2 = 4H$ with mutual coupling $M = 0.1H$ and an operating frequency $f = 100Hz$. What are the voltages developed across the inductors when: (a) $\boldsymbol{I_1} = 2\angle 0°\text{ A}$ and $\boldsymbol{I_2} = 3\angle 30°\text{ A}$ (b)$\boldsymbol{I_1} = 5\angle 60°\text{ A}$ and the second inductor is open-circuited. Assume positive coupling between the inductors

Solution:

13.4 Coupling Coefficient

- The coupling coefficient k is a measure of how much the magnetic flux of one inductor threads the coils of the other inductor.

- k is defined by

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

- The coupling coefficient is a dimensionless constant and is bounded between

$$0 \leq k \leq 1$$

- When $k = 0$, we say that the two coils (inductors) are not coupled
- When $k \ll 1$, we say that the two coils are weakly coupled.
- When $k \approx 1$, we say that the two coils are strongly coupled.
- When $k = 1$, we say that we have an ideal transformer (Section 13.7).

Ex. 13.2 Determine the coupling coefficient for two coils $L_1 = 1H$ and $L_2 = 5H$ whose mutual inductance is $M = 2H$.

Solution:

13.5 The Dot Convention

- Above we stated that when two inductors are coupled, the induced voltages are given by the equations,

$$\boldsymbol{V}_1 = j\omega L_1 \boldsymbol{I}_1 \pm j\omega M \boldsymbol{I}_2$$

$$\boldsymbol{V}_2 = j\omega L_2 \boldsymbol{I}_2 \pm j\omega M \boldsymbol{I}_1$$

- To know which sign to give the induced voltage due to the mutual coupling, we use a **dot convention** in accordance with the following rules,

1. **When both the currents in the two inductors enter the dotted terminals, the signs of the M terms (mutual inductance) are the same as the signs of the L terms (self inductance)**

2. **When both current in the two inductors leave the dotted terminal. the signs of the M terms are the same sign as the L terms.**

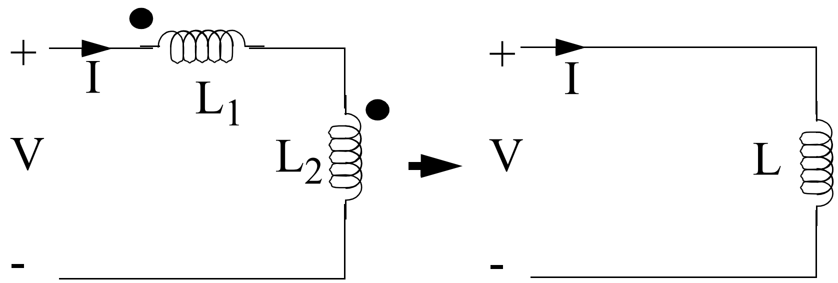
3. **When one current enters the dotted terminal while the other leaves the dotted terminal, the signs of the M terms are opposite to the signs of the L terms.**

- There is an alternate way to phrase the dot convention which proves useful in some situation:

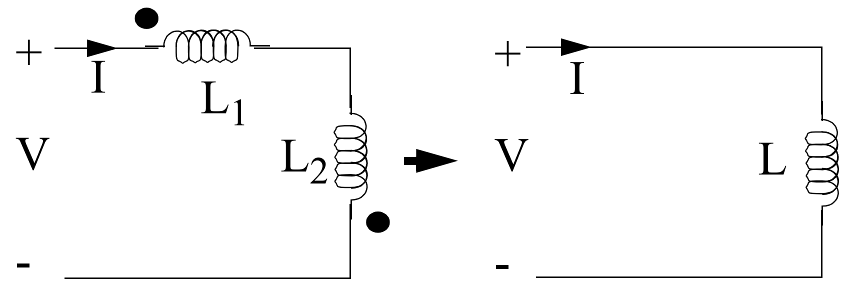
    1. When a current enters the dotted terminal of an inductor, it induces a positive voltage on the dotted terminal of the coupled inductor.

    2. When a current enters the undotted terminal of an inductor, it induces a positive voltage on the undotted terminal of the coupled inductor.

Ex. 13.3 Find the equivalent inductances for the two circuits shown below.



(a)

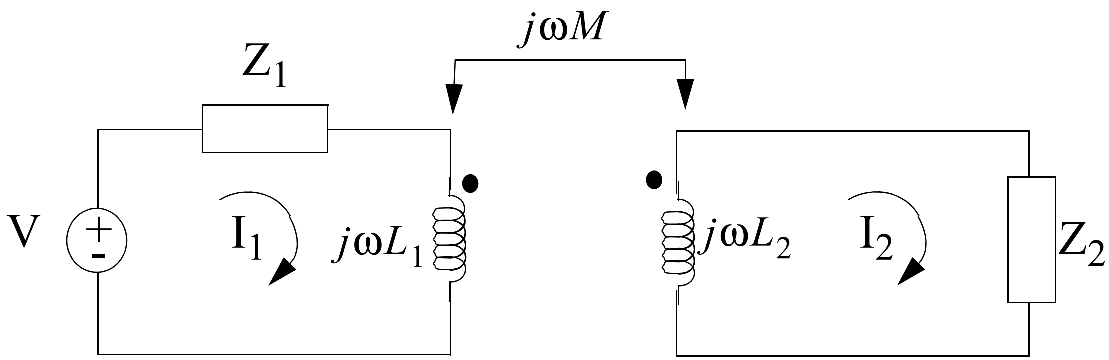


(b)

Solution:

13.6 Coupled Circuits

- Consider the circuit shown below (where the double headed arrow denotes coupled inductors).



- Taking the dot convention into account (I_1 enters a dotted terminal and I_2 enters a non-dotted terminal so the mutual inductances will have the opposite sign of the self inductances), the two loop equations are,

$$V - I_1(Z_1 + j\omega L_1) + j\omega M I_2 = 0$$

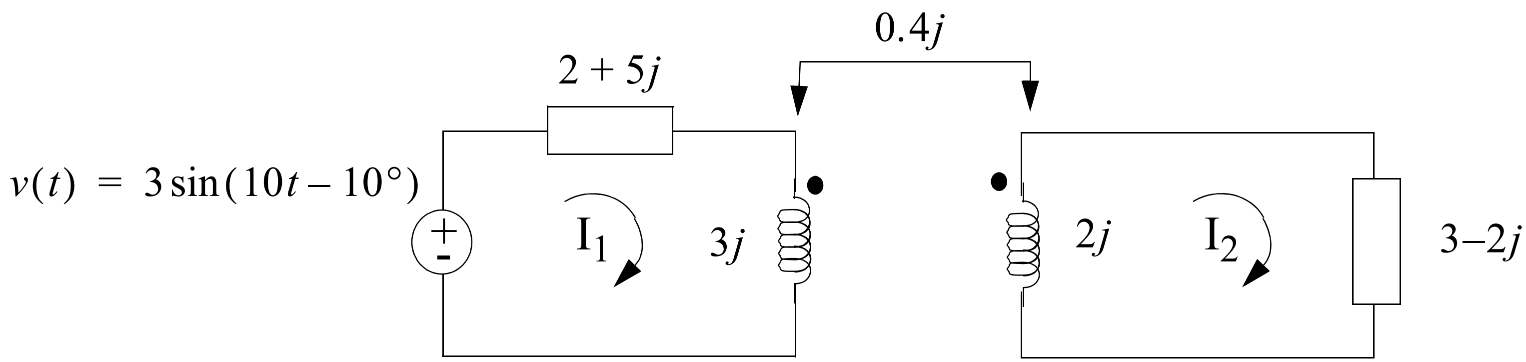
$$-I_2(Z_2 + j\omega L_2) + j\omega M I_1 = 0$$

- Placing in matrix form we have,

$$\begin{bmatrix} Z_1 + j\omega L_1 & -j\omega M \\ -j\omega M & Z_2 + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

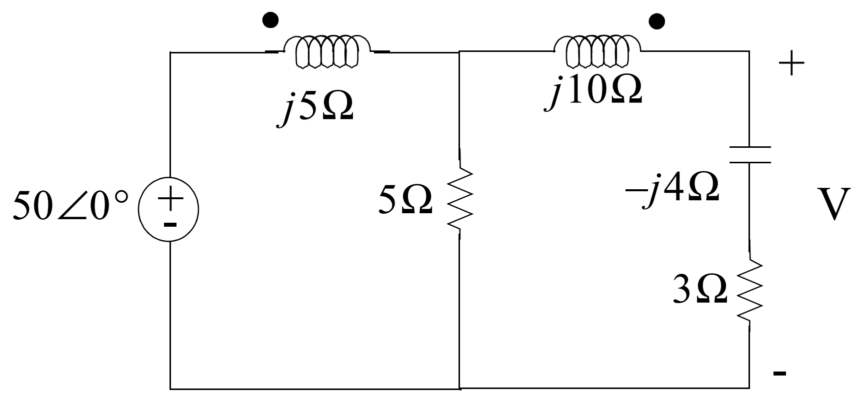
- We can now solve for the mesh currents using standard matrix algebra techniques.

Ex. 13.4 For the circuit below what is the value of the loop currents and what is the value of the mutual inductance?



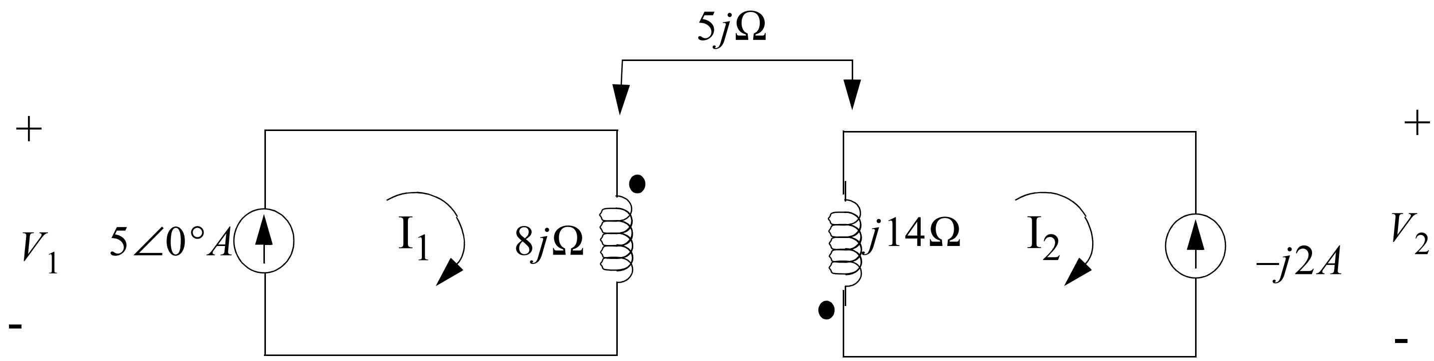
Solution:

Ex. 13.5 Find the voltage V for the circuit below, given the coupling coefficient $k = 0.8$.



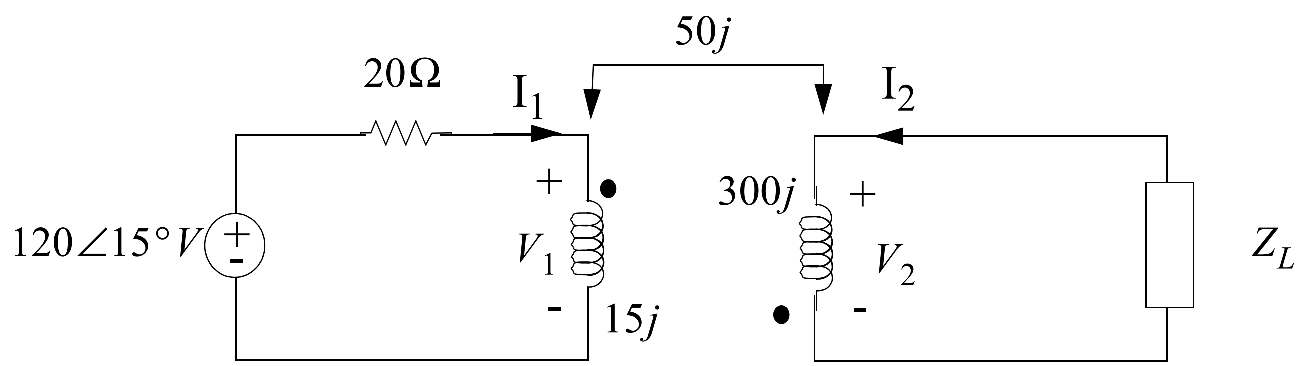
Solution:

Ex. 13.6 Find the unknown voltages in the circuit below.



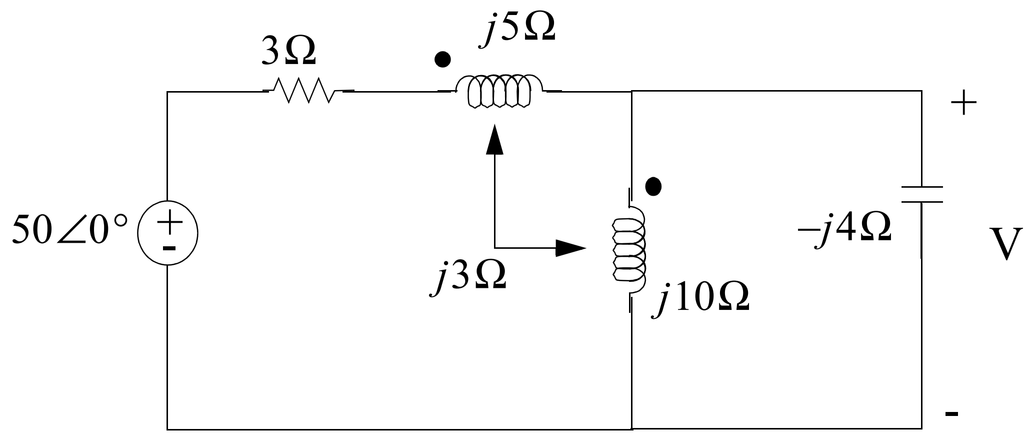
Solution:

Ex. 13.7 Find the Thevenin and Norton equivalent circuits for the circuit shown below.



Solution:

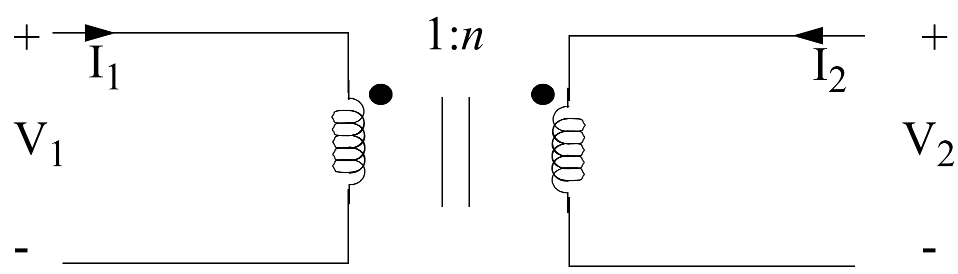
Ex. 13.8 Find the mesh currents in the circuit below.



Solution:

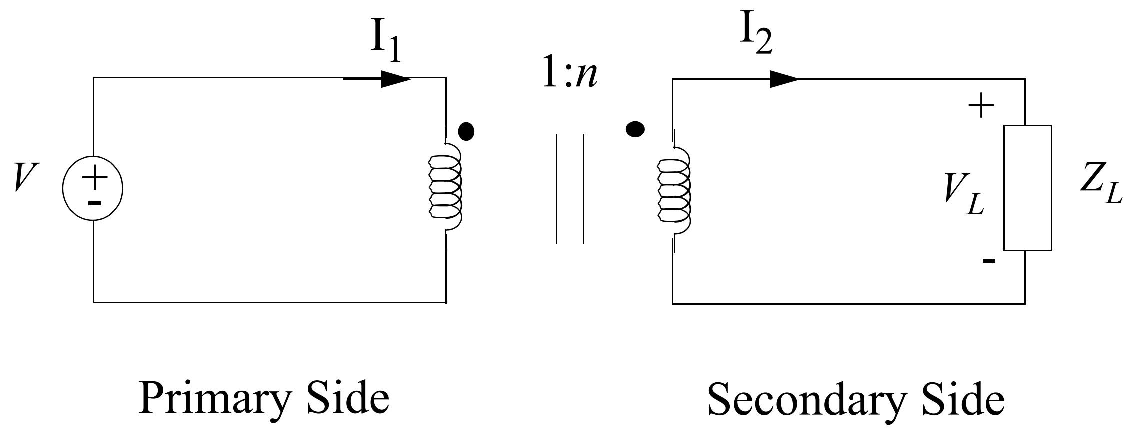
13.7 The Ideal Transformer

- The ideal transformer consists of two inductors that are tightly coupled with $k = 1$.
  - The circuit symbol for an ideal transformer is



  - The vertical lines between the inductors indicate that an iron core is used to confine the magnetic flux to the two coils, thus ensuring a very tight coupling.
  - The ideal transformer is a two-port device with a primary input and a secondary output.
  - The **primary side** is the side that connect to the power supply.

- The **secondary side** is the side which connects to the load.



- A transformer is used to change the voltage, current, and impedance of the primary and secondary sides.

- The main parameter of a transformer is the **turns ration** *n*

  - This is the ration between the number of secondary coil turns to primary coil turns

$$n = \text{Turns Ration} = \frac{\text{Number of secondary turns}}{\text{Number of primary turns}}$$

- In an ideal transformer the secondary to primary voltage ratio is

$$\frac{\boldsymbol{V}_2}{\boldsymbol{V}_1} = n$$

- The secondary to primary current ration is

$$\frac{\boldsymbol{I}_2}{\boldsymbol{I}_1} = \frac{1}{n}$$

- The above two equations are independent of the operating frequency or the load connected the transformer, but they only hold for ideal transformers.

- We know from the previous Chapter that the complex power supplied by the source to the primary coil is given by

$$\boldsymbol{S}_1 = \boldsymbol{V}_1\boldsymbol{I}_1^*$$

  - where it is assumed that $\boldsymbol{V}_1$ and $\boldsymbol{I}_1$ represent rms values.

- We also know that the complex power for the secondary coil will be given by

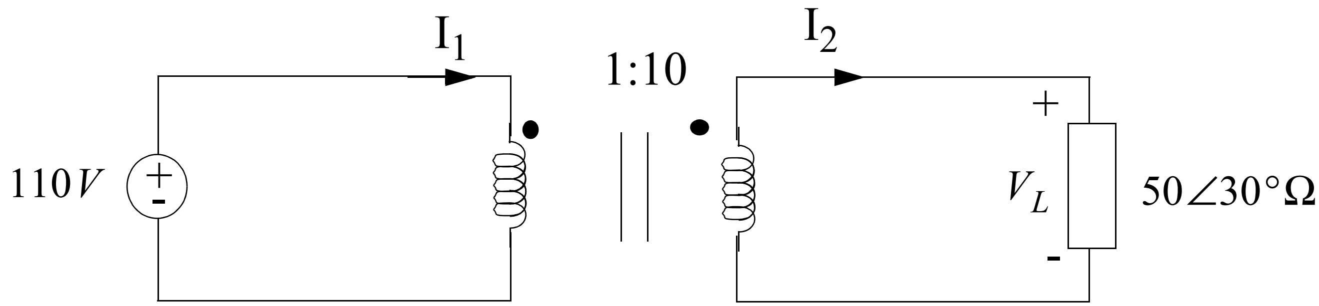
$$\boldsymbol{S}_2 = \boldsymbol{V}_2\boldsymbol{I}_2^*$$

- Using the above equations relating the primary and secondary currents and voltages in ideal transformers we can relate the primary and secondary complex power equations.

$$\boldsymbol{S}_2 = \boldsymbol{V}_2\boldsymbol{I}_2^* = n\boldsymbol{V}_1\frac{\boldsymbol{I}_1^*}{n} = \boldsymbol{S}_1$$

  - Thus the power delivered to the transformer's primary coil exactly equals the power delivered by the transformer's secondary coil to the load.
  - The ideal transformer does not absorb any of the power delivered to it; it delivers all of the power to the load.
  - Obviously, this is an ideal case and for real transformers some power will be lost within the transformer.

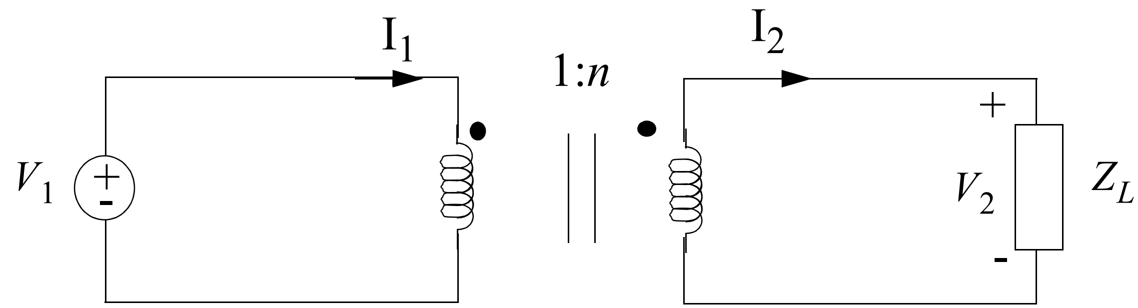
Ex. 13.9 Consider the ideal transformer circuit shown below. Find the secondary voltage, the current in the primary and secondary sides, and the power delivered by the source, and the power consumed by the load.



Solution:

13.7.1 Reflected Impedance

- Consider the ideal transformer shown below



- We are interested in finding the equivalent impedance of the load as it appears to the source on the primary circuit.
- The impedance as seen by the primary side is called the **reflected impedance** and is given by

$$\boldsymbol{Z}_r = \frac{\boldsymbol{V}_1}{\boldsymbol{I}_1}$$

- Since we are dealing with an ideal transformer we can write

$$\boldsymbol{V}_1 = \frac{1}{n}\boldsymbol{V}_2$$

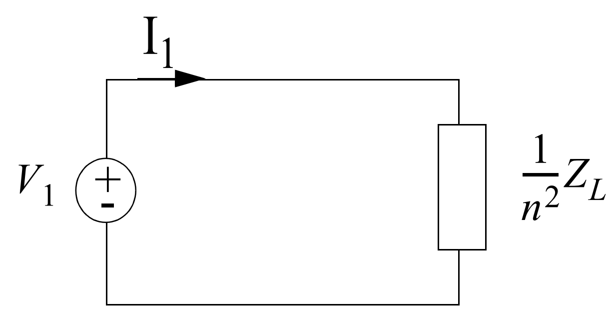
$$\boldsymbol{I}_1 = n\boldsymbol{I}_2$$

- The reflected impedance can then be given in terms of the load impedance as

$$\boldsymbol{Z}_r = \frac{\boldsymbol{V}_1}{\boldsymbol{I}_1} = \frac{1}{n^2}\frac{\boldsymbol{V}_2}{\boldsymbol{I}_2} = \frac{1}{n^2}\boldsymbol{Z}_L$$

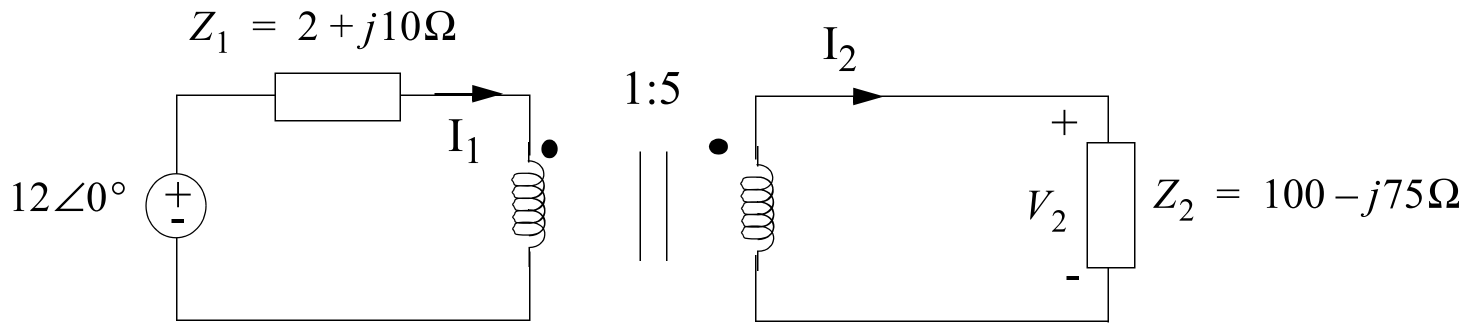
- Thus a load $\boldsymbol{Z}_L$ connected to a secondary side of an ideal transformer appears to the source as if it is an impedance of the value $\frac{1}{n^2}\boldsymbol{Z}_L$.

- Hence, the circuit above, as seen from the primary source, is equivalent to the circuit shown below



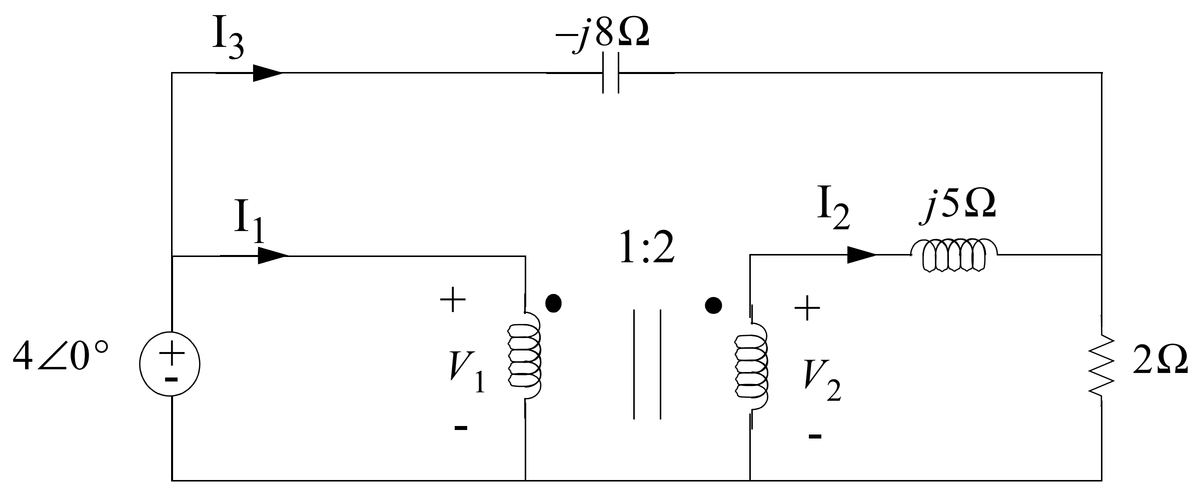
- The reflected impedance will help us simplify transformer circuits by replacing the transformer and the load impedance by their equivalent reflected impedance.

Ex. 13.10 Consider the circuit shown below. Find the primary and secondary voltages and currents, and the power delivered by the source and consumed in $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$.



Solution:

Ex. 13.11 Find the power consumed in the capacitor in the circuit shown below. Such a circuit can occur in practice when an unintentional capacitor is formed through an unintentional connecting of the primary and secondary circuits.



Assignment #13

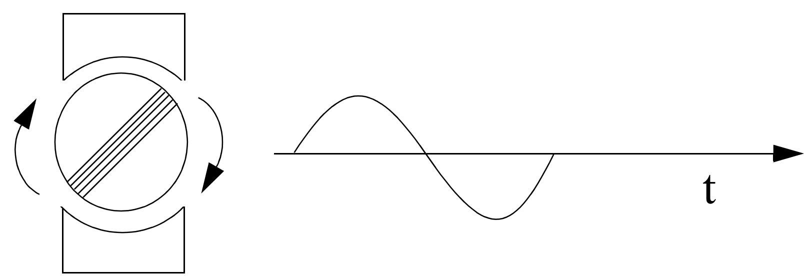
Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 14
Balanced Three-Phase Circuits

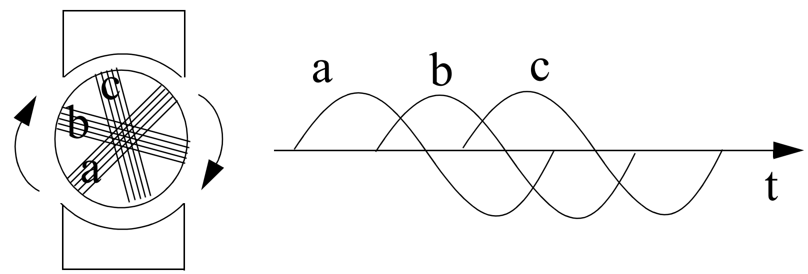
14.1 Introduction

- Up to this point, we have looked at circuits which utilize ac or dc power sources, but we have not looked at the circuits which are used for power generation.
- In power generation, the objective is to maximize the power produced from the generators by using the lightest and most compact generators.
  - This is particularly, true for certain applications such as aviation power sources were weight and size must be minimized.
  - A rotor with a single winding when placed within a fixed magnetic field and rotated about the rotor's axis, produces a sinusiodal voltage waveform.



- To generate more power from the same generator, three equally spaced windings can be placed around the rotor, were equally spaced is relative to the angular spacing (i.e. $120°$ apart)

- Each winding will then gives rise to a sinusiodal voltage waveform, but the three waveforms will be shifted in phase by $120°$ with respect to each other.



14.2 Wye and Delta Connections

- In three-phase power generation systems we have three voltage generators corresponding to three windings $a - a'$, $b - b'$, and $c - c'$.

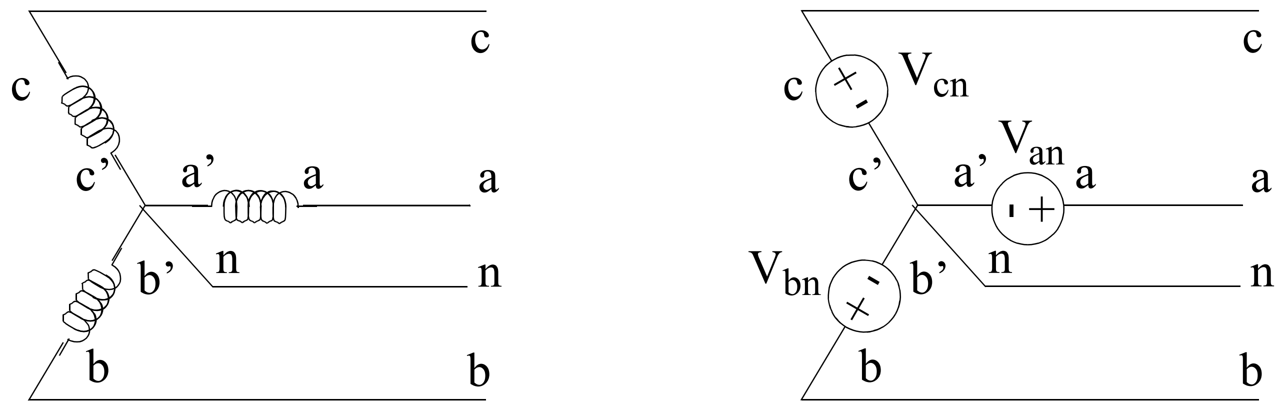
- These three windings could be connected together to form either a **wye** or a **delta** connection

14.2.1 Wye Connection

- The **wye connection** is when the leads a', b', and c' are connected together to form a common neutral n.

- In this case we have four wires coming out of the generator corresponding to the *a*, *b*, and *c* leads of the windings and the neutral lead *n*.

- Such a system is called a **three-phase**, **four-wire system** or a **Y-connected three-phase source**.



Actual Winding Connections

Equivalent Voltage Sources

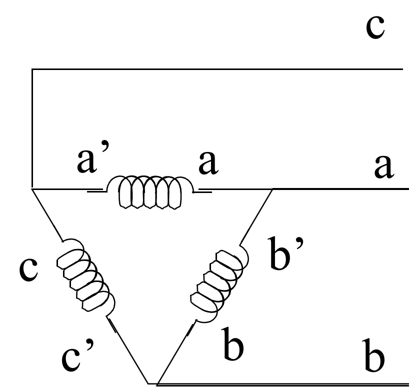
14.2.2 Delta Connection

- The delta connection is when the three windings are connected end to end to form a delta

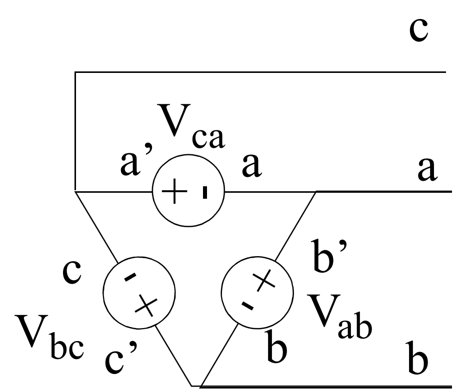
  - In this case we have three wires coming out of the generator corresponding to the *a*, *b*, and *c* leads of the windings.

  - There is **NO** neutral wire in the delta connection

- Such a system is called a **three-phase, three-wire system** or a **Δ-connected three-phase source**.



Actual Winding Connections



Equivalent Voltage Sources

- Note that just having only three wires coming out of the generator is not enough to have a **Δ-connected three-phase source**. It is possible to have a wye-connected source where the neutral wire never comes out of the generator (i.e. a three-wire wye-connected source).

14.3 Phase Voltage

- The voltage generated in each winding of a three-phase generator is called the **phase voltage**.
  - This voltage represents the voltage difference in each of the windings (a-a', b-b', c-c').
- Consider the case of a four-wire, three-phase wye-connected generator

- For this generator the phase voltages can be written in polar form as (with V_{an} assumed to be the reference voltage - i.e. the phasor with 0 phase shift).

$$\boldsymbol{V}_{an} = V_p\angle 0°$$

$$\boldsymbol{V}_{bn} = V_p\angle -120°$$

$$\boldsymbol{V}_{cn} = V_p\angle 120°$$

- where V_p is the **rms phase voltage** and the phases are displaced by $120°$.

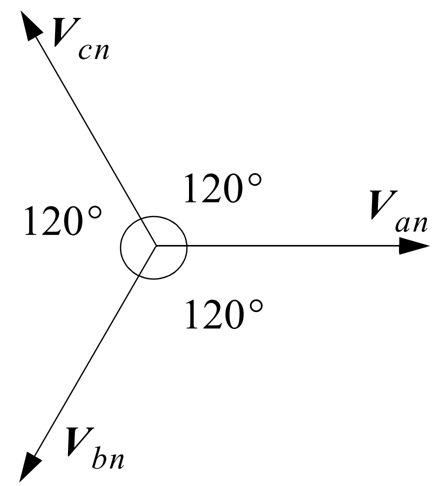
- The phasor $\boldsymbol{V}_{an}$ is the selected reference phasor

- Each of the phase voltages is delayed by $120°$ because of the physical layout of the coils around the generator motor (i.e the coils are physically separated by $120°$ so the phase voltages have a $120°$ phase shift with respect to each other).

- The magnitudes of all of the phase voltages are equal since it is assumed that all of the coils have the same number of turns about the generate rotor.

- The set of phase voltages $\boldsymbol{V}_{an}$, $\boldsymbol{V}_{bn}$, and $\boldsymbol{V}_{cn}$ are said to be a **balanced set** because all of the phase voltages have the same amplitude V_p and are equally displaced by $120°$.

- Graphically, the voltage phasors can be drawn in the complex plane as,



- This ordering of phase voltages is called a **positive sequence** since the phasors are arranged clockwise in the order a, b, then c.

Ex. 14.1 A there-phase ($3-\phi$) four-wire system has an rms phase voltage of 120V at 60Hz. Write down the phasors for the phase voltages and write down their time domain representations.

Solution:

14.4 Line Voltages

- Phase voltages give us the equivalent power source associated with each of the coils on the generator's rotor.

- But, we can also ask what are the voltage differences between the generator's output lines a, b and c.

    - These voltage differences between the lines are termed the **line voltages** and are given as $\boldsymbol{V}_{ab}$, $\boldsymbol{V}_{bc}$, and $\boldsymbol{V}_{ca}$.

    - The expressions for the line voltages can be easily obtained from the expressions for the phase voltages.

    - For example, if it is assumed that we have a we-connected generator then the line voltages will be

$$\boldsymbol{V}_{ab} = \boldsymbol{V}_{an} - \boldsymbol{V}_{bn}$$

$$\boldsymbol{V}_{ab} = V_p\angle 0° - V_p\angle(-120)°$$

$$\boldsymbol{V}_{ab} = \sqrt{3}V_p\angle 30°$$

$$\boldsymbol{V}_{ab} = \sqrt{3}V_p\angle 30°$$

- where $\boldsymbol{V}_L$ is the amplitude of the line voltage

- Similarly, the line voltages $\boldsymbol{V}_{bc}$, and $\boldsymbol{V}_{ca}$. are given by

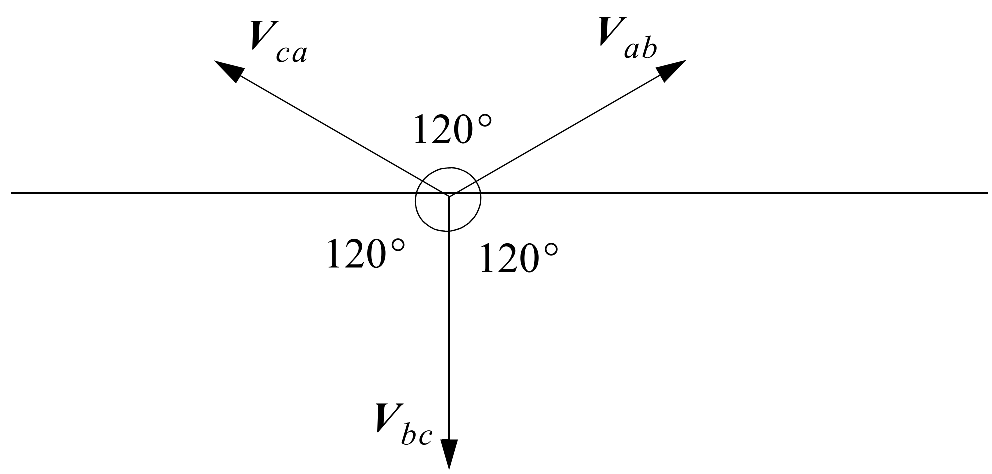
$$V_{bc} = \sqrt{3}V_p\angle{-90°} = \sqrt{3}V_p\angle{-90°}$$

$$V_{ca} = \sqrt{3}V_p\angle 150° = \sqrt{3}V_p\angle 150°$$

- The amplitude of the line voltage V_L is related to the amplitude of the phase voltage V_p by

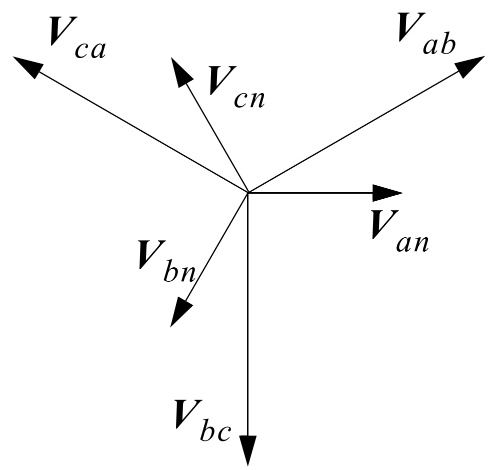
$$V_L = \sqrt{3}V_p$$

- Graphically, these line voltages can be drawn in the complex plane as



- Note that these line voltages also form a positive sequence and the reference phasor $\boldsymbol{V}_{ab}$ is rotated $30°$ above the real line (since the real line was the reference direction we chose for $\boldsymbol{V}_{an}$).

- The relationship between the phase voltages and the line voltages in the complex plane can therefore be given by

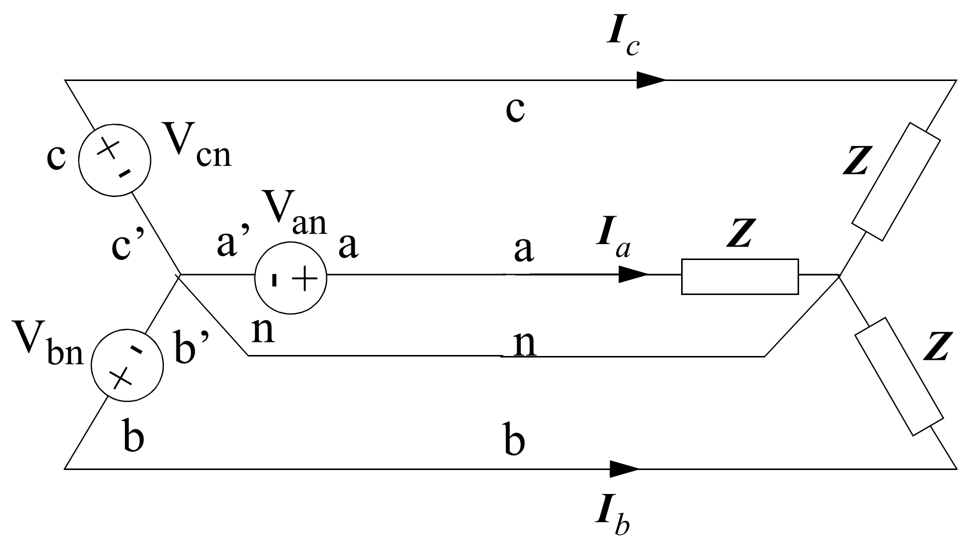


Ex. 14.2 A three-phase $(3-\phi)$ four-wire system has a rms phase voltage of 120V at 60 Hz. Write down the line voltage phasors and their time domain representations.

Solution:

14.5 Wye-Load

- In three-phase systems the loads consist of three impedances, each impedance being connected to one of the lines.
  - Each load impedance the three-phase systems is called a **phase impedance.**
  - When the phase impedances are equal this is termed having a **balanced load**.
- Consider the following system in which a three-phase, four-wire wye-connected generator is connected to a wye-connected four-wye load



- If we assume that each of the phase impedances is equal and is given by

$$\boldsymbol{Z}_p = |\boldsymbol{Z}_p|\angle\theta$$

- The we can obtain expressions for the **line currents** (the currents flowing out of the generators lines a, b and c) by applying KVL around each loop.

$$\boldsymbol{I}_a = \frac{\boldsymbol{V}_{an}}{\boldsymbol{Z}_p} = \frac{V_p\angle 0°}{Z_p\angle\theta} = \frac{V_P}{Z_P}\angle -\theta = I_L\angle -\theta$$

$$\boldsymbol{I}_b = \frac{\boldsymbol{V}_{bn}}{\boldsymbol{Z}_p} = \frac{V_p\angle -120°}{Z_p\angle\theta} = I_L\angle -120° - \theta$$

$$\boldsymbol{I}_c = \frac{\boldsymbol{V}_{cn}}{\boldsymbol{Z}_p} = \frac{V_p\angle 120°}{Z_p\angle\theta} = I_L\angle 120° - \theta$$

- where V_p is the amplitude of the phase voltage, Z_L is the magnitude of the phase impedance, and

$$I_L = \frac{V_P}{|\boldsymbol{Z}_p|}$$

- is the amplitude of the line current (assuming equal phase impedances and rms phase voltages).

- We can also ask what are the **phase currents** (i.e. the currents which flow through each of the phase impedances)

  - it is obvious, for a three-phase four-wire wye-connected generator connected to a four-wire balanced wye-connected load that the phase currents will equal the line currents.

$$\boldsymbol{I}_L = \boldsymbol{I}_p \text{ for Y-load}$$

- Using the result we obtained previously for a 3-phase 4-wire wye-connected sources we have that the line voltage and phase voltages are related by

$$V_L = \sqrt{3}V_p$$

- So for a three-phase four-wire wye connected generator attached to a four-wire balanced load we have that

$$\boldsymbol{I}_L = \boldsymbol{I}_p$$

$$V_L = \sqrt{3}V_p$$

- We can also ask what is the total complex power consumed by the load.

$$\boldsymbol{S} = \boldsymbol{S}_a + \boldsymbol{S}_b + \boldsymbol{S}_c$$

- substituting in the phase currents and phase voltages we have that

$$\boldsymbol{S} = \boldsymbol{V}_{an}\boldsymbol{I}_a^* + \boldsymbol{V}_{bn}\boldsymbol{I}_b^* + \boldsymbol{V}_{cn}\boldsymbol{I}_c^*$$

$$\boldsymbol{S} = 3V_pI_p\angle\theta$$

- or expressing the total power in terms of the line voltages and line currents.

$$\boldsymbol{S} = \sqrt{3}V_LI_L\angle\theta$$

Ex. 14.3 A three-phase four-wire system has $\boldsymbol{V}_p = 120V$ rms and feeds a balanced Y-load with $\boldsymbol{Z}_p = 3 + 4j\ \Omega$. Find the phase currents and voltage and the total power supplied to the load.

Solution:

Ex. 14.4 A three-phase four-wire system has a phase voltage of 120V rms connected to an unbalanced load of $\boldsymbol{Z}_a = 8 - 4j\ \Omega$, $\boldsymbol{Z}_b = 9 - 20j\ \Omega$, and $\boldsymbol{Z}_c = 2j\ \Omega$. Find the phase currents, line currents, and the total power consumed by the unbalanced load.

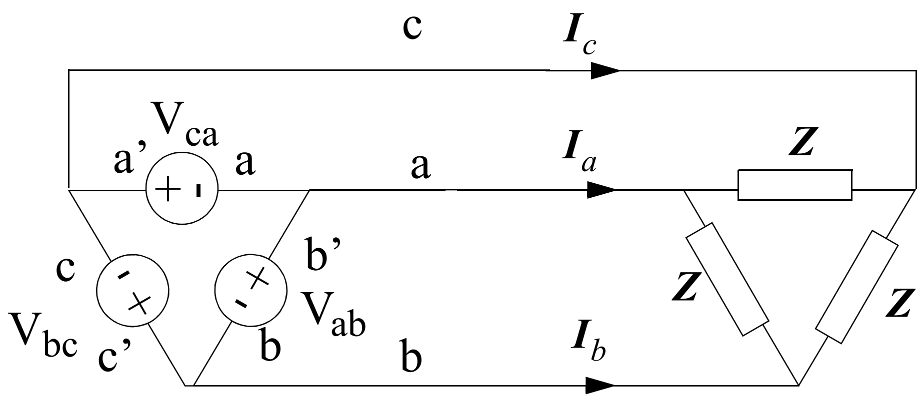
Solution:

Ex. 14.5 A three-phase **three**-wire system has a phase voltage of 120V rms connected to an unbalanced load of $\boldsymbol{Z}_a = 8 - 4j\ \Omega$, $\boldsymbol{Z}_b = 9 - 20j\ \Omega$, and $\boldsymbol{Z}_c = 2j\ \Omega$. Find the phase currents, line currents, and the total power consumed by the unbalanced load. (In this case the neutral wire has been removed from the wye-connected generator)

Solution:

14.6 Delta-Connected Load (Δ-Load)

- Consider the system shown below were a three-phase, three wire Δ-connected generator is connected to a Δ-connected load consisting of three phase impedances.



- As before, when the phase impendances are equal this is termed a balanced load.

$$Z_p = |Z_p| \angle\theta$$

- As before, there are **line currents** flowing in the lines leaving the generator and their are phase currents flowing through the **phase impedances**.

- Notice that for Δ-connected generator its **line voltages** and **phase voltages** will be equal.

$$V_L = V_p$$

- Hence, when the Δ-load is attached the phase voltages across the loads phase impedances will be given by

$$\boldsymbol{V}_{ab} = V_L\angle 30°$$

$$\boldsymbol{V}_{bc} = V_L\angle -90°$$

$$\boldsymbol{V}_{ca} = V_L\angle 150°$$

- Applying Ohm's law at each of the phase impedances gives the phase currents as

$$\boldsymbol{I}_{ab} = \frac{\boldsymbol{V}_{ab}}{\boldsymbol{Z}_p} = I_p\angle(30° - \theta)$$

$$\boldsymbol{I}_{bc} = \frac{\boldsymbol{V}_{bc}}{\boldsymbol{Z}_p} = I_p\angle(-90° - \theta)$$

$$\boldsymbol{I}_{ca} = \frac{\boldsymbol{V}_{ca}}{\boldsymbol{Z}_p} = I_p\angle(150° - \theta)$$

  - where $I_p = \frac{V_p}{|\boldsymbol{Z}_p|}$ is the magnitude of the phase current through the each of the balanced loads.

- We can also ask what are the line currents in a Δ-connected load. (Note that in this case they will **NOT** equal the phase currents)

$$\boldsymbol{I}_a = \boldsymbol{I}_{ab} - \boldsymbol{I}_{ac} = I_p\angle(30°-\theta) - I_p\angle(150°-\theta) = \sqrt{3}I_p\angle-\theta$$

$$\boldsymbol{I}_b = \sqrt{3}I_p\angle-120°-\theta$$

$$\boldsymbol{I}_c = \sqrt{3}I_p\angle120°-\theta$$

- Thus for a Δ-connected balanced load we can write

$$V_L = V_p$$

$$I_L = \sqrt{3}I_P$$

- The total power consumed by the load is given by

$$\boldsymbol{S} = \boldsymbol{S}_a + \boldsymbol{S}_b + \boldsymbol{S}_c$$

  - substituting in the phase currents and phase voltages we have that

$$\boldsymbol{S} = \boldsymbol{V}_{ab}\boldsymbol{I}^*_{ab} + \boldsymbol{V}_{bc}\boldsymbol{I}^*_{bc} + \boldsymbol{V}_{ca}\boldsymbol{I}^*_{ca}$$

$$\boldsymbol{S} = 3V_pI_p\angle\theta$$

  - or expressing the total power in terms of the line voltages and line currents.

$$\boldsymbol{S} = \sqrt{3}V_LI_L\angle\theta$$

Ex. 14.6 A three-phase, three-wire Δ-connected system has $V_L = 240\text{ V rms}$ and feeds a load with $\boldsymbol{Z}_p = 3 + j4\ \Omega$. Find the phase and line currents and the total power supplied to the circuit.

Solution:

Assignment #14

Refer to Elec 250 course web site for assigned problems.

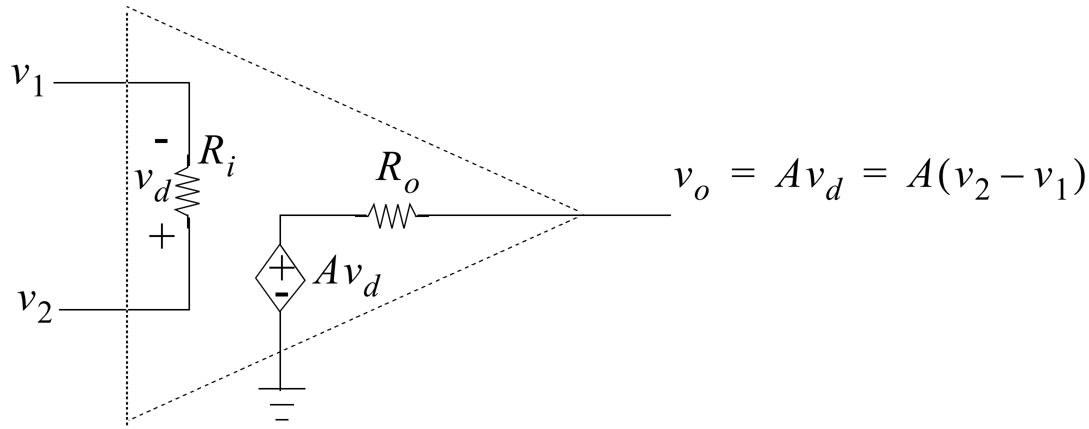
- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.

Chapter 15
Operational Amplifiers

15.1 Introduction

- Up to this point, we have looked at circuits built from the very basic core circuit elements of resistors, inductors, capacitors, and independent and dependent power sources.
- In general, there is a large group of commonly used circuits that are packaged into integrated circuits (IC's) such that they can be easily used within a wider variety of circuit designs
  - Examples of such IC's are range from basic logic gates (i.e., AND, OR, NOT, NAND, NOR, XOR, etc.), timer circuits (i.e., the 555 timer IC's), up to microcontrollers (i.e, 68HC11, etc.), and complete superscalar microprocessors (i.e., dual-core Intel Xeon processors, etc.).
- Operational amplifiers are one extremely common type of IC which can be used to implement a number of useful analog circuits.
- Operational amplifiers are active devices - meaning that they provide power to the circuit, (i.e., they must be connected up to a power supply to work).
- Operational amplifiers are made up from the basic circuit elements we have already studied

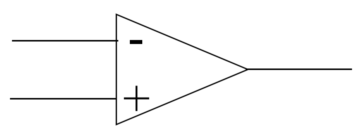
- Specifically, an operational amplifier is comprised of the following combination of resistors with a dependent voltage source



- The value A is termed the op amp's open loop gain.

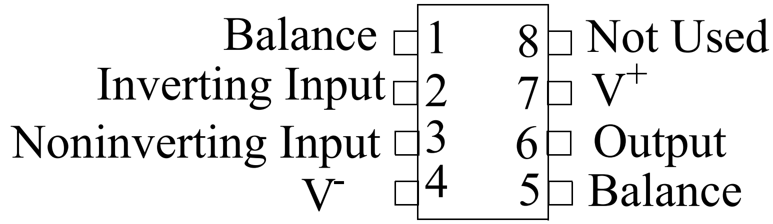
- Ostensibly, an op amp output is an amplified version of the voltage differential across its two input terminals.

- The standard circuit symbol for an op amp is,



  - Note that typically the power supplies to the op amp are not drawn but assumed.

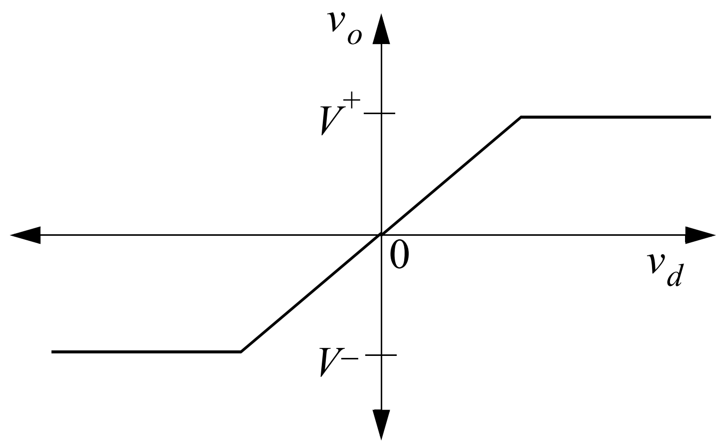
- A common physical packaging of an op amp is as an 8-pin IC package (i.e, more specifically, a 741 op amp),



- Typical ranges for an op amp's parameters are,

Parameter	Typical Range	Ideal Values
Open-loop gain, A	10^5 to 10^8	∞
Input Resistance, R_i	10^5 to 10^{13} Ω	∞ Ω
Output Resistance, R_o	10 to 100 Ω	0 Ω
Supply Voltages, V^-, V^+	±5 to ±24 V	

- The standard graph relating v_d to v_o for an op amp is,



- If we keep the op amp within its linear region then all of our linear circuit theorems apply.

- The areas outside of this linear region are termed the areas where the op amp is s*aturated*, (i.e.,when the limits are reached of its ability to provide power to the circuit)

- Within the context of this course we will assume op amps are in their linear regions unless otherwise stated.

- Obviously for real-world circuits, one must check to ensure that the circuit always stays within the op amps' linear regions and none of the circuit's op amps ever saturate.

• Op amps become really useful circuit elements when we introduce a feedback path from the output terminal back to the inverting input terminal.

- Once we have this feedback path, we can build a number of useful circuits to perform mathematical operations such as,

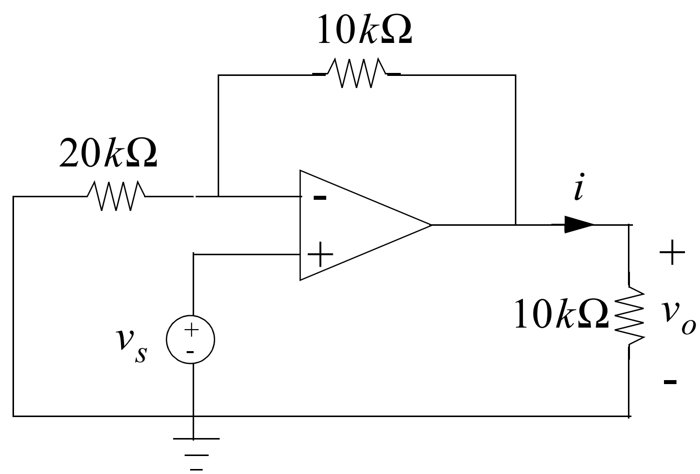
    - Signal Inversion
    - Summation
    - Difference Amplifier
    - Analog Signal Integrator
    - Analog Signal Differentiator

- The high input impedance of op amps can be used to construct voltage followers which isolated one part of a circuit from another part of the circuit.

    - This allows circuits to be cascaded together to form more complex circuits

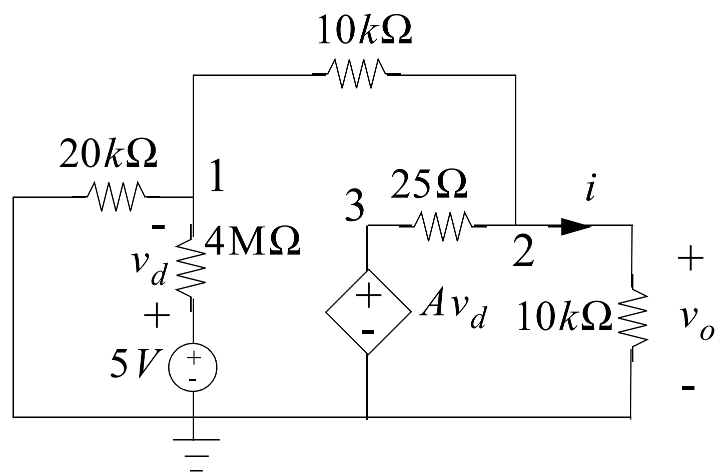
    - Because of the isolation obtained provided by the voltage followers each piece of these complex cascaded circuits can be analyzed individually since their internal operation does not affect the cascade sections which follow.

Ex. 15.1 A 741 op amp below has an open loop gain of $A = 4.5 \times 10^6$, $R_i = 4\ \text{M}\Omega$, and $R_o = 25\Omega$. Find the close-loop gain $\frac{v_o}{v_i}$. Determine the current i when $v_s = 5V$.



Solution:

First, converting the circuit to the form we are more familiar with,



From nodal analysis we have that,

$$\frac{v_1}{20k} + \frac{v_1 - 5}{4M} + \frac{v_1 - v_2}{10k} = 0$$

$$\frac{v_2 - v_1}{10k} + \frac{v_2 - v_3}{25} + \frac{v_2}{10k} = 0$$

$$v_3 = Av_d = (4, 500, 000)(v_1 - 5)$$

Solving for v_2,

$$v_2 = v_o = 7.500002514$$

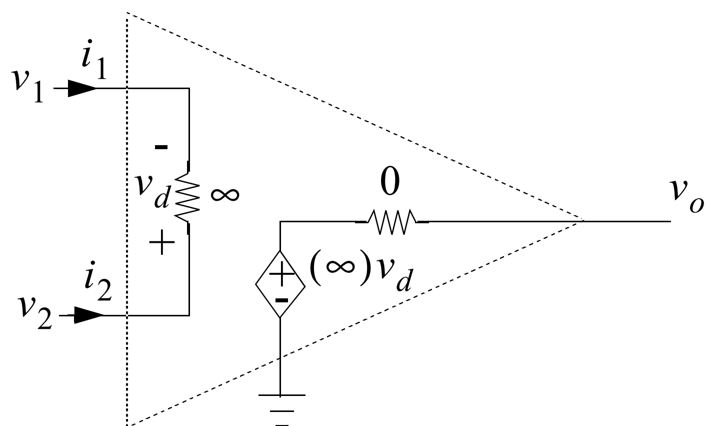
Giving a gain of,

$$g = \frac{v_0}{v_s} = \frac{7.500002514}{5} = 1.5000005028$$

and a current of $i = 0.7500002514mA$

15.2 Ideal Op Amps

- The size of the typical parameter values for op amps makes solving the circuit questions fairly tedious and we have to be quite careful that we do not drop a zero somewhere.
    - What happens if we simplify our analysis by just assuming that the op amp actually has its ideal values for its various parameters, (i.e., $R_i = \infty$, $R_o = 0$, and $A = \infty$)?



- Because of the assumed infinite input resistance $R_i = \infty$, the two input terminal will look like an open circuit

  - Hence,

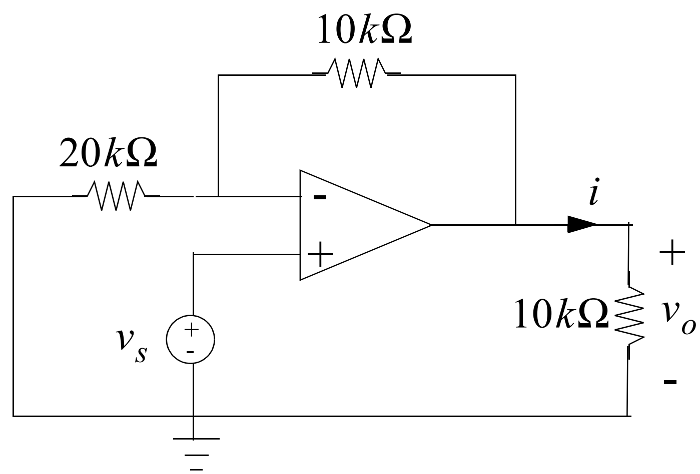
$$i_1 = 0 \text{ and } i_2 = 0$$

  - It must then be the case that $v_d = 0$ and, therefore, that

$$v_1 = v_2$$

  - These two basic assumptions define an ideal op amp.

Ex. 15.2 For the ideal op amp determine the gain $\frac{v_o}{v_s}$ and the current i when $v_s = 5V$



Solution:

- Since we are now assuming that $v_d = 0$, we have that

$$v_s = \left[\frac{20k}{10k+20k}\right]v_o \Rightarrow g = \frac{v_o}{v_s} = 1.5$$

- Applying KCL at node O,

$$i_o = \frac{v_0}{10k+20k} + \frac{v_o}{10k}$$

- When $v_s = 5V$, $v_0 = 7.5V$ and $i_o = 0.75mA$.

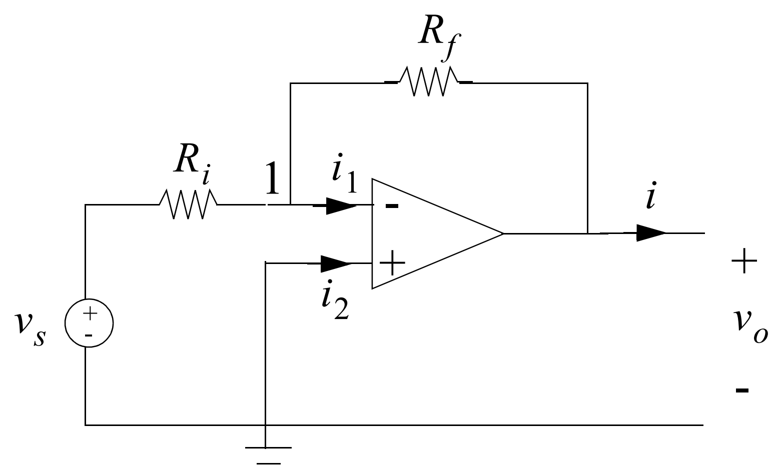
- These answers are very close to the true answers (within 6 significant digits in this particular case)
- Hence, assuming that the op amp is an ideal op amp is adequate is most cases and simplifies the analysis considerably.

15.3 Useful Op Amp Circuits

- Now that we have the basics we can build a number of mathematically useful circuits.

15.3.1 The Inverting Amplifier

- The following circuit is an inverting amplifier
  - The output voltage is a scaled inverted version of the input voltage that is applied to the op amp's inverting input



- Since $i_1 = i_2 = 0$,

$$\frac{v_s - v_1}{R_i} = \frac{v_1 - v_0}{R_f}$$

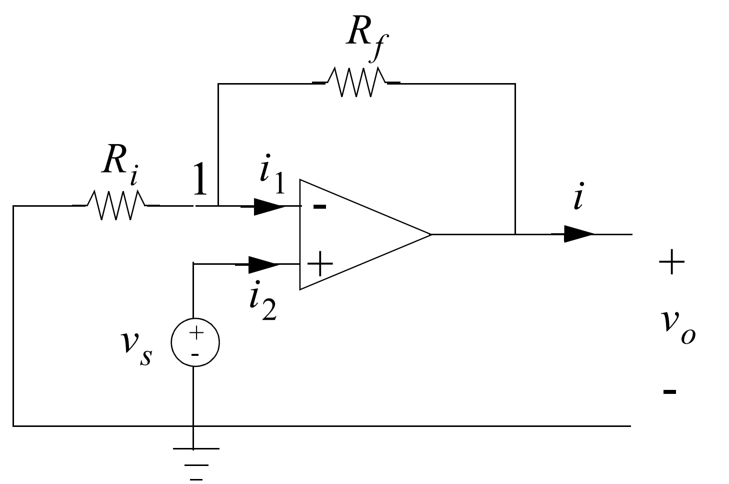
- But, $v_d = v_2 - v_1 = 0$ and v_2 is grounded, hence $v_1 = 0$ so,

$$\frac{v_s}{R_i} = \frac{-v_0}{R_f} \text{ or } v_o = -\frac{R_f}{R_i} v_s$$

- The degree of amplification is controlled by the ratio of the input and output resistors.

15.3.2 The Non-Inverting Amplifier

- The following circuit is a non-inverting amplifier
  - The output voltage is a scaled non-inverted version of the input voltage that is applied to the op amp's inverting input



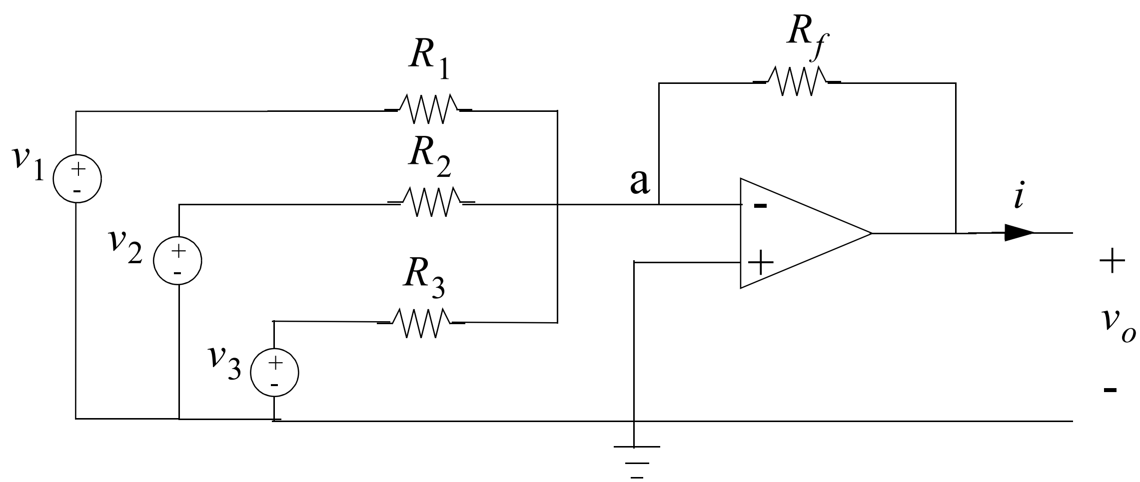
  - Since, $v_d = v_2 - v_1 = 0$, we know that $v_1 = v_s$ and we know that $i_1 = 0$ so,

$$\frac{v_s}{R_i} + \frac{v_s - v_0}{R_f} = 0 \text{ or that } v_o = \left(1 + \frac{R_f}{R_i}\right)v_s$$

  - The degree of amplification is still controlled by the ratio of the input and output resistors, but this time there is no inversion of the input violate (or signal).

15.3.3 The Summing Amplifier

- The following circuit performs voltage summation
  - The output voltage is a scaled inverted version of the sum of the input voltages



- Since $i_1 = i_2 = 0$,

$$\frac{v_a - v_1}{R_1} + \frac{v_a - v_2}{R_2} + \frac{v_a - v_3}{R_3} = \frac{v_a - v_0}{R_f}$$

- But, $v_d = 0$ hence $v_a = 0$ so,

$$v_o = -\left(\frac{R_f}{R_1}v_1 + \frac{R_f}{R_2}v_2 + \frac{R_f}{R_3}v_3\right)$$

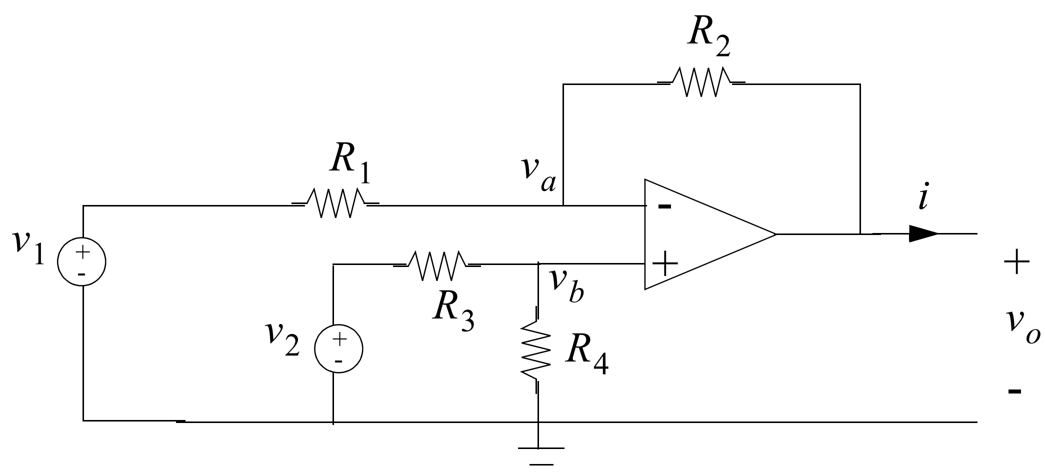
- The degree of amplification of each input is controlled by the ratio of that input's resistor and the output resistor.

- This provides a means of *weighting* the inputs.

15.3.4 The Difference Amplifier

- The following circuit performs voltage summation

  - The output voltage is a scaled inverted version of the sum of the input voltages



- We have that, $\frac{v_a - v_1}{R_1} = \frac{v_a - v_0}{R_2}$ and that $\frac{v_b - v_2}{R_3} = \frac{v_b}{R_4}$

- But, $v_a - v_b = 0$ so, via substitution and simplification we get that,

$$v_o = \left[\frac{R_2\left(1 + \frac{R_1}{R_2}\right)}{R_1\left(1 + \frac{R_3}{R_4}\right)}\right]v_2 - \left[\frac{R_2}{R_1}\right]v_1$$

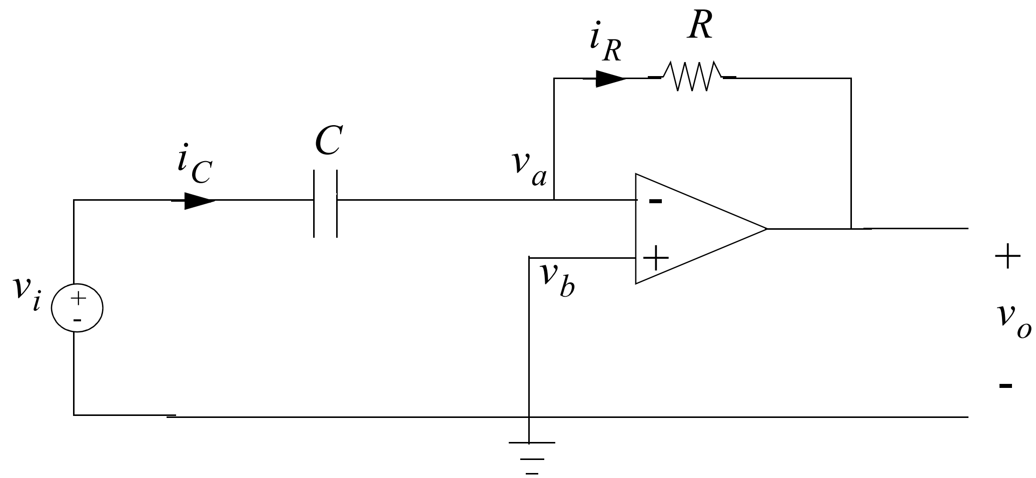
- So v_o is a scaled version of the difference between the two input signals.

- If $R_1 = R_2 = R_3 = R_4 = R$ then we get that,

$$v_o = v_2 - v_1$$

15.3.5 The Differentiator

- The following circuit performs voltage differentiation.
  - The output voltage is a proportional to the time domain derivative of the input voltage.



- For this circuit we have that,

$$i_c = C\frac{d(v_i - v_a)}{dt}$$

- and that

$$i_R = \frac{v_a - v_o}{R}$$

- We also known that $v_a = v_b = 0$

  - So applying KCL at node a and simplifying we get that,

$$v_o = -CR\frac{dv_i}{dt}$$

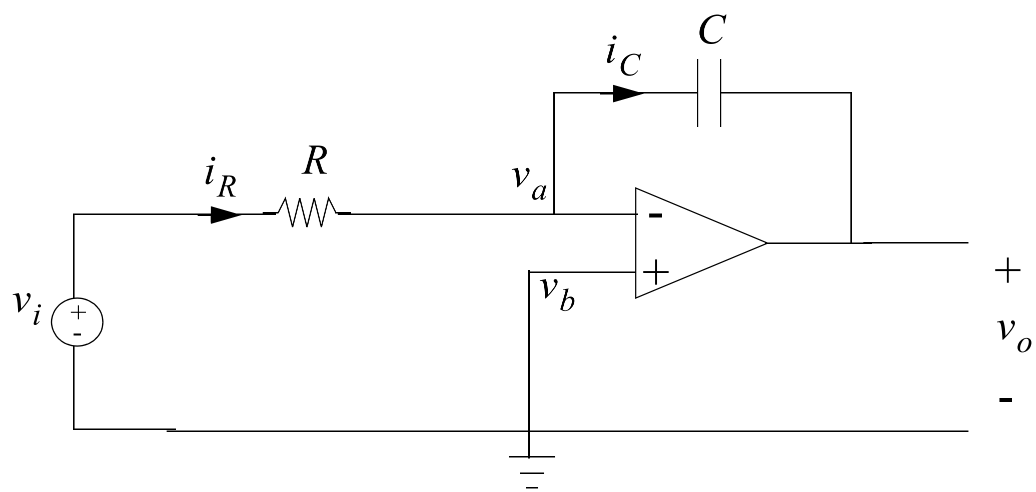
- In practice, differentiator circuits tend to be unstable because variations in v_i due to noise get amplified as v_i passes through the differentiator.

  - This amplification of noise is common whenever differentials are taken of noisy signal.

- Because of this effect, differentiator circuits are seldom used in practice.

15.3.6 The Integrator

- The following circuit performs integration.
  - The output voltage is proportional to a time domain integral of the input voltage.



- Now, we have that, $i_R = \frac{v_i - v_a}{R}$ and that $i_C = C\frac{d(v_a - v_o)}{dt}$

- But, $v_a = v_b = 0$ so, via substitution and simplification we get that,

$$\frac{dv_0}{dt} = -\frac{1}{RC}v_i dt$$

- Integrating both sides of this equation gives over $\tau \in [0, t]$

$$v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau$$

- If we ensure that the capacitor is fully discharged before we start the circuit then,

$$v_o(t) = -\frac{1}{RC}\int_0^t v_i(\tau)d\tau$$

- This analog integration circuit is much more commonly used that its differentiation counter-part
Note that the only difference between these two circuits is where the capacitor is place.

  - Also care must be taken to ensure that the amplifier does not saturate since $v_o(t)$ can never become larger than the power supplied to the operational amplifier itself.

15.4 An Analog Computer

- With the above circuits we can perform a number of useful mathematical operations.
    - Can we for example design an analog circuit which can solve,

$$\frac{d^2 v_0(t)}{dt^2} + 4\frac{dv_0(t)}{dt} + 3v_0(t) = 10\cos(3t - 40°)$$

when $v_o(0) = 2V$ and $v_0'(0) = -3V$ (Note: We need 2 initial conditions if we are to uniquely solve a 2nd order differential equation)

Solution:

- We want $v_0(t)$ so we will need to perform two integrations

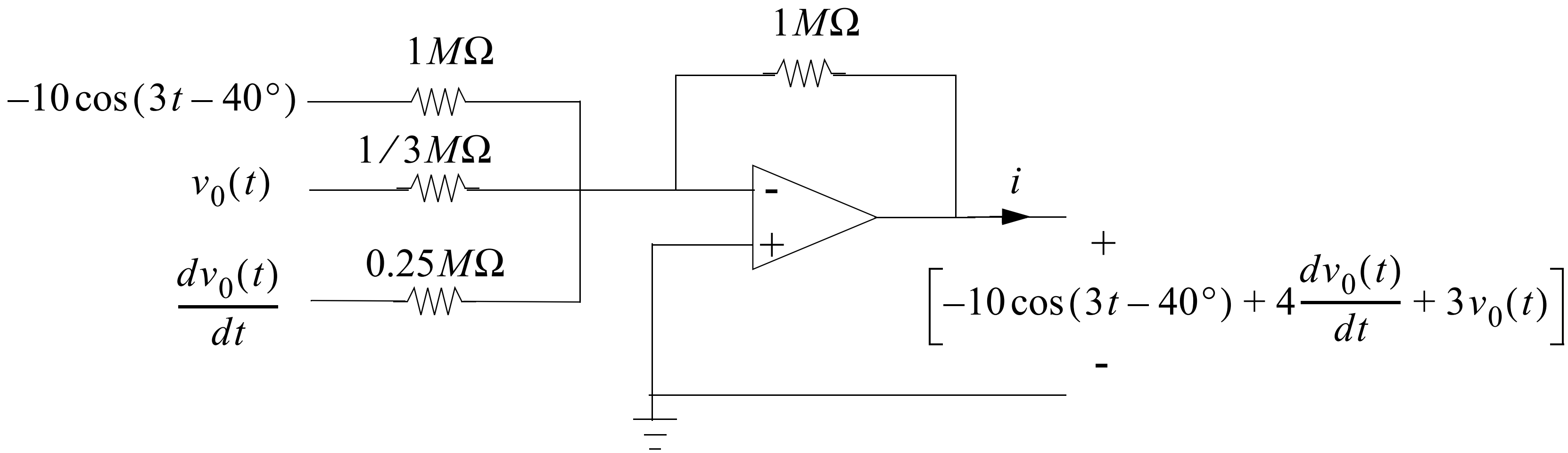
- Re-writing the DE in terms of $\frac{d^2 v_0(t)}{dt^2}$ we have that,

$$\frac{d^2 v_0(t)}{dt^2} = 10\cos(3t - 40°) - 4\frac{dv_0(t)}{dt} - 3v_0(t)$$

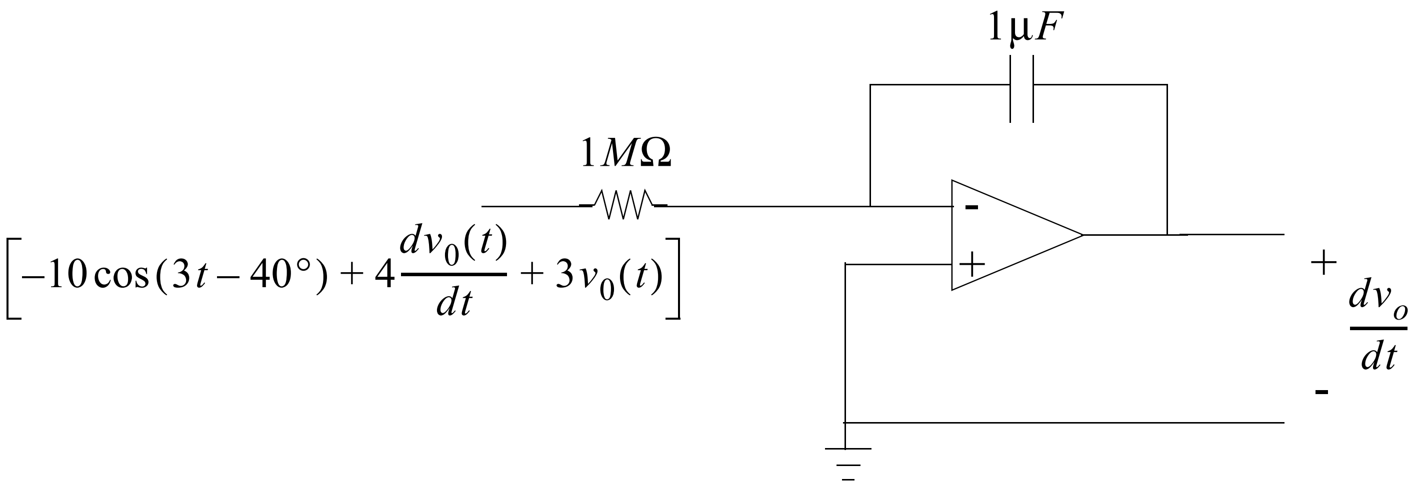
- Integrating both sides and pulling a minus sign out front gives,

$$\frac{dv_0(t)}{dt} = -\int_0^t \left[-10\cos(3t - 40°) + 4\frac{dv_0(t)}{dt} + 3v_0(t)\right] + v_0'(0)$$

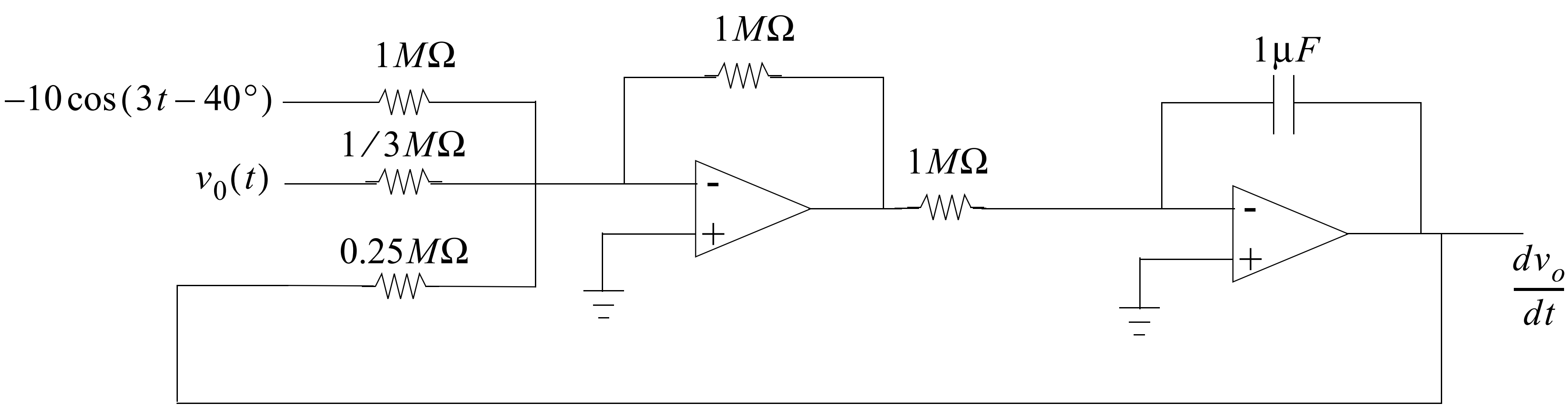
- We can build a circuit to do this summation by the circuit,



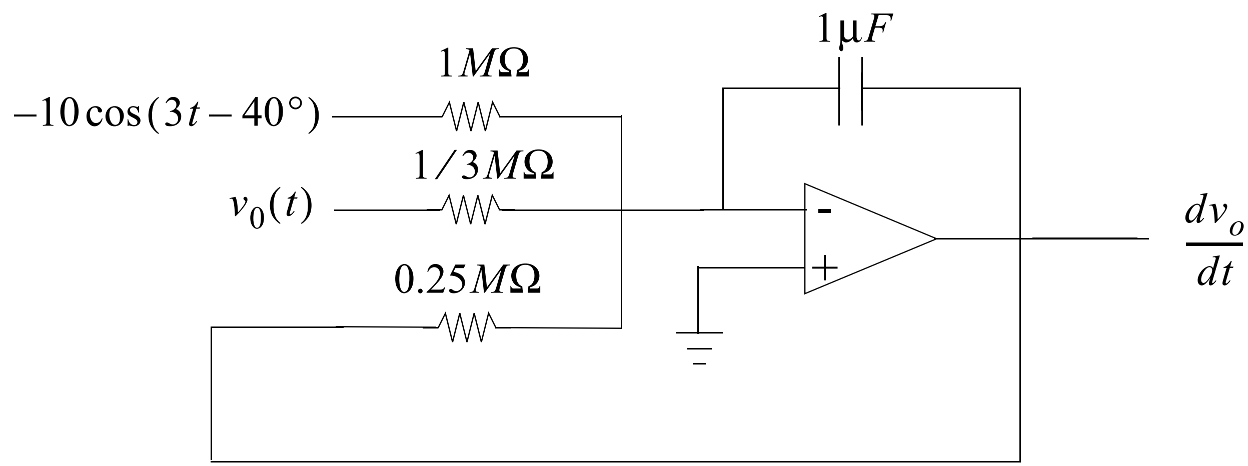
- We can then do the integration by feeding this output into the circuit,



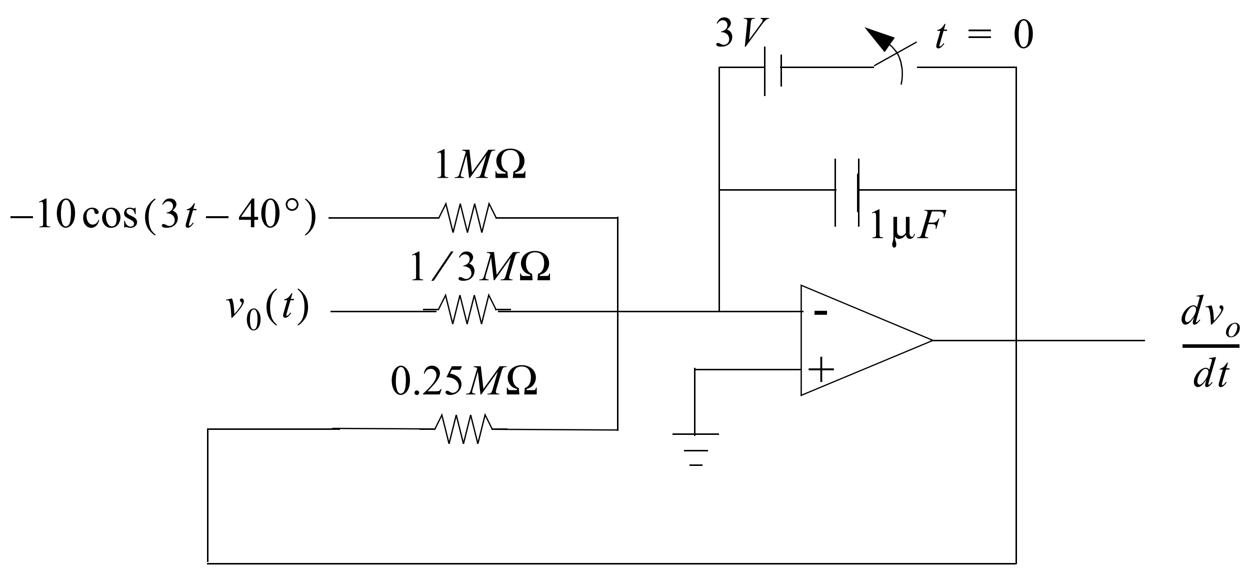
- This gives the combined circuit,



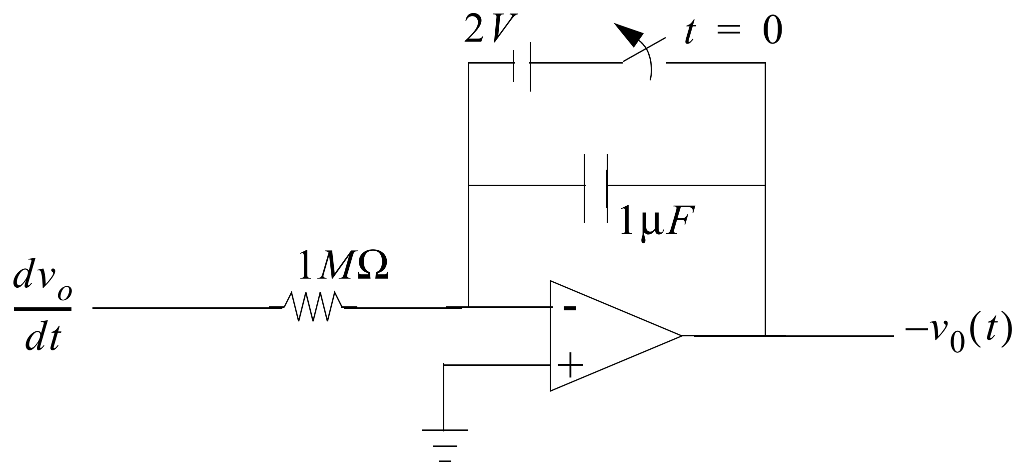
- This circuit can actually be simplified as to a single op amp circuit as,



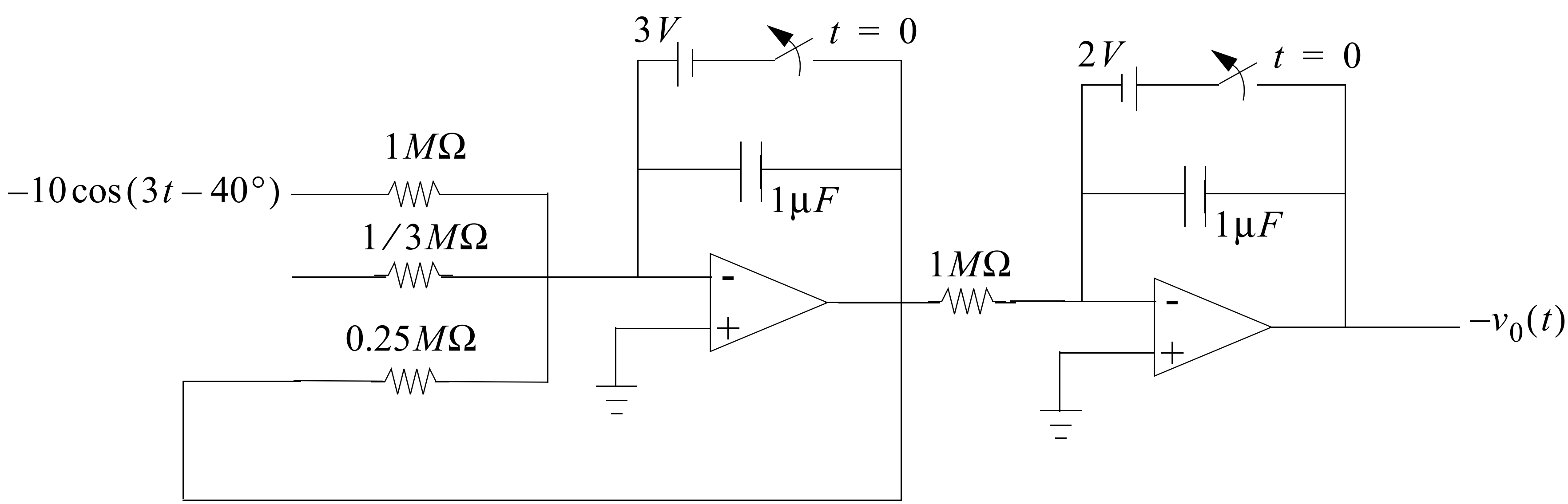
- To satisfy the initial condition of $v_0'(0) = -3V$ we can just add a DC power supply and a switch which open at $t = 0$,



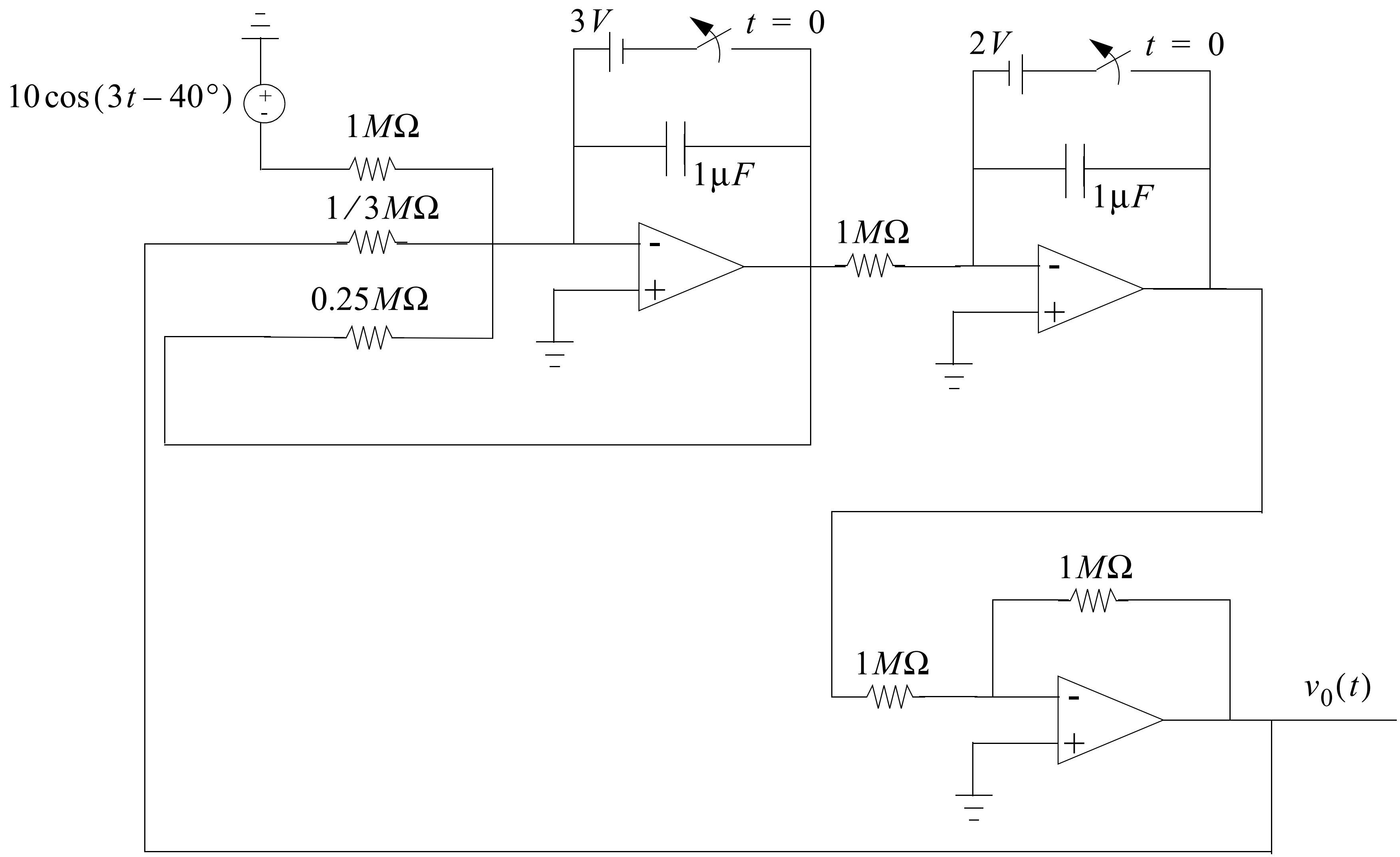
- Now we can do the second integration with an op amp integration circuit and take care of the second initial condition ($v_o(0) = 2V$) in the same way,



- Combining these two circuits gives,



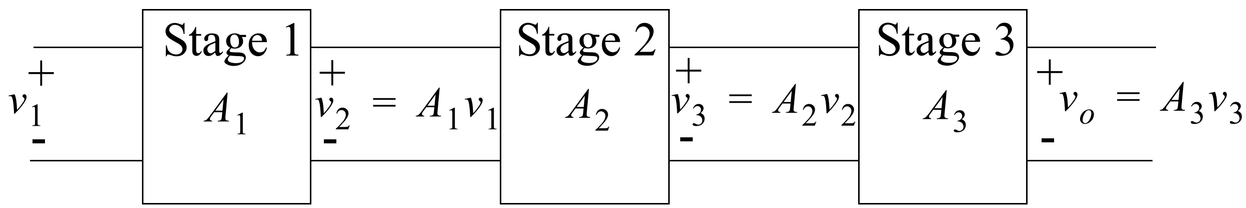
- Now all we need to do is to add a simple op amp inverter at the end to get $v_0(t)$ instead of $-v_0(t)$,



- Prior to digital computers, engineers would build special purpose analog circuits to solve problems which were not easily solved by hand, (i.e., by slide rules).

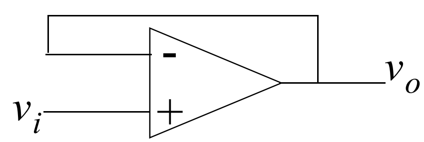
15.5 Cascaded Op Amp Circuits

- The high input impedance of op amps means that they can be cascaded in series without affecting the operation of circuits earlier in the cascade.
  - This allows the analysis of the circuit to proceed one stage at a time.



  - We have already used this cascading property in the analog computer example above,
    - Without it we would have had to re-analyze the behavior of the complete circuit each time we added a new stage.

- This leads the useful voltage follower op amp circuit where $v_o = v_i$



  - The main purpose of this circuit is electrically isolated two stages of a circuit and eliminate inter-stage loading (i.e., power being draw from one stage to service another stage's power needs).

Assignment #15

Refer to Elec 250 course web site for assigned problems.

- Due 1 week from today @ 5pm in the Elec 250 Assignment Drop box.