$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

Continuous-Time Signals and Systems
(Last Revised: January 11, 2012)

by

Michael D. Adams

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\, dt$$
$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\, ds$$

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Continuous-Time Signals and Systems
(Last Revised: January 11, 2012)

by

Michael D. Adams

Department of Electrical and Computer Engineering
University of Victoria, Victoria, BC, Canada

To my students, past, present, and future

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Preface

This book is primarily intended to be used as a text for undergraduate students in engineering (and related) disciplines. The book provides a basic introduction to continuous-time signals and systems. Since many engineering curricula use MATLAB as a teaching tool, the book also includes a detailed tutorial on MATLAB as an appendix. This book evolved from a detailed set of lecture notes that the author prepared in order to teach two different undergraduate courses on signals and systems at the University of Victoria. The first version of these lectures notes was developed while the author was teaching ELEC 260 in the Summer 2003 term. These lecture notes were then augmented in order to accommodate the teaching of ELEC 255. Over time, the lecture notes underwent many changes, eventually evolving into the book that you are now reading.

More recently, the author has started to develop some material on discrete-time signals and systems. With a little luck, the author will complete this draft material, and include it in a future version of this book.

Acknowledgments

I would like to thank my colleague, Dr. Wu-Sheng Lu, for many interesting technical discussions that helped to clarify some of the finer points of the mathematics behind signals and systems. Also, I would like to thank my past students for their feedback regarding earlier revisions of this manuscript. They have helped me to eliminate numerous errors in this manuscript that would have otherwise gone undetected. In particular, I would like to thank Ioana Sevcenco for her careful proofreading of much of an earlier version of the manuscript.

Michael Adams
Victoria, BC
January 11, 2012

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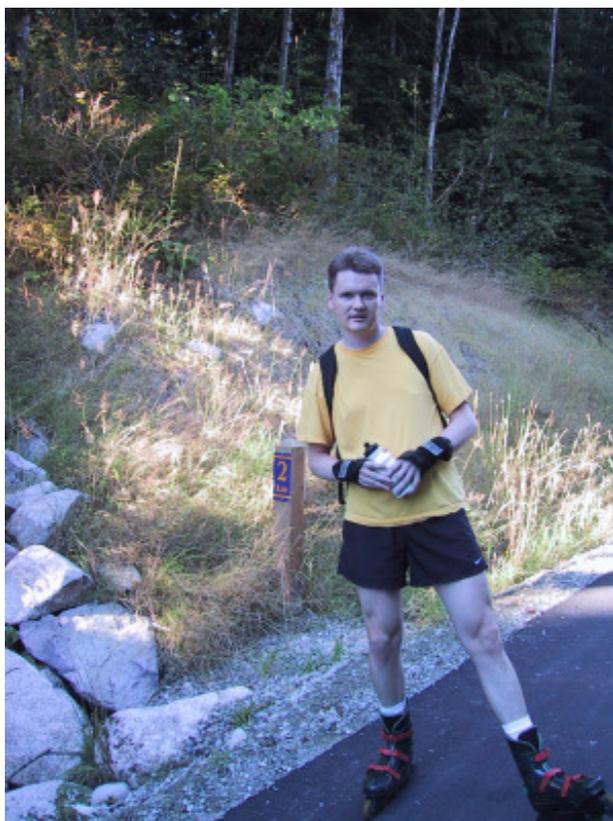
Michael Adams received the B.A.Sc. degree in computer engineering from the University of Waterloo, Waterloo, ON, Canada in 1993, the M.A.Sc. degree in electrical engineering from the University of Victoria, Victoria, BC, Canada in 1998, and the Ph.D. degree in electrical engineering from the University of British Columbia, Vancouver, BC, Canada in 2002. From 1993 to 1995, Michael was a member of technical staff at Bell-Northern Research in Ottawa, ON, Canada, where he developed real-time software for fiber-optic telecommunication systems. Since 2003, Michael has been on the faculty of the Department of Electrical and Computer Engineering at the University of Victoria, Victoria, BC, Canada, first as an Assistant Professor and currently as an Associate Professor.

Michael is the recipient of a Natural Sciences and Engineering Research Council (of Canada) Postgraduate Scholarship. He is a voting member of the Canadian Delegation to ISO/IEC JTC 1/SC 29 (i.e., Coding of Audio, Picture, Multimedia and Hypermedia Information), and has been an active participant in the JPEG-2000 standardization effort, serving as co-editor of the JPEG-2000 Part-5 standard and principal author of one of the first JPEG-2000 implementations (i.e., JasPer). His research interests include multimedia signal processing, image compression, multiresolution signal/geometry processing (e.g., filter banks, wavelets, and subdivision surfaces), and computational geometry.

Chapter 1

Introduction

1.1 Signals

A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon.

1.1.1 Dimensionality of Signals

Signals can be classified based on the number of independent variables with which they are associated. A signal that is a function of only one variable is said to be **one dimensional** (1D). Similarly, a signal that is a function of two or more variables is said to be **multidimensional**. Human speech is an example of a 1D signal. In this case, we have a signal associated with fluctuations in air pressure as a function of time. An example of a 2D signal is a monochromatic image. In this case, we have a signal that corresponds to a measure of light intensity as a function of horizontal and vertical displacement.

1.1.2 Continuous-Time and Discrete-Time Signals

A signal can also be classified on the basis of whether it is a function of continuous or discrete variables. A signal that is a function of continuous variables (e.g., a real variable) is said to be **continuous time**. Similarly, a signal that is a function of discrete variables (e.g., an integer variable) is said to be **discrete time**. Although the independent variable need not represent time, for matters of convenience, much of the terminology is chosen as if this were so. For example, a digital image (which consists of a rectangular array of pixels) would be referred to as a discrete-time signal, even though the independent variables (i.e., horizontal and vertical position) do not actually correspond to time.

If a signal is a function of discrete variables (i.e., discrete-time) and the value of the function itself is also discrete, the signal is said to be **digital**. Similarly, if a signal is a function of continuous variables, and the value of the function itself is also continuous, the signal is said to be **analog**.

Many phenomena in our physical world can be described in terms of continuous-time signals. Some examples of continuous-time signals include: voltage or current waveforms in an electronic circuit; electrocardiograms, speech, and music recordings; position, velocity, and acceleration of a moving body; forces and torques in a mechanical system; and flow rates of liquids or gases in a chemical process. Any signals processed by digital computers (or other digital devices) are discrete-time in nature. Some examples of discrete-time signals include digital video, digital photographs, and digital audio data.

A discrete-time signal may be inherently discrete or correspond to a sampled version of a continuous-time signal. An example of the former would be a signal corresponding to the Dow Jones Industrial Average stock market index (which is only defined on daily intervals), while an example of the latter would be the sampled version of a (continuous-time) speech signal.
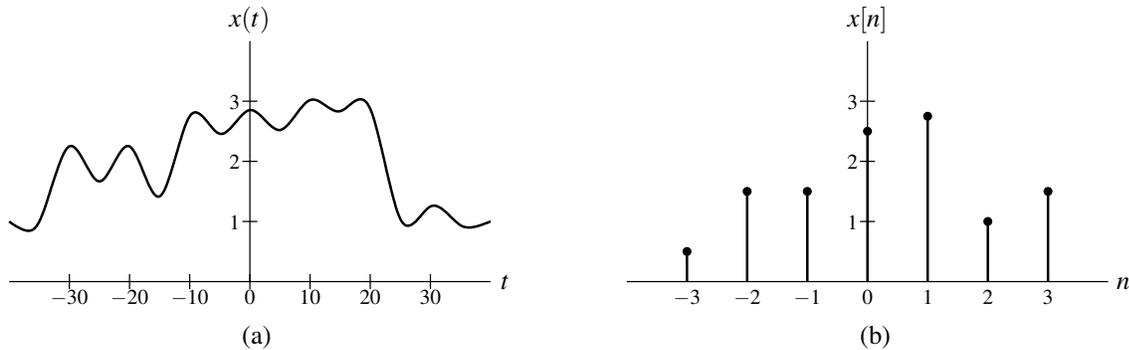
Figure 1.1: Graphical representations of (a) continuous-time and (b) discrete-time signals.
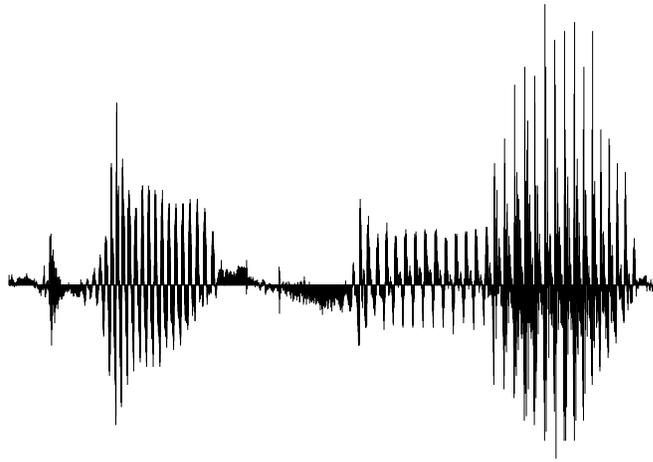
Figure 1.2: Segment of digitized human speech.

1.1.3   Notation and Graphical Representation of Signals

As a matter of notation, we typically write continuous-time signals with their independent variables enclosed in parentheses. For example, a continuous-time signal x with the real-valued independent variable t would be denoted $x(t)$. Discrete-time signals are written with their independent variables enclosed in brackets. For example, a discrete-time signal x with the integer-valued independent variable n would be denoted $x[n]$. This use of parentheses and brackets is a convention followed by much of the engineering literature. In the case of discrete-time signals, we sometimes refer to the signal as a **sequence**. Figure 1.1 shows how continuous-time and discrete-time signals are represented graphically.

1.1.4   Examples of Signals

A number of examples of signals have been suggested previously. Here, we provide some graphical representations of signals for illustrative purposes. Figure 1.2 depicts a digitized speech signal. Figure 1.3 shows an example of a monochromatic image. In this case, the signal represents light intensity as a function of two variables (i.e., horizontal and vertical position).

Figure 1.3: A monochromatic image.
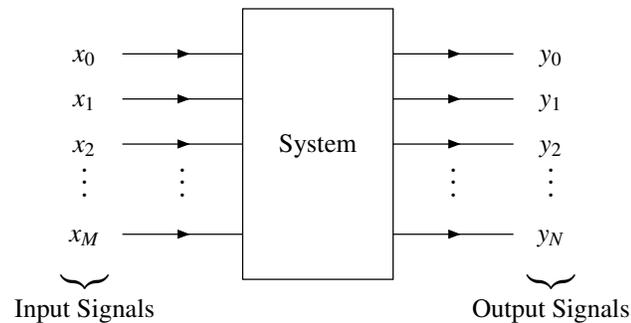
Figure 1.4: System with one or more inputs and one or more outputs.

1.2 Systems

A **system** is an entity that processes one or more input signals in order to produce one or more output signals, as shown in Figure 1.4. Such an entity is represented mathematically by a system of one or more equations.

In a communication system, the input might represent the message to be sent, and the output might represent the received message. In a robotics system, the input might represent the desired position of the end effector (e.g., gripper), while the output could represent the actual position.

1.2.1 Classification of Systems

A system can be classified based on the number of inputs and outputs it employs. A system with only one input is described as **single input**, while a system with multiple inputs is described as **multi-input**. Similarly, a system with only one output is said to be **single output**, while a system with multiple outputs is said to be **multi-output**. Two commonly occurring types of systems are single-input single-output (SISO) and multi-input multi-output (MIMO).

A system can also be classified based on the types of signals with which it interacts. A system that deals with continuous-time signals is called a **continuous-time system**. Similarly, a system that deals with discrete-time signals is said to be a **discrete-time system**. A system that handles both continuous- and discrete-time signals, is sometimes
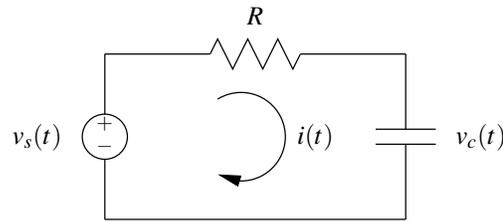
Figure 1.5: A simple RC network.
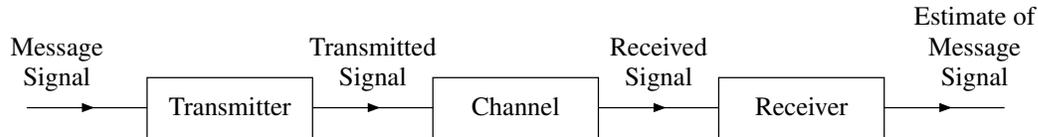
Figure 1.6: Communication system.
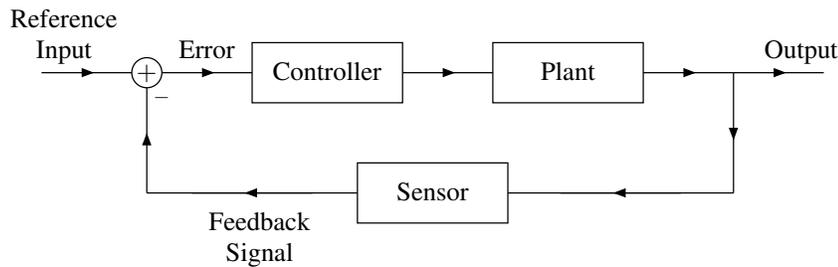
Figure 1.7: Feedback control system.

referred to as a **hybrid system** (or sampled-data system). Similarly, systems that deal with digital signals are referred to as **digital**, while systems that handle analog signals are referred to as **analog**. If a system interacts with 1D signals the system is referred to as **1D**. Likewise, if a system handles multidimensional signals, the system is said to be **multidimensional**.

1.2.2 Examples of Systems

Systems can manipulate signals in many different ways and serve many useful purposes. Sometimes systems serve to extract information from their input signals. For example, in case of speech signals, systems can be used in order to perform speaker identification or voice recognition. A system might analyze electrocardiogram signals in order to detect heart abnormalities. Amplification and noise reduction are other functionalities that systems could offer.

One very basic system is the RC network shown in Figure 1.5. Here, the input would be the source voltage $v_s(t)$ and the output would be the capacitor voltage $v_c(t)$.

Consider the communication system shown in Figure 1.6. This system takes a message at one location and reproduces this message at another location. In this case, the system input is the message to be sent, and the output is the estimate of the original message. Usually, we want the message reproduced at the receiver to be as close as possible to the original message sent by the transmitter.

A system of the general form shown in Figure 1.7 frequently appears in control applications. Often, in such applications, we would like an output to track some reference input as closely as possible. Consider, for example, a robotics application. The reference input might represent the desired position of the end effector, while the output represents the actual position.

1.3 Continuous-Time Signals and Systems

In the remainder of this course, we will focus our attention primarily on the study of 1D continuous-time signals and systems. Moreover, we will mostly concern ourselves with SISO (i.e., single-input single-output) systems. The discrete-time and multidimensional cases will be treated in other courses.

1.4 Why Study Signals and Systems?

As can be seen from the earlier examples, there are many practical situations in which we need to develop systems that manipulate/process signals. In order to do this, we need a formal mathematical framework for the study of such systems. The goal of this course is to provide the student with such a framework.

Chapter 2

Continuous-Time Signals and Systems

2.1 Transformations of the Independent Variable

An important concept in the study of signals and systems is the transformation of a signal. Here, we introduce several elementary signal transformations. Each of these transformations involves a simple modification of the independent variable.

2.1.1 Time Reversal

The first type of signal transformation that we shall consider is known as **time reversal**. Time reversal maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(-t). \tag{2.1}$$

In other words, the output signal $y(t)$ is formed by replacing t with $-t$ in the input signal $x(t)$. Geometrically, the output signal $y(t)$ is a reflection of the input signal $x(t)$ about the (vertical) line $t = 0$.

To illustrate the effects of time reversal, an example is provided in Figure 2.1. Suppose that we have the signal in Figure 2.1(a). Then, time reversal will yield the signal in Figure 2.1(b).

2.1.2 Time Scaling

Another type of signal transformation is called **time scaling**. Time scaling maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(at), \tag{2.2}$$
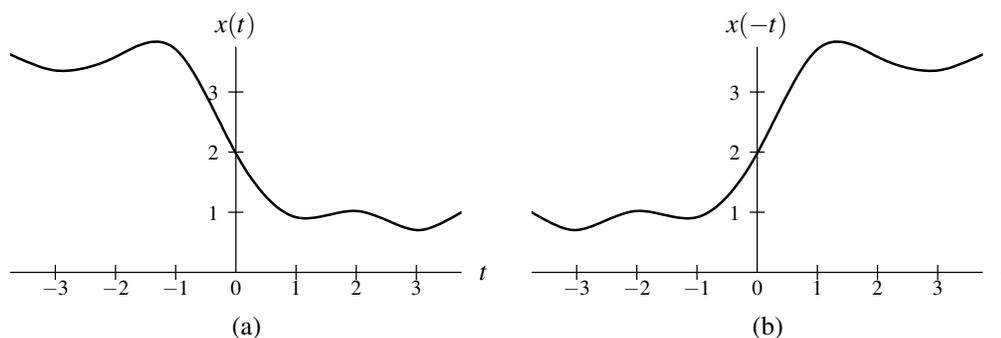
Figure 2.1: Example of time reversal.
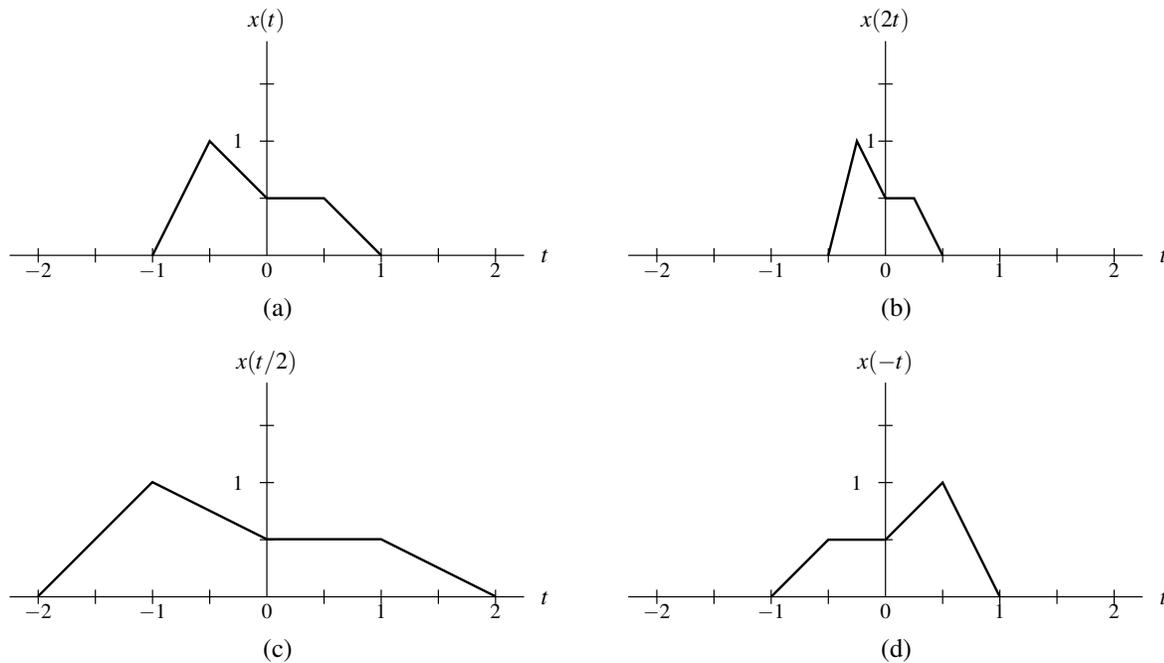
Figure 2.2: Example of time scaling.

where a is a scaling constant. In other words, the output signal $y(t)$ is formed by replacing t with at in the original signal $x(t)$. Geometrically, this transformation is associated with a compression/expansion along the time axis and/or reflection about the (vertical) line $t = 0$. If $|a| < 1$, the signal is expanded (i.e., stretched) along the time axis. If $|a| > 1$, the signal is instead compressed. If $|a| = 1$, the signal is neither expanded nor compressed. Lastly, if $a < 0$, the signal is reflected about the vertical line $t = 0$. Observe that time reversal is simply a special case of time scaling with $a = -1$.

To demonstrate the behavior of time scaling, we provide an example in Figure 2.2. Suppose that we have the signal $x(t)$ as shown in Figure 2.2(a). Then, using time scaling, we can obtain the signals in Figures 2.2(b), (c), and (d).

2.1.3 Time Shifting

The next type of signal transformation that we consider is called **time shifting**. Time shifting maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(t - b), \tag{2.3}$$

where b is a shifting constant. In other words, the output signal $y(t)$ is formed by replacing t by $t - b$ in input signal $x(t)$. Geometrically, this operation shifts the signal (to the left or right) along the time axis. If b is positive, $y(t)$ is shifted to the right relative to $x(t)$ (i.e., delayed in time). If b is negative, $y(t)$ is shifted to the left relative to $x(t)$ (i.e., advanced in time).

The effects of time shifting are illustrated in Figure 2.3. Suppose that we have the input signal $x(t)$ as shown in Figure 2.3(a). Then, the signals in Figures 2.3(b) and (c) can be obtained through time shifting.

2.1.4 Combining Time Scaling and Time Shifting

In the preceding sections, we introduced the time reversal, time scaling, and time shifting transformations. Moreover, we observed that time reversal is simply a special case of time scaling. Now, we introduce a more general transformation that combines the effects of time scaling, time shifting, and time reversal. This new transformation maps the
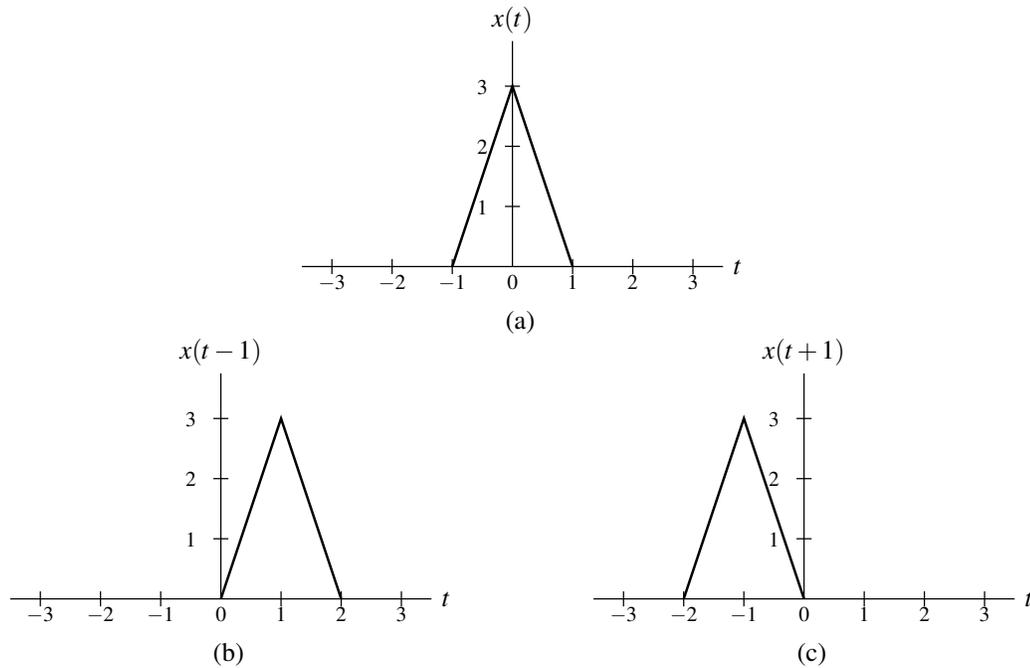
Figure 2.3: Example of time shifting.

input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(at - b), \tag{2.4}$$

where a is a scaling constant and b is a shifting constant. In other words, the output signal $y(t)$ is formed by replacing t with $at - b$ in the input signal $x(t)$. One can show that such a transformation is equivalent to first time shifting $x(t)$ by b, and then time scaling the resulting signal by a. Geometrically, this transformation preserves the shape of $x(t)$ except for a possible expansion/compression along the time axis and/or a reflection about the vertical line $t = 0$. If $|a| < 1$, the signal is stretched along the time axis. If $|a| > 1$, the signal is instead compressed. If $a < 0$, the signal is reflected about the vertical line $t = 0$.

The above transformation has two distinct but equivalent interpretations. That is, it is equivalent to each of the following:

1. first, time shifting $x(t)$ by b, and then time scaling the result by a.

2. first, time scaling $x(t)$ by a, and then time shifting the result by b/a.

One can easily confirm that both of the above equivalences hold, as shown below.

Consider the first case. First, we time shift $x(t)$ by b. Let us denote the result of this operation as $v(t)$, so that $v(t) = x(t - b)$. Next, we time scale $v(t)$ by a. This yields the result $v(at)$. From the definition of $v(t)$, however, we have $v(at) = x(at - b) = y(t)$.

Consider the second case. First, we time scale $x(t)$ by a. Let us denote the result of this operation as $v(t)$, so that $v(t) = x(at)$. Next, we time shift $v(t)$ by b/a. This yields the result $v(t - \frac{b}{a})$. From the definition of $v(t)$, however, we can write $v(t - \frac{b}{a}) = x\left(a(t - \frac{b}{a})\right) = x(at - b) = y(t)$.

Example 2.1. To illustrate the equivalent interpretations, we consider a simple example. Consider the signal $x(t)$ shown in Figure 2.4(a). Let us now determine the transformed signal $y(t) = x(at - b)$ where $a = 2$ and $b = 1$.

Solution. First, we consider the shift-then-scale method. In this case, we first shift the signal $x(t)$ by b (i.e., 1). This yields the signal in Figure 2.4(b). Then, we scale this new signal by a (i.e., 2) in order to obtain $y(t)$ as shown in Figure 2.4(c).
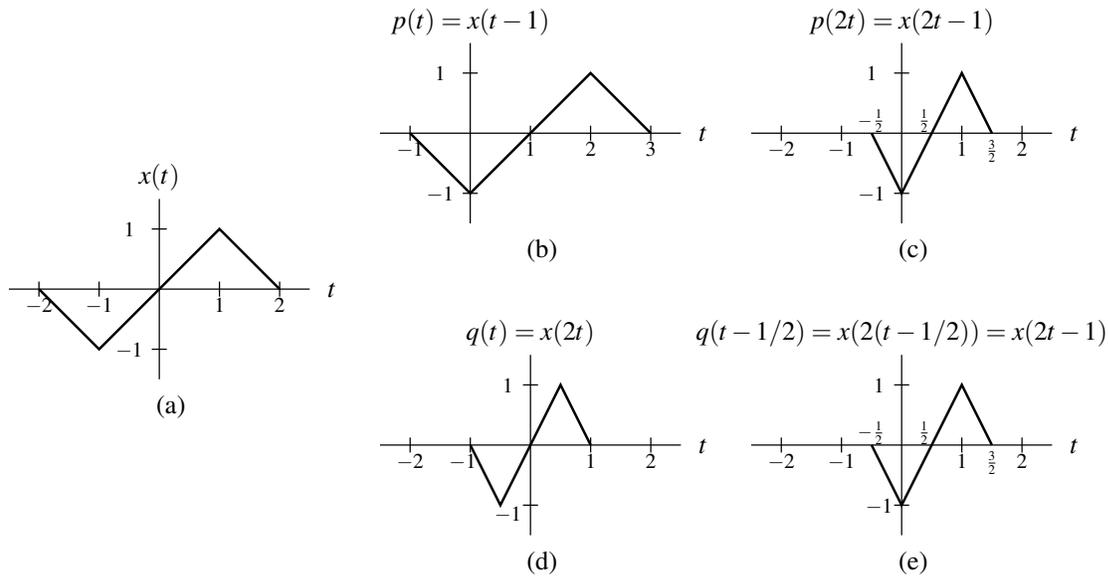
Figure 2.4: Two different interpretations of combined shifting and scaling transformation. (a) Original signal. Results obtained by shifting followed by scaling: (b) intermediate result and (c) final result. Results obtained by scaling followed by shifting: (d) intermediate result and (e) final result.

Second, we consider the scale-then-shift method. In this case, we first scale the signal $x(t)$ by a (i.e., 2). This yields the signal in Figure 2.4(d). Then, we shift this new signal by $\frac{b}{a}$ (i.e., $\frac{1}{2}$) in order to obtain $y(t)$ as shown in Figure 2.4(e).

□

2.2   Transformations of the Dependent Variable

In the preceding sections, we examined several transformations of the independent variable. Now, we consider similar transformations of the dependent variable.

2.2.1   Amplitude Scaling

The first transformation that we consider is referred to as **amplitude scaling**. Amplitude scaling maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = ax(t), \tag{2.5}$$

where a is a scaling constant. Geometrically, the output signal $y(t)$ is expanded/compressed in amplitude and/or reflected about the time axis. An amplifier is an example of a device that performs amplitude scaling.

To illustrate the effects of amplitude scaling, an example is given in Figure 2.5. Suppose that we have the signal $x(t)$ as shown in Figure 2.5(a). Then, amplitude scaling can be used to obtain the signals in Figures 2.5(b), (c), and (d).

2.2.2   Amplitude Shifting

The next transformation that we consider is referred to as **amplitude shifting**. Amplitude shifting maps the input signal $x(t)$ to the output signal $y(t)$ as given by

$$y(t) = x(t) + b, \tag{2.6}$$

where b is a shifting constant. Geometrically, amplitude shifting adds a vertical displacement to $x(t)$.
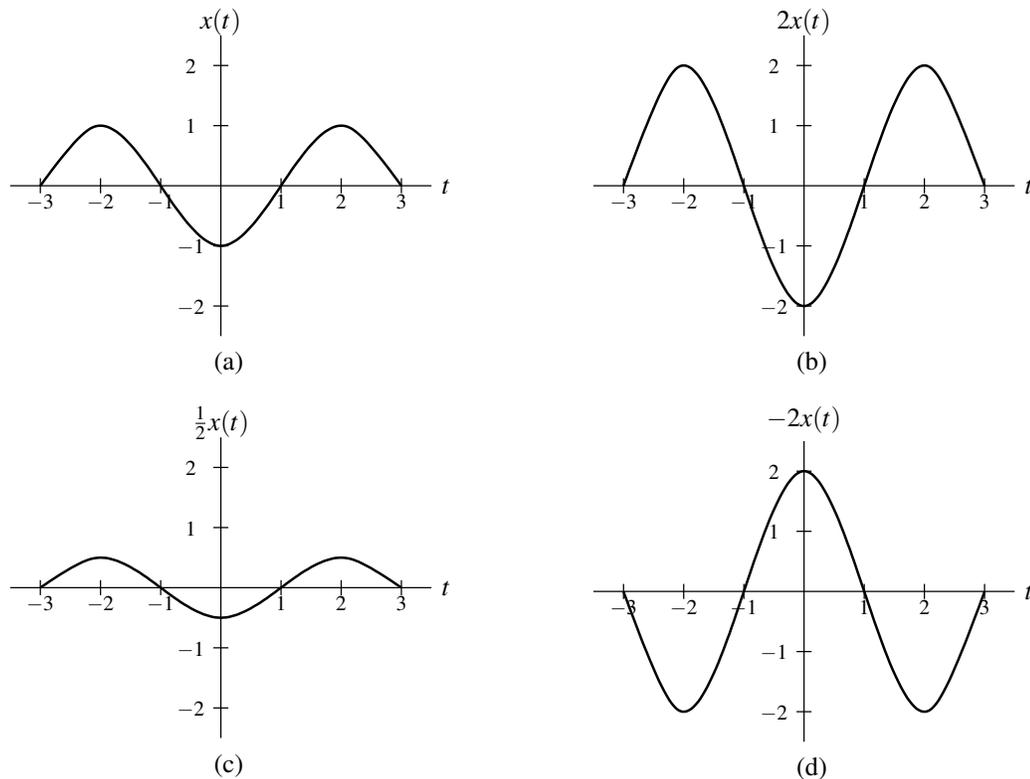
Figure 2.5: Example of amplitude scaling.

The effects of amplitude shifting are illustrated by Figure 2.6. The original signal $x(t)$ is given in Figure 2.6(a), and an amplitude-shifted version in Figure 2.6(b).

2.2.3 Combining Amplitude Scaling and Shifting

In the previous sections, we considered the amplitude-scaling and amplitude-shifting transformations. We can define a new transformation that combines the effects amplitude-scaling and amplitude-shifting. This transformation maps the input signal $x(t)$ to the output signal $y(t)$, as given by

$$y(t) = ax(t) + b, \tag{2.7}$$

where a is a scaling constant and b is a shifting constant. One can show that this transformation is equivalent to first scaling $x(t)$ by a and then shifting the resulting signal by b.

2.3 Signal Properties

Signals can possess a number of interesting properties. In what follows, we define several such properties. These properties are frequently useful in the analysis of signals and systems.

2.3.1 Even and Odd Signals

A signal $x(t)$ is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t. \tag{2.8}$$

(a)

(b)
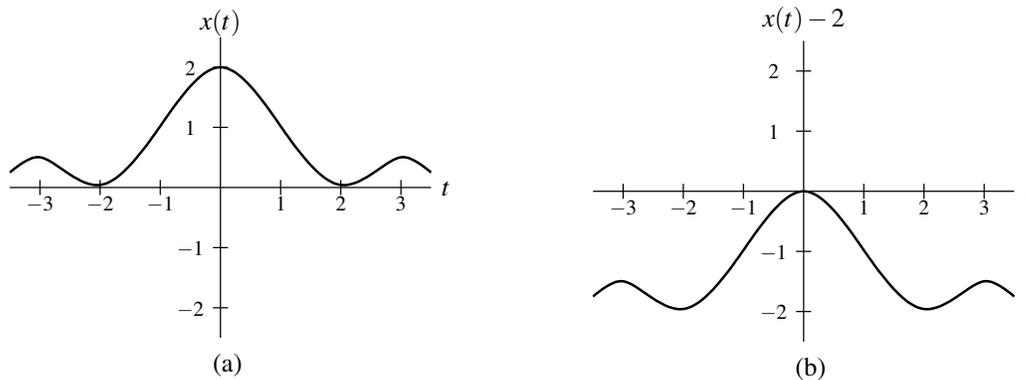
Figure 2.6: Example of amplitude shifting.
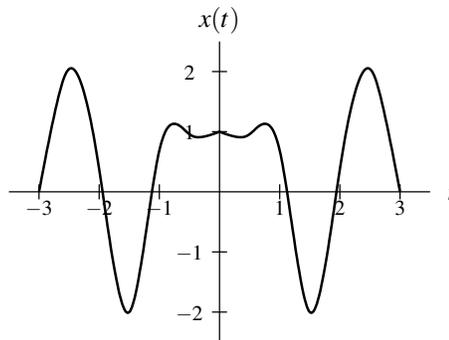
Figure 2.7: Example of an even signal.

Geometrically, an even signal is symmetric about the vertical line $t = 0$. Some common examples of even signals include the cosine, absolute value, and square functions. Another example of an even signal is given in Figure 2.7.

A signal $x(t)$ is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t. \tag{2.9}$$

Geometrically, an odd signal is antisymmetric about $t = 0$. In other words, the signal has symmetry with respect to the origin. One can easily show that an odd signal $x(t)$ must be such that $x(0) = 0$. Some common examples of odd signals include the sine, signum, and cube (i.e., $x(t) = t^3$) functions. Another example of an odd signal is given in Figure 2.8.

Any signal $x(t)$ can be represented as the sum of the form

$$x(t) = x_e(t) + x_o(t) \tag{2.10}$$

where $x_e(t)$ and $x_o(t)$ are even and odd, respectively, and given by

$$x_e(t) = \tfrac{1}{2}(x(t) + x(-t)) \quad \text{and} \tag{2.11}$$

$$x_o(t) = \tfrac{1}{2}(x(t) - x(-t)). \tag{2.12}$$

Often, we denote the even and odd parts of $x(t)$ as $\text{Even}\{x(t)\}$ and $\text{Odd}\{x(t)\}$, respectively. From (2.11) and (2.12), we can easily confirm that $x_e(t) + x_o(t) = x(t)$ as follows:

$$\begin{aligned}
x_e(t) + x_o(t) &= \tfrac{1}{2}(x(t) + x(-t)) + \tfrac{1}{2}(x(t) - x(-t)) \\
&= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) + \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \\
&= x(t).
\end{aligned}$$
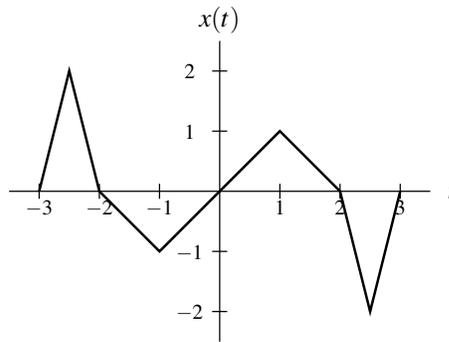
Figure 2.8: Example of an odd signal.

Furthermore, we can easily verify that $x_e(t)$ is even and $x_o(t)$ is odd. From the definition of $x_e(t)$ in (2.11), we can write:

$$x_e(-t) = \tfrac{1}{2}(x(-t) + x(-(-t)))$$
$$= \tfrac{1}{2}(x(t) + x(-t))$$
$$= x_e(t).$$

Thus, $x_e(t)$ is even. From the definition of $x_o(t)$ in (2.12), we can write:

$$x_o(-t) = \tfrac{1}{2}(x(-t) - x(t))$$
$$= \tfrac{1}{2}(-x(t) + x(-t))$$
$$= -x_o(t).$$

Thus, $x_o(t)$ is odd.

Sums involving even and odd signals have the following properties:

- The sum of two even signals is even.

- The sum of two odd signals is odd.

- The sum of an even signal and odd signal is neither even nor odd.

Products involving even and odd signals have the following properties:

- The product of two even signals is even.

- The product of two odd signals is even.

- The product of an even signal and an odd signal is odd.

2.3.2 Periodic Signals

A signal $x(t)$ is said to be **periodic** if it satisfies

$$x(t) = x(t+T), \quad \text{for all } t \text{ and some constant } T, T > 0. \tag{2.13}$$

The quantity T is referred to as the **period** of the signal. Two quantities closely related to the period are the **frequency** and **angular frequency**, denoted as f and ω, respectively. These quantities are defined as

$$f = \tfrac{1}{T} \quad \text{and} \quad \omega = 2\pi f = \tfrac{2\pi}{T}.$$
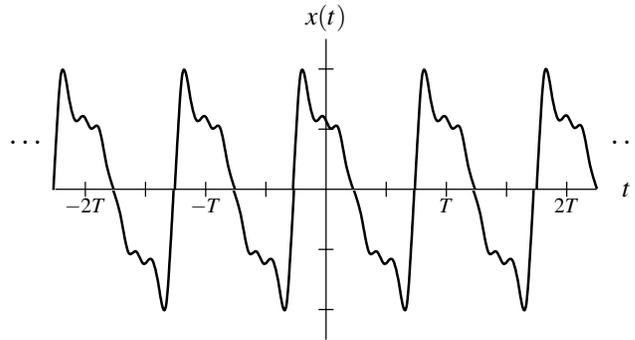
Figure 2.9: Example of a periodic signal.

A signal that is not periodic is said to be **aperiodic**.

Examples of periodic signals include the cosine and sine functions. Another example of a periodic signal is shown in Figure 2.9.

A signal that satisfies (2.13) must also satisfy

$$x(t) = x(t + NT) \tag{2.14}$$

for any integer value N. Therefore, a signal that is periodic with period T is also periodic with period NT. In most cases, we are interested in the smallest (positive) value of T for which (2.13) is satisfied. We refer to this value as the **fundamental period** of the signal.

Theorem 2.1 (Sum of periodic functions). *Suppose that $x_1(t)$ and $x_2(t)$ are periodic signals with fundamental periods T_1 and T_2, respectively. Then, the sum $y(t) = x_1(t) + x_2(t)$ is a periodic signal if and only if the ratio T_1/T_2 is a rational number (i.e., the quotient of two integers). Suppose that $T_1/T_2 = q/r$ where q and r are integers and coprime (i.e., have no common factors), then the fundamental period of $y(t)$ is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)*

In passing, we note that the above result can be extended to the more general case of the sum of N periodic signals. The sum of N periodic signals $x_1(t), x_2(t), \ldots, x_N(t)$ with periods $T_1, T_2, \ldots, T_N$, respectively, is periodic if and only if the ratios of the periods are rational numbers (i.e., T_1/T_k is rational for $k = 2, 3, \ldots, N$). If the sum is periodic, then the fundamental period is simply the least common multiple of $\{T_1, T_2, \ldots, T_N\}$. (Note that the **least common multiple** of the set of positive real numbers $\{T_1, T_2, \ldots, T_N\}$ is the smallest positive real number T that is an integer multiple of each T_k for $k = 1, 2, \ldots, N$.)

Example 2.2. Let $x_1(t) = \sin \pi t$ and $x_2(t) = \sin t$. Determine whether the signal $y(t) = x_1(t) + x_2(t)$ is periodic.

Solution. Denote the fundamental periods of $x_1(t)$ and $x_2(t)$ as T_1 and T_2, respectively. We then have $T_1 = 2\pi/\pi = 2$ and $T_2 = 2\pi/1 = 2\pi$. (Recall that the fundamental period of $\sin \alpha t$ is $2\pi/|\alpha|$.) Consider the quotient $T_1/T_2 = 2/(2\pi) = 1/\pi$. Since π is an irrational number, this quotient is not rational. Therefore, $y(t)$ is not periodic. □

Example 2.3. Let $x_1(t) = \cos(2\pi t + \frac{\pi}{4})$ and $x_2(t) = \sin(7\pi t)$. Determine whether the signal $y(t) = x_1(t) + x_2(t)$ is periodic.

Solution. Let T_1 and T_2 denote the fundamental periods of $x_1(t)$ and $x_2(t)$, respectively. Thus, we have $T_1 = \frac{2\pi}{2\pi} = 1$ and $T_2 = \frac{2\pi}{7\pi} = 2/7$. We can express T_1/T_2 as the rational number $7/2$. Therefore, $y(t)$ is periodic. Furthermore, the period is $2T_1 = 7T_2 = 2$ (since 2 and 7 are coprime). □

Example 2.4. Let $x_1(t) = \cos(6\pi t)$ and $x_2(t) = \sin(30\pi t)$. Determine whether the signal $y(t) = x_1(t) + x_2(t)$ is periodic.
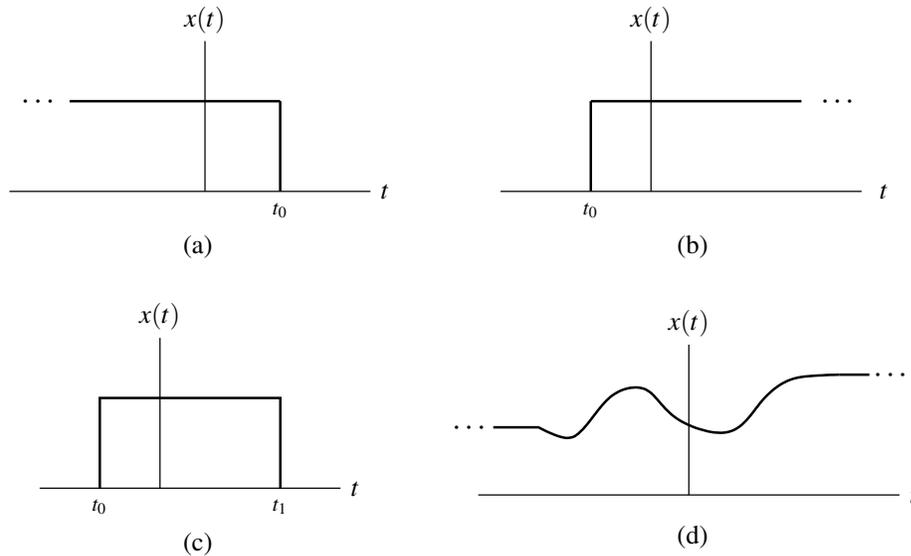
Figure 2.10: Examples of (a) left-sided, (b) right-sided, (c) finite-duration, and (d) two-sided signals.

Solution. Let T_1 and T_2 denote the fundamental periods of $x_1(t)$ and $x_2(t)$, respectively. Thus, we have $T_1 = \frac{2\pi}{6\pi} = \frac{1}{3}$ and $T_2 = \frac{2\pi}{30\pi} = \frac{1}{15}$. We can express T_1/T_2 as the rational number $5/1$ (where 5 and 1 are coprime). Therefore, $y(t)$ is periodic. Furthermore, the period is $T = 1T_1 = 5T_2 = \frac{1}{3}$ (since 5 and 1 are coprime). $\square$

2.3.3 Support of Signals

We can classify signals based on the interval over which their function value is nonzero. (This is sometimes referred to as the support of a signal.)

A signal $x(t)$ is said to be **left sided** if, for some finite constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0.$$

In other words, the signal is only potentially nonzero to the left of some point. A signal $x(t)$ is said to be **right sided** if, for some finite constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0.$$

In other words, the signal is only potentially nonzero to the right of some point. A signal that is both left sided and right sided is said to be **time limited** or **finite duration**. A signal that is neither left sided nor right sided is said to be **two sided**. Examples of left-sided, right-sided, finite-duration, and two-sided signals are shown in Figure 2.10.

A signal $x(t)$ is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

A causal signal is a special case of a right-sided signal. Similarly, a signal $x(t)$ is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

An anticausal signal is a special case of a left-sided signal.

2.3.4 Signal Energy and Power

The **energy** E contained in the signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

As a matter of terminology, a signal $x(t)$ with finite energy is said to be an **energy signal**.

The **average power** P contained in the signal $x(t)$ is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

As a matter of terminology, a signal $x(t)$ with (nonzero) finite average power is said to be a **power signal**.

2.3.5 Examples

Example 2.5. Suppose that we have a signal $x(t)$ with the following properties:

$$x(t-3) \text{ is causal, and}$$
$$x(t) \text{ is odd.}$$

Determine for what values of t the signal $x(t)$ must be zero.

Solution. Since $x(t-3)$ is causal, we know that $x(t-3) = 0$ for $t < 0$. This implies that

$$x(t) = 0 \text{ for } t < -3. \tag{2.15}$$

Since $x(t)$ is odd, we know that

$$x(t) = -x(-t) \text{ for all } t. \tag{2.16}$$

From (2.15) and (2.16), we have $-x(-t) = 0$ for $t < -3$ which implies that

$$x(t) = 0 \text{ for } t > 3. \tag{2.17}$$

Substituting $t = 0$ into (2.16) yields $x(0) = -x(0)$ which implies that

$$x(0) = 0. \tag{2.18}$$

Combining (2.15), (2.17), and (2.18), we conclude that $x(t)$ must be zero for

$$t < -3, \quad t > 3, \quad \text{or} \quad t = 0.$$

$\square$

Example 2.6. Suppose that we have a signal $x(t)$ with the following properties:

$$x(t-3) \text{ is even,}$$
$$x(t) = t + 5 \text{ for } -5 \le t \le -3,$$
$$x(t-1) \text{ is anticausal, and}$$
$$x(t-5) \text{ is causal.}$$

Find $x(t)$ for all t.

Solution. Since $x(t-1)$ is anticausal, we know that

$$x(t-1) = 0 \text{ for } t > 0.$$

This implies that

$$x(t) = 0 \text{ for } t > -1. \tag{2.19}$$

Since $x(t-5)$ is causal, we know that

$$x(t-5) = 0 \text{ for } t < 0.$$

This implies that

$$x(t) = 0 \text{ for } t < -5. \tag{2.20}$$

We are given that

$$x(t) = t+5 \text{ for } -5 \leq t \leq -3. \tag{2.21}$$

Combining (2.19), (2.20), and (2.21), we have

$$x(t) = \begin{cases} 0 & \text{for } t < -5 \\ t+5 & \text{for } -5 \leq t \leq -3 \\ 0 & \text{for } t > -1. \end{cases} \tag{2.22}$$

So, we only need to determine $x(t)$ for $-3 \leq t \leq -1$. Since $x(t-3)$ is even, we know that

$$x(t-3) = x(-t-3) \text{ for all } t.$$

This implies that

$$x(t) = x(-t-6) \text{ for all } t. \tag{2.23}$$

Combining (2.21) and (2.23), we have

$$\begin{aligned} x(t) &= x(-t-6) \\ &= (-t-6)+5 \text{ for } -5 \leq -t-6 \leq -3 \\ &= -t-1 \text{ for } 1 \leq -t \leq 3 \\ &= -t-1 \text{ for } -3 \leq t \leq -1. \end{aligned}$$

Combining this with our earlier result (2.22), we have

$$x(t) = \begin{cases} 0 & \text{for } t < -5 \\ t+5 & \text{for } -5 \leq t < -3 \\ -t-1 & \text{for } -3 \leq t < -1 \\ 0 & \text{for } t \geq -1. \end{cases}$$

$\square$

2.4 Elementary Signals

A number of elementary signals are particularly useful in the study of signals and systems. In what follows, we introduce some of the more beneficial ones for our purposes.

2.4.1 Real Sinusoidal Signals

One important class of signals is the real sinusoids. A **real sinusoidal** signal $x(t)$ has the general form

$$x(t) = A\cos(\omega t + \theta), \tag{2.24}$$

where A, ω, and θ are real constants. Such a signal is periodic with period $T = \frac{2\pi}{|\omega|}$, and has a plot resembling that shown in Figure 2.11.
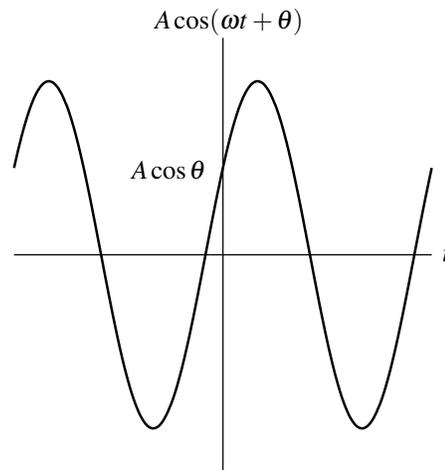
Figure 2.11: Real sinusoidal signal.
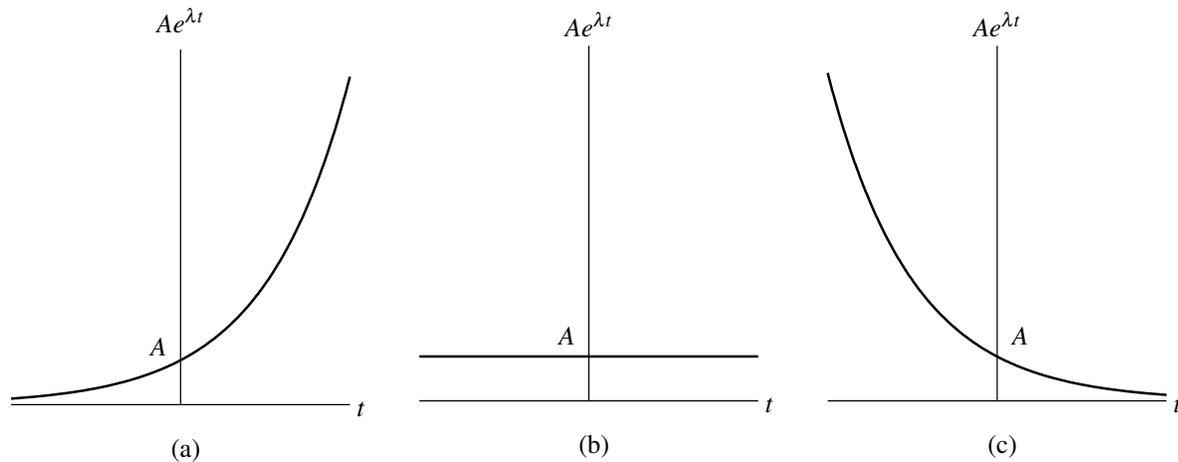
Figure 2.12: Real exponential signal for (a) $\lambda > 0$, (b) $\lambda = 0$, and (c) $\lambda < 0$.

2.4.2   Complex Exponential Signals

Another important class of signals is the complex exponentials signals. A **complex exponential** signal $x(t)$ has the general form

$$x(t) = Ae^{\lambda t}, \tag{2.25}$$

where A and λ are complex constants. Complex exponentials are of fundamental importance to systems theory, and also provide a convenient means for representing a number of other classes of signals. A complex exponential can exhibit one of a number of distinctive modes of behavior, depending on the values of its parameters A and λ. In what follows, we examine some special cases of complex exponentials, in addition to the general case.

Real Exponential Signals

The first special case of the complex exponential signals to be considered is the **real exponential** signals. In this case, we restrict A and λ in (2.25) to be real. A real exponential can exhibit one of three distinct modes of behavior, depending on the value of λ, as illustrated in Figure 2.12. If $\lambda > 0$, the signal $x(t)$ increases exponentially as time increases (i.e., a growing exponential). If $\lambda < 0$, the signal $x(t)$ decreases exponentially as time increases (i.e., a decaying or damped exponential). If $\lambda = 0$, the signal $x(t)$ simply equals the constant A.
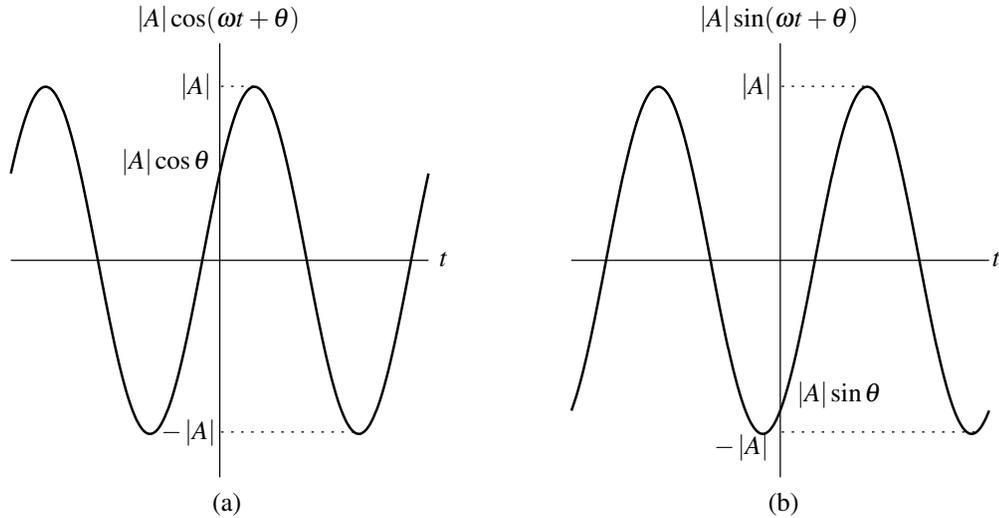
Figure 2.13: Complex sinusoidal signal. (a) Real and (b) imaginary parts.

Complex Sinusoidal Signals

The second special case of the complex exponential signals that we shall consider is the **complex sinusoidal** signals. In this case, the parameters in (2.25) are such that A is complex and λ is purely imaginary (i.e., $\text{Re}\{\lambda\} = 0$). For convenience, let us re-express A in polar form and λ in Cartesian form as follows:

$$A = |A|e^{j\theta} \quad \text{and} \quad \lambda = j\omega,$$

where θ and ω are real constants. Using Euler's relation (A.6), we can rewrite (2.25) as

$$\begin{aligned}
x(t) &= Ae^{\lambda t} \\
&= |A|e^{j\theta}e^{j\omega t} \\
&= |A|e^{j(\omega t + \theta)} \\
&= |A|\cos(\omega t + \theta) + j|A|\sin(\omega t + \theta).
\end{aligned}$$

From the above equation, we can see that $x(t)$ is periodic with period $T = \frac{2\pi}{|\omega|}$. Similarly, the real and imaginary parts of $x(t)$ are also periodic with the same period. To illustrate the form of a complex sinusoid, we plot its real and imaginary parts in Figure 2.13. The real and imaginary parts are the same except for a phase difference.

General Complex Exponential Signals

Lastly, we consider general complex exponential signals. That is, we consider the general case of (2.25) where A and λ are both complex. For convenience, let us re-express A in polar form and λ in Cartesian form as

$$A = |A|e^{j\theta} \quad \text{and} \quad \lambda = \sigma + j\omega,$$

where θ, σ, and ω are real constants. Substituting these expressions for A and λ into (2.25), we obtain

$$\begin{aligned}
x(t) &= Ae^{\lambda t} \\
&= |A|e^{j\theta}e^{(\sigma + j\omega)t} \\
&= |A|e^{\sigma t}e^{j(\omega t + \theta)} \\
&= |A|e^{\sigma t}\cos(\omega t + \theta) + j|A|e^{\sigma t}\sin(\omega t + \theta).
\end{aligned}$$
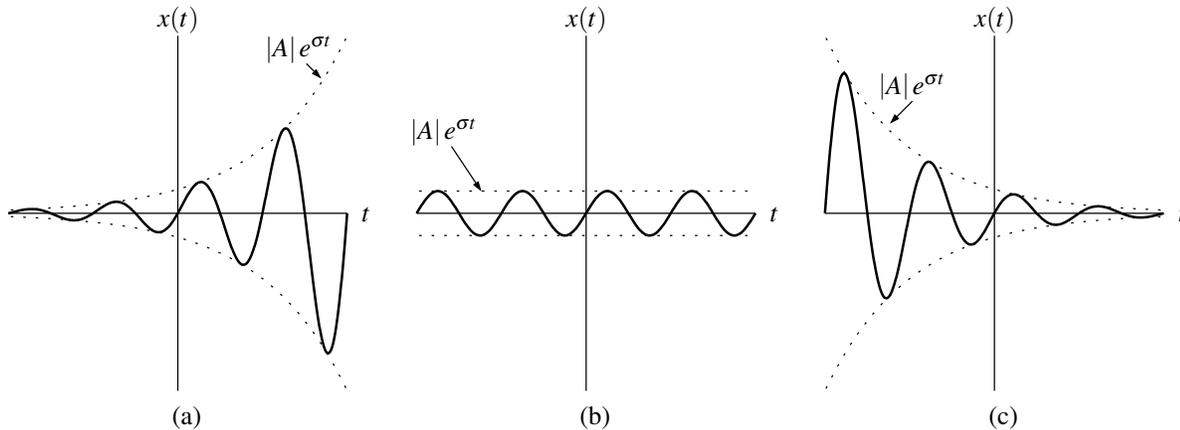
Figure 2.14: Real part of a general complex exponential for (a) $\sigma > 0$, (b) $\sigma = 0$, and (c) $\sigma < 0$.

We can see that $\text{Re}\{x(t)\}$ and $\text{Im}\{x(t)\}$ have a similar form. Each is the product of a real exponential and real sinusoidal function. One of three distinct modes of behavior is exhibited by $x(t)$, depending on the value of σ. If $\sigma = 0$, $\text{Re}\{x(t)\}$ and $\text{Im}\{x(t)\}$ are real sinusoids. If $\sigma > 0$, $\text{Re}\{x(t)\}$ and $\text{Im}\{x(t)\}$ are each the product of a real sinusoid and a growing real exponential. If $\sigma < 0$, $\text{Re}\{x(t)\}$ and $\text{Im}\{x(t)\}$ are each the product of a real sinusoid and a decaying real exponential. These three cases are illustrated for $\text{Re}\{x(t)\}$ in Figure 2.14.

2.4.3   Relationship Between Complex Exponential and Real Sinusoidal Signals

A real sinusoid can be expressed as the sum of two complex sinusoids using the identity

$$A\cos(\omega t + \theta) = \frac{A}{2}\left(e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}\right) \quad \text{and} \tag{2.26}$$

$$A\sin(\omega t + \theta) = \frac{A}{2j}\left(e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}\right). \tag{2.27}$$

(This result follows from Euler's relation and is simply a restatement of (A.7).)

2.4.4   Unit-Step Function

Another elementary signal often used in systems theory is the unit-step function. The **unit-step function**, denoted $u(t)$, is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0. \end{cases} \tag{2.28}$$

A plot of this function is given in Figure 2.15. Clearly, the unit-step function is discontinuous at $t = 0$. Unfortunately, there is very little agreement in the existing literature as to how to define $u(t)$ at $t = 0$. The most commonly used values are 0, 1, and 1/2. In our treatment of the subject, we simply choose to leave the value of $u(t)$ unspecified at $t = 0$ with the implicit assumption that the value (whatever it happens to be) is finite. As it turns out, for most practical purposes, it does not matter how the unit-step function is defined at the origin (as long as its value is finite). (The unit-step function is also sometimes referred to as the **Heaviside function**.)
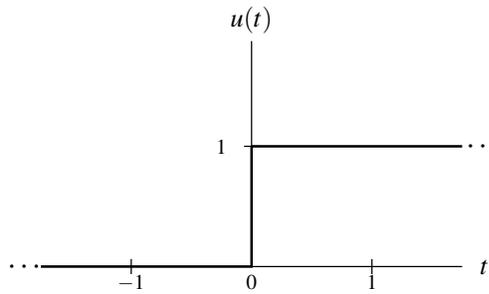
Figure 2.15: Unit-step function.
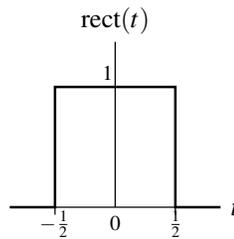
Figure 2.16: Unit rectangular pulse.

2.4.5 Unit Rectangular Pulse

Another useful signal is the unit rectangular pulse. The **unit rectangular pulse** function is denoted as $\text{rect}(t)$ and is given by

$$\text{rect}(t) = \begin{cases} 1 & \text{for } |t| < \frac{1}{2} \\ 0 & \text{for } |t| > \frac{1}{2}. \end{cases}$$

A plot of this signal is shown in Figure 2.16.

Example 2.7 (Extracting part of a function with a rectangular pulse)**.** Use the unit rectangular pulse to extract one period of the waveform $x(t)$ shown in Figure 2.17(a).

Solution. Let us choose to extract the period of $x(t)$ for $-\frac{T}{2} < t \leq \frac{T}{2}$. In order to extract this period, we want to multiply $x(t)$ by a function that is one over this interval and zero elsewhere. Such a function is simply $\text{rect}(t/T)$ as shown in Figure 2.17(b). Multiplying $\text{rect}(t/T)$ and $x(t)$ results in the function shown in Figure 2.17(c). $\square$

2.4.6 Unit Triangular Pulse

Another useful elementary function is the **unit triangular pulse** function, which is denoted as $\text{tri}(t)$ and defined as

$$\text{tri}(t) = \begin{cases} 1 - 2\,|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

A plot of this function is given in Figure 2.18.

(a)

(b)

(c)
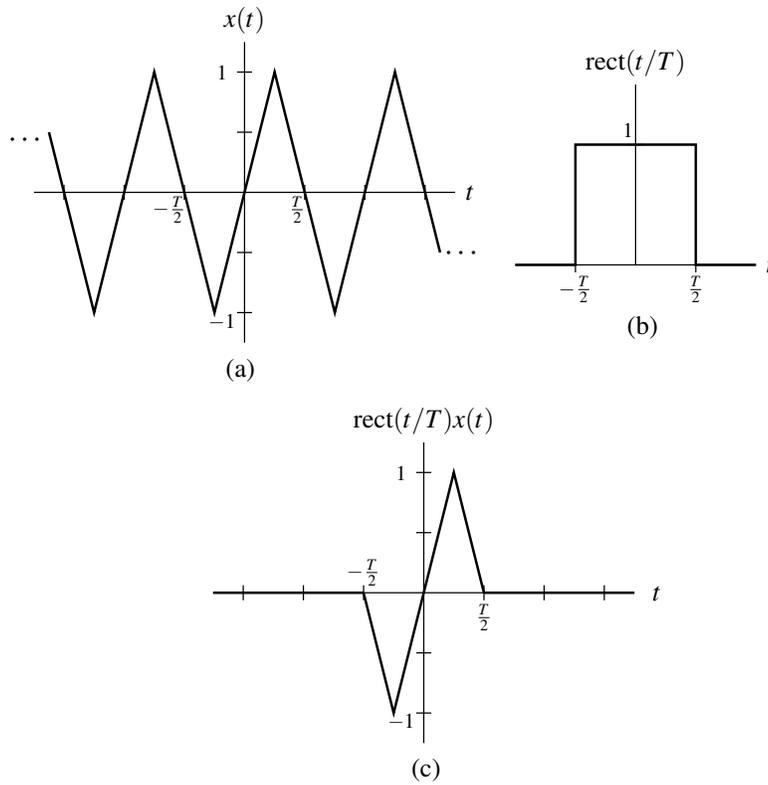
Figure 2.17: Using the unit rectangular pulse to extract one period of a periodic waveform.
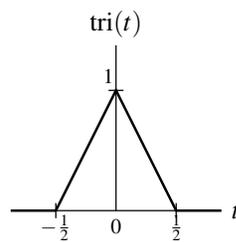
Figure 2.18: Unit triangular pulse.

2.4.7 Cardinal Sine Function

In the study of signals and systems, a function of the form

$$x(t) = \frac{\sin t}{t}$$

frequently appears. Therefore, as a matter of convenience, we give this particular function a special name, the **cardinal sine** or sinc function. That is, we define

$$\operatorname{sinc} t \triangleq \frac{\sin t}{t}.$$

(Note that the symbol "$\triangleq$" simply means "is defined as".) By using l'Hopital's rule, one can confirm that $\operatorname{sinc} t$ is well defined for $t = 0$. That is, $\operatorname{sinc} 0 = 1$. In passing, we note that the name "sinc" is simply a contraction of the function's full Latin name "sinus cardinalis" (cardinal sine).

2.4.8 Unit-Impulse Function

In systems theory, one elementary signal of fundamental importance is the unit-impulse function. We denote this function as $\delta(t)$. Instead of defining this function explicitly, it is defined in terms of its properties. In particular, the **unit-impulse function** is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and} \tag{2.29a}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1. \tag{2.29b}$$

From these properties, we can see that the function is zero everywhere, except at $t = 0$ where it is undefined. Indeed, this is an unusual function. Although it is zero everywhere except at a single point, it has a nonzero integral. Technically, the unit-impulse function is not a function in the ordinary sense. Rather, it is what is known as a **generalized function**. Sometimes the unit-impulse function is also referred to as the **Dirac delta function** or **delta function**.

Graphically, we represent the impulse function as shown in Figure 2.19. Since the function assumes an infinite value at $t = 0$, we cannot plot the true value of the function. Instead, we use a vertical arrow to represent this infinite spike in the value of the signal. To show the strength of the impulse, its weight is also indicated. In Figure 2.20, we plot a scaled and shifted version of the unit impulse function.

We can view the unit-impulse function as a limiting case involving a rectangular pulse. More specifically, let us define the following rectangular pulse function

$$g(t) = \begin{cases} 1/\varepsilon & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

This function is plotted in Figure 2.21. Clearly, the area under the curve is unity for any choice of ε. The unit impulse function $\delta(t)$ is obtained by taking the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g(t).$$

Thus, the unit-impulse function can be viewed as a limiting case of a rectangular pulse where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

Informally, one can also think of the unit-impulse function $\delta(t)$ as the derivative of the unit-step function $u(t)$. Strictly speaking, however, the derivative of $u(t)$ does not exist in the ordinary sense, since $u(t)$ is discontinuous at $t = 0$. To be more precise, $\delta(t)$ is what is called the **generalized derivative** of $u(t)$. The generalized derivative is essentially an extension of the notion of (ordinary) derivative, which can be well defined even for functions with discontinuities.
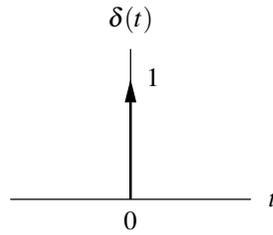
Figure 2.19: Unit impulse function.
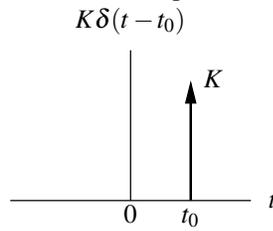
Figure 2.20: Scaled and shifted unit impulse function.
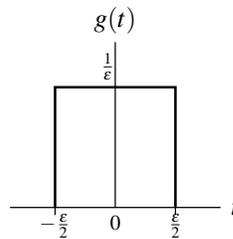
Figure 2.21: Unit-impulse function as limit of rectangular pulse.
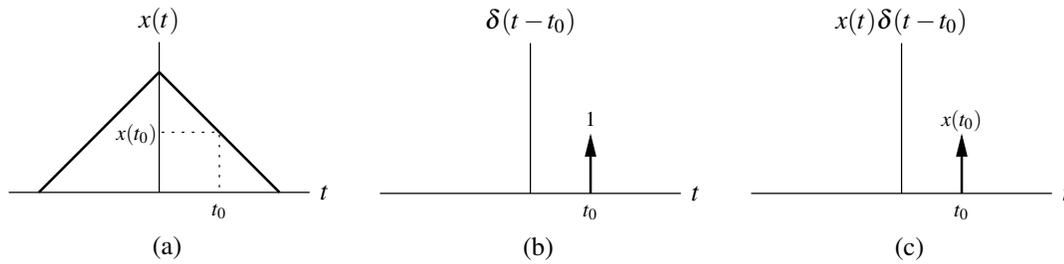
Figure 2.22: Graphical interpretation of equivalence property.

The unit-impulse function has two important properties that follow from its operational definition in (2.29). The first is sometimes referred to as the **equivalence property**. This property states that for any signal $x(t)$ that is continuous at $t = t_0$,

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0). \qquad (2.30)$$

This property is illustrated graphically in Figure 2.22. This result can be used to help derive the second property of interest. From (2.29b), we can write

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1.$$

Multiplying both sides of the preceding equation by $x(t_0)$ yields

$$\int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0).$$

Then, by using the equivalence property in (2.30), we can write

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0). \qquad (2.31)$$

This result (i.e., (2.31)) is known as the **sifting property** of the unit-impulse function. As we shall see later, this property is of great importance. In passing, we also note two other identities involving the unit-impulse function which are occasionally useful:

$$\delta(t) = \delta(-t) \quad \text{and}$$
$$\delta(at) = \tfrac{1}{|a|}\delta(t),$$

where a is a nonzero real constant. (Lastly, note that it follows from the definition of $\delta(t)$ that integrating the function over any interval not containing the origin will result in the value of zero.)

Example 2.8 (Sifting property example). Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin t]\delta(t - \pi/4)dt.$$

Solution. Using the sifting property of the unit impulse function, we can write:

$$\int_{-\infty}^{\infty} [\sin t]\delta(t - \pi/4)dt = \sin \pi/4$$
$$= \tfrac{1}{\sqrt{2}}.$$

$\square$

Example 2.9 (Sifting property example). Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin 2\pi t]\delta(4t - 1)dt.$$

Solution. First, we observe that the integral to be evaluated does not quite have the same form as (2.31). So, we need to perform a change of variable. Let $\tau = 4t$ so that $t = \tau/4$ and $dt = d\tau/4$. Performing the change of variable, we obtain

$$\int_{-\infty}^{\infty} [\sin 2\pi t] \delta(4t - 1) dt = \int_{-\infty}^{\infty} \tfrac{1}{4} [\sin 2\pi \tau/4] \delta(\tau - 1) d\tau$$
$$= \int_{-\infty}^{\infty} \left[\tfrac{1}{4} \sin \pi\tau/2 \right] \delta(\tau - 1) d\tau.$$

Now the integral has the desired form, and we can use the sifting property of the unit-impulse function to write

$$\int_{-\infty}^{\infty} [\sin 2\pi t] \delta(4t - 1) dt = \left. \left[\tfrac{1}{4} \sin \pi\tau/2 \right] \right|_{\tau=1}$$
$$= \tfrac{1}{4} \sin \pi/2$$
$$= \tfrac{1}{4}.$$

$\square$

Example 2.10. Evaluate the integral $\int_{-\infty}^{t} (\tau^2 + 1) \delta(\tau - 2) d\tau$.

Solution. Using the equivalence property of the delta function given by (2.30), we can write

$$\int_{-\infty}^{t} (\tau^2 + 1) \delta(\tau - 2) d\tau = \int_{-\infty}^{t} (2^2 + 1) \delta(\tau - 2) d\tau$$
$$= 5 \int_{-\infty}^{t} \delta(\tau - 2) d\tau.$$

Using the defining properties of the delta function given by (2.29), we have that

$$\int_{-\infty}^{t} \delta(\tau - 2) d\tau = \begin{cases} 1 & \text{for } t > 2 \\ 0 & \text{for } t < 2. \end{cases}$$

Therefore, we conclude that

$$\int_{-\infty}^{t} (\tau^2 + 1) \delta(\tau - 2) d\tau = \begin{cases} 5 & \text{for } t > 2 \\ 0 & \text{for } t < 2 \end{cases}$$
$$= 5u(t - 2).$$

$\square$

2.5 Signal Representation Using Elementary Signals

In the earlier sections, we introduced a number of elementary signals. Often in signal analysis, it is convenient to represent arbitrary signals in terms of elementary signals. Here, we consider how the unit-step function can be exploited in order to obtain alternative representations of signals.

Example 2.11 (Unit rectangular pulse). Express the function $y(t) = \text{rect}(t)$ in terms of unit-step functions. A plot of $y(t)$ is given in Figure 2.23(c).

Solution. We observe that the unit rectangular pulse is simply the difference of two time-shifted unit-step functions. In particular, we have that

$$y(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2}).$$
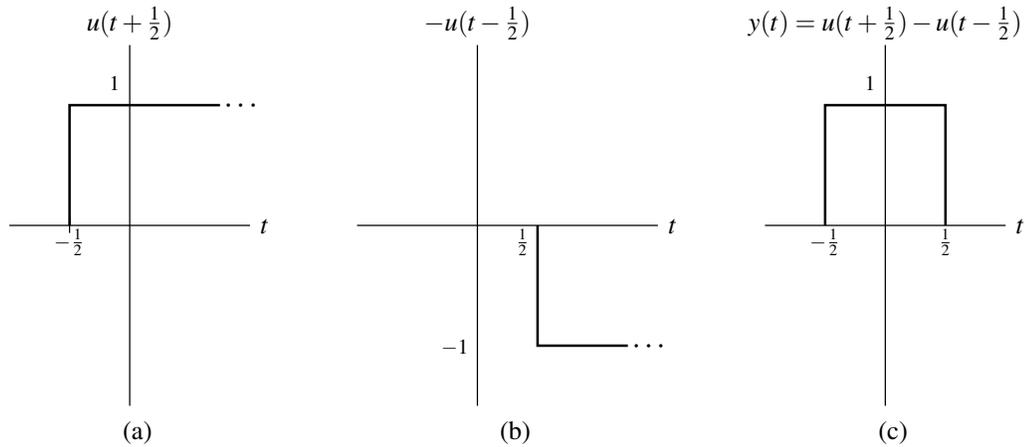
Figure 2.23: Representing the unit rectangular pulse using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their sum (which is the rectangular pulse).

Graphically, we have the scenario depicted in Figure 2.23.

In passing, we note that the above result can be generalized. Suppose that we have a rectangular pulse $x(t)$ of height 1 with a rising edge at $t = a$ and falling edge at $t = b$. One can show that

$$x(t) = u(t-a) - u(t-b)$$
$$= \begin{cases} 1 & \text{for } a < t < b \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$

$\square$

Example 2.12 (Piecewise linear function). Using unit-step functions, find an equivalent representation of the following function:

$$x(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 1 & \text{for } 1 \leq t < 2 \\ 3-t & \text{for } 2 \leq t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

A plot of $x(t)$ can be found in Figure 2.24(a).

Solution. We consider each segment of the piecewise linear function separately. The first segment (i.e., for $0 \leq t < 1$) can be expressed as

$$v_1(t) = t[u(t) - u(t-1)].$$

This function is plotted in Figure 2.24(b). The second segment (i.e., for $1 \leq t < 2$) can be expressed as

$$v_2(t) = u(t-1) - u(t-2).$$

This function is plotted in Figure 2.24(c). The third segment (i.e., for $2 \leq t < 3$) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$
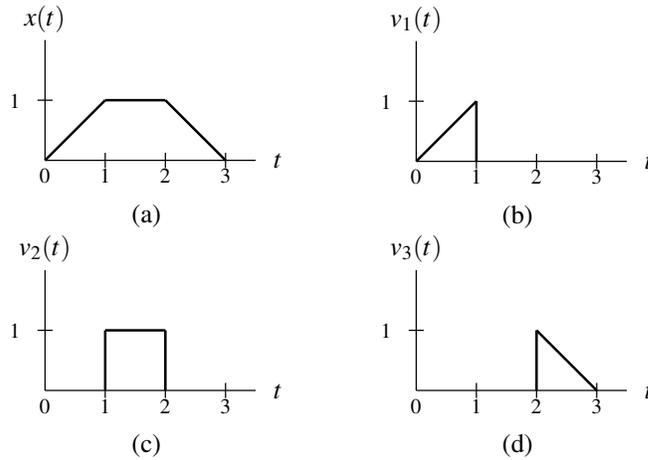
Figure 2.24: Representing piecewise linear function using unit-step functions.

This function is plotted in Figure 2.24(d). Now, we observe that $x(t)$ is simply the sum of the three preceding functions. That is, we have

$$
\begin{aligned}
x(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\
&= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\
&= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3).
\end{aligned}
$$

Thus, we have found an alternative representation of $x(t)$ that uses unit-step functions.   □

Example 2.13 (Piecewise polynomial function). Find an alternative representation of the following function by using unit-step functions:

$$
x(t) = \begin{cases}
1 & \text{for } 0 \le t < 1 \\
(t-2)^2 & \text{for } 1 \le t < 3 \\
4-t & \text{for } 3 \le t < 4 \\
0 & \text{otherwise.}
\end{cases}
$$

A plot of $x(t)$ is shown in Figure 2.25(a).

Solution. We consider each segment of the piecewise polynomial function separately. The first segment (i.e., for $0 \le t < 1$) can be written as

$$
v_1(t) = u(t) - u(t-1).
$$

This function is plotted in Figure 2.25(b). The second segment (i.e., for $1 \le t < 3$) can be written as

$$
v_2(t) = (t-2)^2[u(t-1) - u(t-3)] = (t^2 - 4t + 4)[u(t-1) - u(t-3)].
$$

This function is plotted in Figure 2.25(c). The third segment (i.e., for $3 \le t < 4$) can be written as

$$
v_3(t) = (4-t)[u(t-3) - u(t-4)].
$$
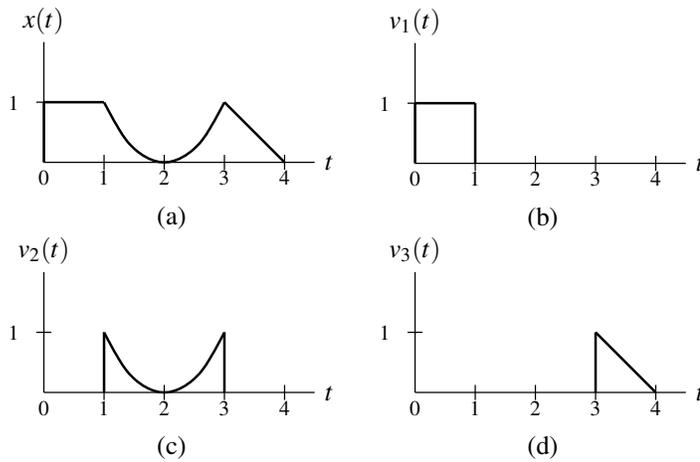
Figure 2.25: Representing piecewise polynomial function using unit-step functions.

This function is plotted in Figure 2.25(d). Now, we observe that $x(t)$ is obtained by summing the three preceding functions as follows:

$$
\begin{aligned}
x(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= [u(t) - u(t-1)] + (t^2 - 4t + 4)[u(t-1) - u(t-3)] + (4-t)[u(t-3) - u(t-4)] \\
&= u(t) + (t^2 - 4t + 4 - 1)u(t-1) + (4 - t - [t^2 - 4t + 4])u(t-3) - (4-t)u(t-4) \\
&= u(t) + (t^2 - 4t + 3)u(t-1) + (-t^2 + 3t)u(t-3) + (t-4)u(t-4).
\end{aligned}
$$

Thus, we have found an alternative representation of $x(t)$ that utilizes unit-step functions. □

Example 2.14 (Periodic function). Consider the periodic function $x(t)$ shown in Figure 2.26(a). Find a representation of $x(t)$ that utilizes unit-step functions.

Solution. We begin by finding an expression for a single period of $x(t)$. Let us denote this expression as $v(t)$. We can then write:

$$
v(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2}).
$$

This function is plotted in Figure 2.26(b). In order to obtain the periodic function $x(t)$, we must repeat $v(t)$ every two units (since the period of $x(t)$ is two). This can be accomplished by adding an infinite number of shifted copies of $v(t)$ as follows:

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} v(t - 2k) \\
&= \sum_{k=-\infty}^{\infty} [u(t + \tfrac{1}{2} - 2k) - u(t - \tfrac{1}{2} - 2k)].
\end{aligned}
$$

Thus, we have found a representation of the periodic function $x(t)$ that makes use of unit-step functions. □

2.6 Continuous-Time Systems

Suppose that we have a system with input $x(t)$ and output $y(t)$. Such a system can be described mathematically by the equation

$$
y(t) = \mathscr{T}\{x(t)\}, \tag{2.32}
$$
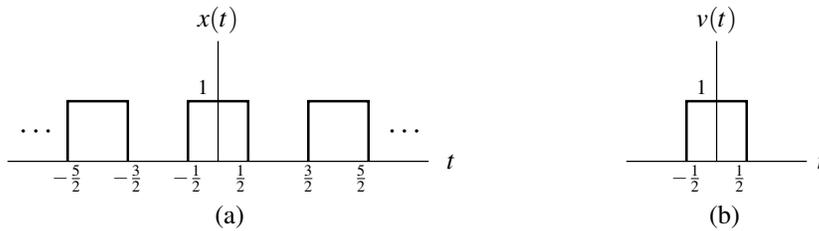
Figure 2.26: Representing periodic function using unit-step functions.
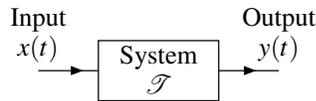
Figure 2.27: Block diagram of system.

where $\mathcal{T}$ denotes an operator (i.e., transformation). The operator $\mathcal{T}$ simply maps the input signal $x(t)$ to the output signal $y(t)$. Such an operator might be associated with a system of differential equations, for example.

Alternatively, we sometimes express the relationship (2.32) using the notation

$$x(t) \xrightarrow{\mathcal{T}} y(t).$$

Furthermore, if clear from the context, the operator $\mathcal{T}$ is often omitted, yielding the abbreviated notation

$$x(t) \rightarrow y(t).$$

Note that the symbols "$\rightarrow$" and "$=$" have very different meanings. For example, the notation $x(t) \rightarrow y(t)$ does not in any way imply that $x(t) = y(t)$. The symbol "$\rightarrow$" should be read as "produces" (not as "equals"). That is, "$x(t) \rightarrow y(t)$" should be read as "the input $x(t)$ produces the output $y(t)$".

2.6.1 Block Diagram Representation

Suppose that we have a system defined by the operator $\mathcal{T}$ and having the input $x(t)$ and output $y(t)$. Often, we represent such a system using a block diagram as shown in Figure 2.27.

2.6.2 Interconnection of Systems

Systems may be interconnected in a number of ways. Two basic types of connections are as shown in Figure 2.28. The first type of connection, as shown in Figure 2.28(a), is known as a **series** or **cascade** connection. In this case, the overall system is defined by

$$y(t) = \mathcal{T}_2 \{\mathcal{T}_1\{x(t)\}\}. \tag{2.33}$$

The second type of connection, as shown in Figure 2.28(b), is known as a **parallel** connection. In this case, the overall system is defined by

$$y(t) = \mathcal{T}_1\{x(t)\} + \mathcal{T}_2\{x(t)\}. \tag{2.34}$$

The system equations in (2.33) and (2.34) cannot be simplified further unless the definitions of the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are known.

2.7 Properties of Continuous-Time Systems

In what follows, we will define a number of important properties that a system may possess. These properties are useful in classifying systems, as well as characterizing their behavior.
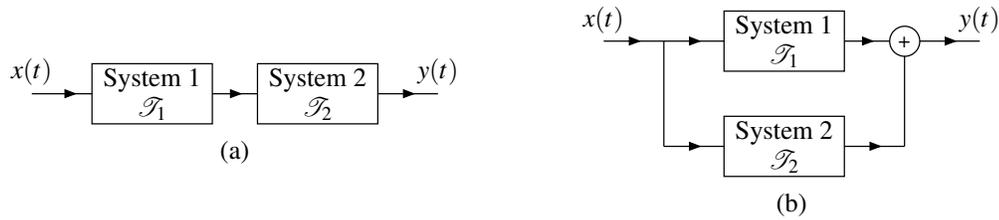
Figure 2.28: Interconnection of systems. (a) Series and (b) parallel interconnections.

2.7.1 Memory

A system is said to have **memory** if its output $y(t)$ at any arbitrary time t_0 depends on the value of its input $x(t)$ at any time other than $t = t_0$. If a system does not have memory, it is said to be **memoryless**.

Example 2.15. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Such a system is commonly referred to as an integrator. Determine whether this system has memory.

Solution. Consider the calculation of $y(t)$ for any arbitrary time $t = t_0$. For $t = t_0$, $y(t)$ depends on $x(t)$ for $-\infty < t \leq t_0$. Thus, $y(t)|_{t=t_0}$ is dependent on $x(t)$ for some $t \neq t_0$. Therefore, this system has memory. $\square$

Example 2.16. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = Ax(t),$$

where A is a real constant. This system is known as an ideal amplifier. Determine whether this system has memory.

Solution. Consider the calculation of $y(t)$ for any arbitrary time $t = t_0$. The quantity $y(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. $\square$

2.7.2 Causality

A system is said to be **causal** if its output $y(t)$ at any arbitrary time t_0 depends only on the values of its input $x(t)$ for $t \leq t_0$.

   If the independent variable represents time, a system must be causal in order to be physically realizable. Noncausal systems can sometimes be useful in practice, however, as the independent variable need not always represent time.

Example 2.17. A system with input $x(t)$ and output $y(t)$ is described by

$$y(t) = x(-t).$$

Determine whether this system is causal.

Solution. For $t < 0$, $y(t)$ depends on $x(t)$ for $t > 0$. So, for example, if $t = -1$, $y(t)$ depends on $x(1)$. In this case, we have that $y(-1)$ depends on $x(t)$ for some $t > -1$ (e.g., for $t = 1$). Therefore, the system is noncausal. $\square$

Example 2.18. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Determine whether this system is causal.

Solution. Consider the calculation of $y(t_0)$ for arbitrary t_0. We can see that $y(t_0)$ depends only on $x(t)$ for $-\infty < t \leq t_0$. Therefore, this system is causal. $\square$

2.7.3 Invertibility

A system is said to be **invertible** if its input $x(t)$ can always be uniquely determined from its output $y(t)$. From this definition, it follows that an invertible system will always produce distinct outputs from any two distinct inputs.

   If a system is invertible, this is most easily demonstrated by finding the inverse system. If a system is not invertible, often the easiest way to prove this is to show that two distinct inputs result in identical outputs.

Example 2.19. A system with input $x(t)$ and output $y(t)$ is described by the equation

$$y(t) = x(t - t_0),$$

where t_0 is a real constant. Determine whether this system is invertible.

Solution. Let $\tau \triangleq t - t_0$ which implies that $t = \tau + t_0$. Using these two relationships, we can rewrite (2.19) as

$$y(\tau + t_0) = x(\tau).$$

Thus, we have solved for $x(t)$ in terms of $y(t)$. Therefore, the system is invertible.   □

Example 2.20. A system with input $x(t)$ and output $y(t)$ is defined by the equation

$$y(t) = \sin[x(t)].$$

Determine whether this system is invertible.

Solution. Consider an input of the form $x(t) = 2\pi k$ where k can assume any integer value. The response to such an input is given by

$$y(t) = \sin[x(t)] = \sin 2\pi k = 0.$$

Thus, we have found an infinite number of distinct inputs (i.e., $x(t) = 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$) that all result in the same output. Therefore, the system is not invertible.   □

2.7.4 Stability

Although stability can be defined in numerous ways, in systems theory, we are often most interested in bounded-input bounded-output (BIBO) stability.

   A system having the input $x(t)$ and output $y(t)$ is **BIBO stable** if, for any $x(t)$, $|x(t)| \leq A < \infty$ for all t implies that $|y(t)| \leq B < \infty$ for all t. In other words, a system is BIBO stable if a bounded input always produces a bounded output.

   To prove that a system is BIBO stable, we must show that every bounded input leads to a bounded output. To show that a system is not BIBO stable, we simply need to find one counterexample (i.e., a single bounded input that leads to an unbounded output).

Example 2.21. A system is characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

where $x(t)$ and $y(t)$ denote the system input and output, respectively. Determine whether this system is BIBO stable.

Solution. Suppose now that we choose the input to be the unit-step function. That is, suppose that $x(t) = u(t)$. Clearly, $u(t)$ is a bounded function for all t (i.e., $|u(t)| \leq 1$ for all t). We can calculate the system response to this input as follows:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{t} u(\tau) d\tau \\
&= \int_{0}^{t} d\tau \\
&= [\tau]\big|_{0}^{t} \\
&= t.
\end{aligned}
$$

From this result, however, we can see that as $t \to \infty$, $y(t) \to \infty$. Thus, the output becomes unbounded for arbitrarily large t, in spite of the input being bounded. Therefore, the system is unstable. □

Example 2.22. A system is described by the equation

$$y(t) = x^2(t),$$

where $x(t)$ and $y(t)$ denote the system input and output, respectively. Determine whether this system is BIBO stable.

Solution. If $|x(t)| \le A < \infty$, then from the above equation it follows that $|y(t)| = |x^2(t)| \le A^2$. Thus, if the input is bounded, the output is also bounded. Therefore, the system is stable. □

2.7.5 Time Invariance

Let $y(t)$ denote the response of a system to the input $x(t)$, and let t_0 denote a shift constant. If, for any choice of $x(t)$ and t_0, the input $x(t - t_0)$ produces the output $y(t - t_0)$, the system is said to be **time invariant**. In other words, a system is time invariant, if a time shift (i.e., advance or delay) in the input signal results in an identical time shift in the output signal.

Example 2.23. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = tx(t).$$

Determine whether this system is time invariant.

Solution. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_1(t - t_0)$, respectively, where t_0 is a real constant. We can easily deduce that

$$y_1(t) = tx_1(t) \quad \text{and}$$
$$y_2(t) = tx_1(t - t_0).$$

So, we have that

$$y_1(t - t_0) = (t - t_0)x_1(t - t_0).$$

Since $y_2(t) \ne y_1(t - t_0)$, the system is not time invariant (i.e., the system is time varying). □

Example 2.24. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = \sin[x(t)].$$

Determine whether this system is time invariant.

Solution. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_1(t - t_0)$, respectively, where t_0 is a real constant. We can easily deduce that

$$y_1(t) = \sin[x_1(t)] \quad \text{and}$$
$$y_2(t) = \sin[x_1(t - t_0)].$$

So, we have that

$$y_1(t - t_0) = \sin[x_1(t - t_0)].$$

Since $y_2(t) = y_1(t - t_0)$, the system is time invariant. □

2.7.6 Linearity

Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively. If, for any choice of $x_1(t)$ and $x_2(t)$, the response to the input $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$, the system is said to possess the **additivity** property.

Let $y(t)$ denote the response of a system to the input $x(t)$, and let a denote a complex constant. If, for any choice of $x(t)$ and a, the response to the input $ax(t)$ is $ay(t)$, the system is said to possess the **homogeneity** property.

If system possesses both the additivity and homogeneity properties, it is said to be **linear**. Otherwise, it is said to be **nonlinear**.

The two linearity conditions (i.e., additivity and homogeneity) can be combined into a single condition known as superposition. Let $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively, and let a and b denote complex constants. If, for any choice of $x_1(t)$, $x_2(t)$, a, and b, the input $ax_1(t) + bx_2(t)$ produces the response $ay_1(t) + by_2(t)$, the system is said to possess the **superposition** property.

To show that a system is linear, we can show that it possesses both the additivity and homogeneity properties, or we can simply show that the superposition property holds.

Example 2.25. A system with input $x(t)$ and output $y(t)$ is characterized by the equation

$$y(t) = tx(t).$$

Determine whether this system is linear.

Solution. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system to the inputs $x_1(t)$ and $x_2(t)$, respectively. Let $y_3(t)$ denote the response to the input $ax_1(t) + bx_2(t)$ where a and b are complex constants. From the definition of the system, we can write

$$y_1(t) = tx_1(t),$$
$$y_2(t) = tx_2(t), \quad \text{and}$$
$$y_3(t) = t(ax_1(t) + bx_2(t)) = atx_1(t) + btx_2(t) = ay_1(t) + by_2(t).$$

Since $y_3(t) = ay_1(t) + by_2(t)$, the superposition property holds and the system is linear. $\square$

Example 2.26. A system is defined by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

where $x(t)$ and $y(t)$ denote the system input and output, respectively. Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we consider the additivity property. Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the system responses to the inputs $x_1(t)$, $x_2(t)$, and $x_1(t) + x_2(t)$, respectively. Thus, we have

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau) d\tau,$$
$$y_2(t) = \int_{-\infty}^{t} x_2(\tau) d\tau, \quad \text{and}$$

$$y_3(t) = \int_{-\infty}^{t} [x_1(\tau) + x_2(\tau)] d\tau$$
$$= \int_{-\infty}^{t} x_1(\tau) d\tau + \int_{-\infty}^{t} x_2(\tau) d\tau$$
$$= y_1(t) + y_2(t).$$

Since $y_3(t) = y_1(t) + y_2(t)$, the system possesses the additivity property.

Second, we consider the homogeneity property. Let $y_1(t)$ and $y_2(t)$ denote the system responses to the inputs $x_1(t)$ and $ax_1(t)$ where a is a complex constant. Thus, we can write

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau \quad \text{and}$$

$$y_2(t) = \int_{-\infty}^{t} ax_1(\tau)d\tau$$
$$= a\int_{-\infty}^{t} x_1(\tau)d\tau$$
$$= ay_1(t).$$

Since $y_2(t) = ay_1(t)$, the system has the homogeneity property.

Lastly, we consider the linearity property. The system is linear since it has both the additivity and homogeneity properties. □

Example 2.27. A system with input $x(t)$ and output $y(t)$ (where $x(t)$ and $y(t)$ are complex valued) is defined by the equation
$$y(t) = \text{Re}\{x(t)\}.$$
Determine whether this system is additive and/or homogeneous. Determine whether this system is linear.

Solution. First, we check if the additivity property is satisfied. Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the system responses to the inputs $x_1(t)$, $x_2(t)$, and $x_1(t) + x_2(t)$, respectively. Thus, we have

$$y_1(t) = \text{Re}\{x_1(t)\},$$
$$y_2(t) = \text{Re}\{x_2(t)\}, \quad \text{and}$$

$$y_3(t) = \text{Re}\{x_1(t) + x_2(t)\}$$
$$= \text{Re}\{x_1(t)\} + \text{Re}\{x_2(t)\}$$
$$= y_1(t) + y_2(t).$$

Since $y_3(t) = y_1(t) + y_2(t)$, the system has the additivity property.

Second, we check if the homogeneity property is satisfied. Let $y_1(t)$ and $y_2(t)$ denote the system responses to the inputs $x_1(t)$ and $ax_1(t)$ where a is a complex constant. Thus, we have

$$y_1(t) = \text{Re}\, x_1(t),$$
$$y_2(t) = \text{Re}\{ax_1(t)\}, \quad \text{and}$$
$$ay_1(t) = a\,\text{Re}\{x_1(t)\}.$$

In order for this system to possess the homogeneity property, $ay_1(t) = y_2(t)$ must hold for any complex a. Suppose that $a = j$. In this case, we have

$$y_2(t) = \text{Re}\{jx_1(t)\}$$
$$= \text{Re}\{j[\text{Re}\{x_1(t)\} + j\,\text{Im}\{x_1(t)\}]\}$$
$$= \text{Re}\{-\text{Im}\{x_1(t)\} + j\,\text{Re}\{x_1(t)\}\}$$
$$= -\text{Im}\{x_1(t)\},$$

and

$$ay_1(t) = j\,\text{Re}\{x_1(t)\}.$$

Thus, the quantities $y_2(t)$ and $ay_1(t)$ are clearly not equal. Therefore, the system does not possess the homogeneity property.

Lastly, we consider the linearity property. Since the system does not possess both the additivity and homogeneity properties, it is not linear. □

2.7.7 Examples

Example 2.28. Suppose that we have the system with the input $x(t)$ and output $y(t)$ given by

$$y(t) = \text{Odd}\{x(t)\} = \tfrac{1}{2}(x(t) - x(-t)).$$

Determine whether the system has each of the following properties: (a) memory, (b) causal, (c) invertible, (d) BIBO stable, (e) time invariant, (f) linear.

Solution. (a) MEMORY. For any $x(t)$ and any real t_0, we have that $y(t)|_{t=t_0}$ depends on $x(t)$ for $t = t_0$ and $t = -t_0$. Since $y(t)|_{t=t_0}$ depends on $x(t)$ for $t \neq t_0$, the system has memory (i.e., the system is not memoryless).

   (b) CAUSALITY. For any $x(t)$ and any real constant t_0, we have that $y(t)|_{t=t_0}$ depends only on $x(t)$ for $t = t_0$ and $t = -t_0$. Suppose that $t_0 = -1$. In this case, we have that $y(t)|_{t=t_0}$ (i.e., $y(-1)$) depends on $x(t)$ for $t = 1$ but $t = 1 > t_0$. Therefore, the system is not causal.

   (c) INVERTIBILITY. Consider the response $y(t)$ of the system to an input $x(t)$ of the form

$$x(t) = \alpha$$

where α is a real constant. We have that

$$\begin{aligned} y(t) &= \tfrac{1}{2}[x(t) - x(-t)] \\ &= \tfrac{1}{2}(\alpha - \alpha) \\ &= 0. \end{aligned}$$

Therefore, any constant input yields the same zero output. This, however, implies that distinct inputs can yield identical outputs. Therefore, the system is not invertible.

   (d) STABILITY. Suppose that $x(t)$ is bounded. Then, $x(-t)$ is also bounded. Since the difference of two bounded functions is bounded, $x(t) - x(-t)$ is bounded. Multiplication of a bounded function by a finite constant yields a bounded result. So, the function $\tfrac{1}{2}[x(t) - x(-t)]$ is bounded. Thus, $y(t)$ is bounded. Since a bounded input must yield a bounded output, the system is BIBO stable.

   (e) TIME INVARIANCE. Suppose that

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) \quad \text{and} \\ x_1(t - t_0) &\rightarrow y_2(t). \end{aligned}$$

The system is time invariant if, for any $x_1(t)$ and any real constant t_0, $y_2(t) = y_1(t - t_0)$. From the definition of the system, we have

$$y_1(t) = \tfrac{1}{2}[x_1(t) - x_1(-t)],$$

$$\begin{aligned} y_2(t) &= \tfrac{1}{2}[x_1(t - t_0) - x_1((-t) - t_0)] \\ &= \tfrac{1}{2}[x_1(t - t_0) - x_1(-t - t_0)], \quad \text{and} \end{aligned}$$

$$\begin{aligned} y_1(t - t_0) &= \tfrac{1}{2}[x_1(t - t_0) - x_1(-(t - t_0))] \\ &= \tfrac{1}{2}[x_1(t - t_0) - x_1(t_0 - t)]. \end{aligned}$$

Since $y_2(t) \neq y_1(t - t_0)$, the system is not time invariant.

   (f) LINEARITY. Suppose that

$$\begin{aligned} x_1(t) &\rightarrow y_1(t), \\ x_2(t) &\rightarrow y_2(t), \quad \text{and} \\ a_1 x_1(t) + a_2 x_2(t) &\rightarrow y_3(t). \end{aligned}$$

The system is linear if, for any $x_1(t)$ and $x_2(t)$ and any complex constants a_1 and a_2, $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$. From the definition of the system, we have

$$y_1(t) = \tfrac{1}{2}[x_1(t) - x_1(-t)],$$

$$y_2(t) = \tfrac{1}{2}[x_2(t) - x_2(-t)], \quad \text{and}$$

$$\begin{aligned}
y_3(t) &= \tfrac{1}{2}[a_1 x_1(t) + a_2 x_2(t) - [a_1 x_1(-t) + a_2 x_2(-t)]] \\
&= \tfrac{1}{2}[a_1 x_1(t) - a_1 x_1(-t) + a_2 x_2(t) - a_2 x_2(-t)] \\
&= a_1 [\tfrac{1}{2}(x_1(t) - x_1(-t))] + a_2 [\tfrac{1}{2}(x_2(t) - x_2(-t))] \\
&= a_1 y_1(t) + a_2 y_2(t).
\end{aligned}$$

Since $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, the system is linear. $\qquad\square$

Example 2.29. Suppose that we have the system with input $x(t)$ and output $y(t)$ given by

$$y(t) = 3x(3t + 3).$$

Determine whether the system has each of the following properties: (a) memory, (b) causal, (c) invertible, (d) BIBO stable, (e) time invariant, (f) linear.

Solution. (a) MEMORY. For any $x(t)$ and any real t_0, we have that $y(t)|_{t=t_0}$ depends on $x(t)$ for $t = 3t_0 + 3$. Since $y(t)|_{t=t_0}$ depends on $x(t)$ for $t \neq t_0$, the system has memory (i.e., the system is not memoryless).
    (b) CAUSALITY. For any $x(t)$ and any real constant t_0, we have that $y(t)|_{t=t_0}$ depends only on $x(t)$ for $t = 3t_0 + 3$. Suppose that $t_0 = 0$. In this case, $y(t)|_{t=t_0}$ depends on $x(t)$ for $t = 3$, but $t = 3 > t_0$. Therefore, the system is not causal.
    (c) INVERTIBILITY. From the definition of the system, we can write

$$\begin{aligned}
y(t) = 3x(3t + 3) &\Rightarrow y(\tfrac{1}{3}\tau - 1) = 3x(\tau) \\
&\Rightarrow x(\tau) = \tfrac{1}{3} y(\tfrac{1}{3}\tau - 1).
\end{aligned}$$

Thus, we have just solved for $x(t)$ in terms of $y(t)$. Therefore, an inverse system exists. Consequently, the system is invertible.
    (d) STABILITY. Suppose that $x(t)$ is bounded such that $|x(t)| \leq A < \infty$ (for all t). Then, $|x(3t + 3)| \leq A$. Furthermore, we have that $|3x(3t + 3)| = 3|x(3t + 3)|$. So, $|3x(3t + 3)| \leq 3A$. Therefore, $|y(t)| \leq 3A$ (i.e., $y(t)$ is bounded). Consequently, the system is BIBO stable.
    (e) TIME INVARIANCE. Suppose that

$$\begin{aligned}
x_1(t) &\to y_1(t) \quad \text{and} \\
x_1(t - t_0) &\to y_2(t).
\end{aligned}$$

The system is time invariant if, for any $x_1(t)$ and any real constant t_0, $y_2(t) = y_1(t - t_0)$. From the definition of the system, we have

$$y_1(t) = 3x_1(3t + 3),$$

$$y_2(t) = 3x_1(3t + 3 - t_0), \quad \text{and}$$

$$\begin{aligned}
y_1(t - t_0) &= 3x_1(3(t - t_0) + 3) \\
&= 3x_1(3t - 3t_0 + 3).
\end{aligned}$$

Since $y_2(t) \neq y_1(t - t_0)$, the system is not time invariant.

(f) LINEARITY. Suppose that

$$x_1(t) \rightarrow y_1(t),$$
$$x_2(t) \rightarrow y_2(t), \quad \text{and}$$
$$a_1 x_1(t) + a_2 x_2(t) \rightarrow y_3(t).$$

The system is linear if, for any $x_1(t)$ and $x_2(t)$, and any complex constants a_1 and a_2, $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$. From the definition of the system, we have

$$y_1(t) = 3x_1(3t + 3),$$

$$y_2(t) = 3x_2(3t + 3), \quad \text{and}$$

$$\begin{aligned}
y_3(t) &= 3[a_1 x_1(3t + 3) + a_2 x_2(3t + 3)] \\
&= a_1[3x_1(3t + 3)] + a_2[3x_2(3t + 3)] \\
&= a_1 y_1(t) + a_2 y_2(t).
\end{aligned}$$

Since $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, the system is linear.   □

2.8 Problems

2.1 Identify the time and/or amplitude transformations that must be applied to the signal $x(t)$ in order to obtain each of the signals specified below. Choose the transformations such that time shifting precedes time scaling and amplitude scaling precedes amplitude shifting. Be sure to clearly indicate the order in which the transformations are to be applied.
(a) $y(t) = x(2t - 1)$;
(b) $y(t) = x(\frac{1}{2}t + 1)$;
(c) $y(t) = 2x(-\frac{1}{2}t + 1) + 3$;
(d) $y(t) = -\frac{1}{2}x(-t + 1) - 1$; and
(e) $y(t) = -3x(2[t - 1]) - 1$.

2.2 Suppose that we have two signals $x(t)$ and $y(t)$ related as

$$y(t) = x(at - b),$$

where a and b are real constants and $a \neq 0$.
(a) Show that $y(t)$ can be formed by first time shifting $x(t)$ by b and then time scaling the result by a.
(b) Show that $y(t)$ can also be formed by first time scaling $x(t)$ by a and then time shifting the result by $\frac{b}{a}$.

2.3 Given the signal $x(t)$ shown in the figure below, plot and label each of the following signals:
(a) $x(t - 1)$;
(b) $x(2t)$;
(c) $x(-t)$;
(d) $x(2t + 1)$; and
(e) $\frac{1}{4}x(-\frac{1}{2}t + 1) - \frac{1}{2}$.
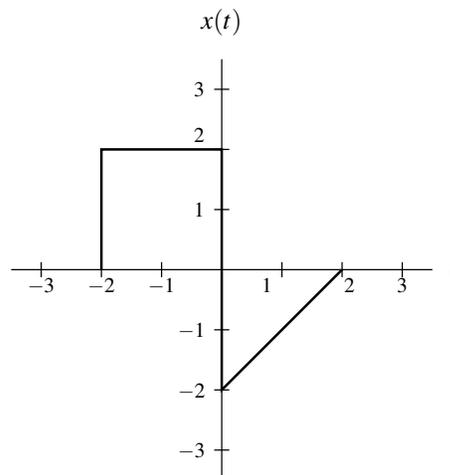
2.4 Determine whether each of the following functions is even, odd, or neither even nor odd:
(a) $x(t) = t^3$;
(b) $x(t) = t^3 |t|$;
(c) $x(t) = |t^3|$;
(d) $x(t) = (\cos 2\pi t)(\sin 2\pi t)$;
(e) $x(t) = e^{j2\pi t}$; and
(f) $x(t) = \frac{1}{2}[e^t + e^{-t}]$.

2.5 Prove each of the following assertions:
(a) The sum of two even signals is even.

(b) The sum of two odd signals is odd.

(c) The sum of an even signal and an odd signal is neither even nor odd.

(d) The product of two even signals is even.

(e) The product of two odd signals is even.

(f) The product of an even signal and an odd signal is odd.

2.6 Show that, if $x(t)$ is an odd signal, then

$$\int_{-A}^{A} x(t)dt = 0,$$

where A is a positive real constant.

2.7 Show that, for any signal $x(t)$,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt,$$

where $x_e(t)$ and $x_o(t)$ denote the even and odd parts of $x(t)$, respectively.

2.8 Suppose $h(t)$ is a causal signal and has the even part $h_e(t)$ given by

$$h_e(t) = t[u(t) - u(t-1)] + u(t-1) \quad \text{for } t > 0.$$

Find $h(t)$ for all t.

2.9 Determine whether each of the signals given below is periodic. If the signal is periodic, find its fundamental period.

(a) $x(t) = \cos 2\pi t + \sin 5t$;

(b) $x(t) = [\cos(4t - \frac{\pi}{3})]^2$;

(c) $x(t) = e^{j2\pi t} + e^{j3\pi t}$; and

(d) $x(t) = 1 + \cos 2t + e^{j5t}$.

2.10 Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} \sin\left(2t + \frac{\pi}{4}\right)\delta(t)dt$;

(b) $\int_{-\infty}^{t}[\cos\tau]\delta(\tau + \pi)d\tau$;

(c) $\int_{-\infty}^{\infty} x(t)\delta(at - b)dt$ where a and b are real constants and $a \neq 0$;

(d) $\int_{0}^{2} e^{j2t}\delta(t-1)dt$; and

(e) $\int_{-\infty}^{t} \delta(\tau)d\tau$.

2.11 Suppose that we have the signal $x(t)$ shown in the figure below. Use unit-step functions to find a single expression for $x(t)$ that is valid for all t.

                                Last Revised: January 11, 2012
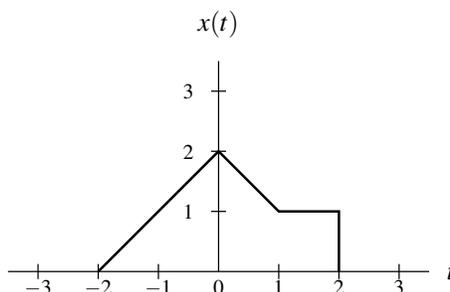
2.12 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is linear:
(a) $y(t) = \int_{t-1}^{t+1} x(\tau)d\tau$;
(b) $y(t) = e^{x(t)}$;
(c) $y(t) = \text{Even}\{x(t)\}$; and
(d) $y(t) = x^2(t)$.

2.13 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is time invariant:
(a) $y(t) = \frac{d}{dt}x(t)$;
(b) $y(t) = \text{Odd}\{x(t)\}$;
(c) $y(t) = \int_t^{t+1} x(\tau - \alpha)d\tau$ where α is a constant;
(d) $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$;
(e) $y(t) = x(-t)$; and
(f) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$.

2.14 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the following equations is causal and/or memoryless:
(a) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(b) $y(t) = \text{Odd}\{x(t)\}$;
(c) $y(t) = x(t-1) + 1$;
(d) $y(t) = \int_t^{\infty} x(\tau)d\tau$; and
(e) $y(t) = \int_{-\infty}^{t} x(\tau)\delta(\tau)d\tau$.

2.15 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the equations given below is invertible. If the system is invertible, specify its inverse.
(a) $y(t) = x(at - b)$ where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{x(t)}$;
(c) $y(t) = \text{Even}\{x(t)\} - \text{Odd}\{x(t)\}$; and
(d) $y(t) = \frac{d}{dt}x(t)$.

2.16 Determine whether the system with input $x(t)$ and output $y(t)$ defined by each of the equations given below is BIBO stable.
(a) $y(t) = \int_t^{t+1} x(\tau)d\tau$;
(b) $y(t) = \frac{1}{2}x^2(t) + x(t)$; and
(c) $y(t) = 1/x(t)$.
[Hint for part (a): For any function $f(x)$, $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$.]

2.17 Show that if a system with input $x(t)$ and output $y(t)$ is either additive or homogeneous, it has the property that if $x(t)$ is identically zero (i.e., $x(t) = 0$ for all t), then $y(t)$ is identically zero (i.e., $y(t) = 0$ for all t).

2.18 Suppose that we have a signal $x(t)$ with the derivative $y(t) = \frac{d}{dt}x(t)$.
(a) Show that if $x(t)$ is even then $y(t)$ is odd.
(b) Show that if $x(t)$ is odd then $y(t)$ is even.

2.19 Given the signals $x_1(t)$ and $x_2(t)$ shown in the figures below, express $x_2(t)$ in terms of $x_1(t)$.

(a)   (b)
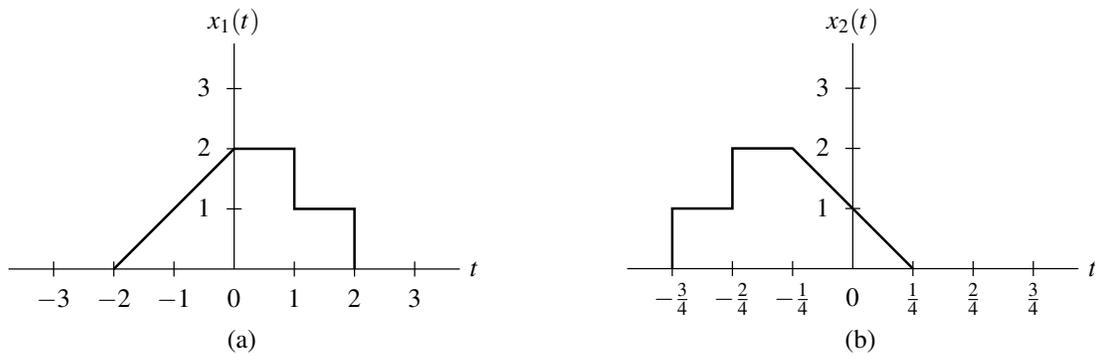
2.20 Given the signal

$$x(t) = u(t+2) + u(t+1) + u(t) - 2u(t-1) - u(t-2),$$

find and sketch $y(t) = x(-4t - 1)$.

2.21 For the system shown in the figure below, express the output $y(t)$ in terms of the input $x(t)$ and the transformations $T_1, T_2, \ldots, T_5$.
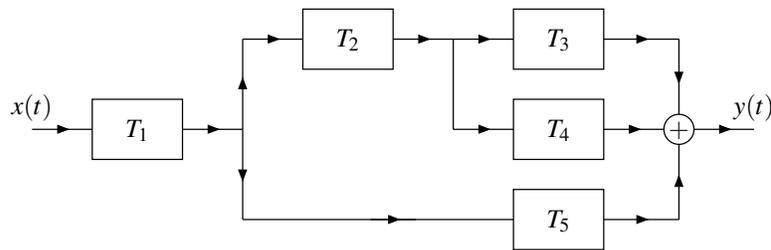
Chapter 3

Continuous-Time Linear Time-Invariant Systems

3.1 Introduction

In the previous chapter, we identified a number of properties that a system may possess. Two of these properties were the linearity and time invariance properties. In this chapter, we focus our attention exclusively on systems with both of these properties. Such systems are referred to as **linear time-invariant** (LTI) systems.

3.2 Continuous-Time Convolution

In the context of LTI systems, we often find an operation known as convolution to be particularly useful. The **convolution** of the functions $x(t)$ and $h(t)$ is denoted as $x(t) * h(t)$ as is defined as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau. \tag{3.1}$$

Throughout the remainder of these notes, the asterisk (or star) symbol (i.e., "$*$") will be used to denote convolution, not multiplication. It is important to make a distinction between convolution and multiplication, since these two operations are quite different and do not generally yield the same result.

One must also be careful when using the abbreviated (i.e., star notation) for convolution, since this notation can sometimes behave in counter-intuitive ways. For example, the expressions $x(t - t_0) * h(t - t_0)$ and $x(\tau) * h(\tau)|_{\tau = t - t_0}$, where t_0 is a nonzero constant, have very different meanings and are almost never equal.

Since the convolution operation is used extensively in system theory, we need some practical means for evaluating a convolution integral. Suppose that, for the given functions $x(t)$ and $h(t)$, we wish to compute

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau.$$

Of course, we could naively attempt to compute $y(t)$ by evaluating a separate convolution integral for each possible value of t. This approach, however, is not feasible, as t can assume an infinite number of values, and therefore, an infinite number of integrals would need to be evaluated. Instead, we consider a slightly different approach. Let us redefine the integrand in terms of the intermediate function $w_t(\tau)$ where

$$w_t(\tau) = x(\tau)h(t - \tau).$$

(Note that $w_t(\tau)$ is implicitly a function of t.) This means that we need to compute

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau.$$

Now, we observe that, for most functions $x(t)$ and $h(t)$ of practical interest, the form of $w_t(\tau)$ typically remains fixed over particular ranges of t. Thus, we can compute the integral $y(t)$ by first identifying each of the distinct expressions for $w_t(\tau)$ and the range over which each expression is valid. Then, for each range, we evaluate a convolution integral. In this way, we typically only need to compute a small number of integrals instead of the infinite number required with the naive approach suggested above.

The above discussion leads us to propose the following general approach for performing the convolution operation:

1. Plot $x(\tau)$ and $h(t-\tau)$ as a function of τ.

2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t-\tau)$ being shifted very far to the left on the time axis.

3. Write the mathematical expression for $w_t(\tau)$.

4. Increase t gradually until the expression for $w_t(\tau)$ changes form. Record the interval over which the expression for $w_t(\tau)$ was valid.

5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t-\tau)$ being shifted very far to the right on the time axis.

6. For each of the intervals identified above, integrate $w_t(\tau)$ in order to find an expression for $y(t)$. This will yield an expression for $y(t)$ for each interval.

7. The results for the various intervals can be combined in order to obtain an expression for $y(t)$ that is valid for all t.

Example 3.1. Compute the convolution $y(t) = x(t) * h(t)$ where

$$
x(t) = \begin{cases} -1 & \text{for } -1 \le t < 0 \\ 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise,} \quad \text{and} \end{cases}
$$
$$
h(t) = e^{-t}u(t).
$$

Solution. We begin by plotting the signals $x(\tau)$ and $h(\tau)$ as shown in Figures 3.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.1(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 3.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 3.1(e), we can see that

$$
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.2}
$$

Second, we consider the case of $-1 \le t < 0$. From Figure 3.1(f), we can see that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{t} \\
&= -e^{-t}[e^{t} - e^{-1}] \\
&= e^{-t-1} - 1. \tag{3.3}
\end{aligned}
$$

Third, we consider the case of $0 \leq t < 1$. From Figure 3.1(g), we can see that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\
&= -e^{-t}[1-e^{-1}] + e^{-t}[e^{t}-1] \\
&= e^{-t}[e^{-1}-1+e^{t}-1] \\
&= 1 + (e^{-1}-2)e^{-t}.
\end{aligned}
\tag{3.4}
$$

Fourth, we consider the case of $t \geq 1$. From Figure 3.1(h), we can see that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\
&= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\
&= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\
&= e^{-t}[e^{-1}-1+e-1] \\
&= (e-2+e^{-1})e^{-t}.
\end{aligned}
\tag{3.5}
$$

Combining the results of (3.2), (3.3), (3.4), and (3.5), we have that

$$
x(t) * h(t) = \begin{cases}
0 & \text{for } t < -1 \\
e^{-t-1}-1 & \text{for } -1 \leq t < 0 \\
(e^{-1}-2)e^{-t}+1 & \text{for } 0 \leq t < 1 \\
(e-2+e^{-1})e^{-t} & \text{for } 1 \leq t.
\end{cases}
$$

The convolution result $x(t) * h(t)$ is plotted in Figure 3.1(i).   □

Example 3.2. Compute the convolution $y(t) = x(t) * h(t)$ where

$$
x(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}
$$

$$
h(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Solution. We begin by plotting the signals $x(\tau)$ and $h(\tau)$ as shown in Figures 3.2(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.2(c). Second, we time-shift the resulting signal by t to obtain $h(t-\tau)$ as shown in Figure 3.2(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.2(e) to (h).

First, we consider the case of $t < 0$. From Figure 3.2(e), we can see that

$$
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.
\tag{3.6}
$$

Figure 3.1: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < -1$, (f) $-1 \leq t < 0$, (g) $0 \leq t < 1$, and (h) $t \geq 1$, (i) The convolution result $x(t) * h(t)$.

Second, we consider the case of $0 \leq t < 1$. From Figure 3.2(f), we can see that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{0}^{t} (t-\tau)d\tau \\
&= [t\tau - \tfrac{1}{2}\tau^2]|_{0}^{t} \\
&= t^2 - \tfrac{1}{2}t^2 \\
&= \tfrac{1}{2}t^2.
\end{aligned}
\tag{3.7}
$$

Third, we consider the case of $1 \leq t < 2$. From Figure 3.2(g), we can see that

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^{1} (t-\tau)d\tau \\
&= [t\tau - \tfrac{1}{2}\tau^2]_{t-1}^{1} \\
&= t - \tfrac{1}{2}(1)^2 - [t(t-1) - \tfrac{1}{2}(t-1)^2] \\
&= t - \tfrac{1}{2} - [t^2 - t - \tfrac{1}{2}(t^2 - 2t + 1)] \\
&= -\tfrac{1}{2}t^2 + t.
\end{aligned}
\tag{3.8}
$$

Fourth, we consider the case of $t \geq 2$. From Figure 3.2(h), we can see that

$$
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0.
\tag{3.9}
$$

Combining the results of (3.6), (3.7), (3.8), and (3.9), we have that

$$
x(t) * h(t) = \begin{cases}
0 & \text{for } t < 0 \\
\tfrac{1}{2}t^2 & \text{for } 0 \leq t < 1 \\
-\tfrac{1}{2}t^2 + t & \text{for } 1 \leq t < 2 \\
0 & \text{for } t \geq 2.
\end{cases}
$$

The convolution result $x(t) * h(t)$ is plotted in Figure 3.2(i).   □

Example 3.3. Compute the quantity $y(t) = x(t) * h(t)$, where

$$
x(t) = \begin{cases}
0 & \text{for } t < 0 \\
t & \text{for } 0 \leq t < 1 \\
-t + 2 & \text{for } 1 \leq t < 2 \\
0 & \text{for } t > 2,
\end{cases}
\quad \text{and}
$$

$$
h(t) = u(t) - u(t-1).
$$

Solution. Due to the somewhat ugly nature of the expressions for $x(t)$ and $h(t)$, this problem can be more easily solved if we use the graphical interpretation of convolution to guide us. We begin by plotting the signals $x(\tau)$ and $h(\tau)$, as shown in Figures 3.3(a) and (b), respectively.

Next, we need to determine $h(t - \tau)$, the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.3(c). Second, we time-shift the resulting signal by t to obtain $h(t - \tau)$ as shown in Figure 3.3(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t - \tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.3(e) to (i).

Figure 3.2: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < 0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, and (h) $t \ge 2$, (i) The convolution result $x(t) * h(t)$.

First, we consider the case of $t < 0$. From Figure 3.3(e), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{3.10}$$

Second, we consider the case of $0 \le t < 1$. From Figure 3.3(f), we can see that

$$\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_0^t \tau d\tau \\
&= [\tfrac{1}{2}\tau^2]_0^t \\
&= \tfrac{1}{2}t^2. \tag{3.11}
\end{aligned}$$

Third, we consider the case of $1 \le t < 2$. From Figure 3.3(g), we can see that

$$\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^1 \tau d\tau + \int_1^t (-\tau+2)d\tau \\
&= [\tfrac{1}{2}\tau^2]_{t-1}^1 + [-\tfrac{1}{2}\tau^2 + 2\tau]_1^t \\
&= \tfrac{1}{2} - [\tfrac{1}{2}(t-1)^2] - \tfrac{1}{2}t^2 + 2t - [-\tfrac{1}{2}+2] \\
&= -t^2 + 3t - \tfrac{3}{2}. \tag{3.12}
\end{aligned}$$

Fourth, we consider the case of $2 \le t < 3$. From Figure 3.3(h), we can see that

$$\begin{aligned}
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{t-1}^2 (-\tau+2)d\tau \\
&= [-\tfrac{1}{2}\tau^2 + 2\tau]_{t-1}^2 \\
&= 2 - [-\tfrac{1}{2}t^2 + 3t - \tfrac{5}{2}] \\
&= \tfrac{1}{2}t^2 - 3t + \tfrac{9}{2}. \tag{3.13}
\end{aligned}$$

Lastly, we consider the case of $t \ge 3$. From Figure 3.3(i), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau) = 0. \tag{3.14}$$

Combining the results of (3.10), (3.11), (3.12), (3.13), and (3.14) together, we have that

$$x(t) * h(t) = \begin{cases}
0 & \text{for } t < 0 \\
\tfrac{1}{2}t^2 & \text{for } 0 \le t < 1 \\
-t^2 + 3t - \tfrac{3}{2} & \text{for } 1 \le t < 2 \\
\tfrac{1}{2}t^2 - 3t + \tfrac{9}{2} & \text{for } 2 \le t < 3 \\
0 & \text{for } t \ge 3.
\end{cases}$$

The convolution result (i.e., $x(t) * h(t)$) is plotted in Figure 3.3(j). $\qquad\square$

Example 3.4. Compute the convolution $y(t) = x(t) * h(t)$ where

$$\begin{aligned}
x(t) &= e^{-at}u(t), \quad \text{and} \\
h(t) &= u(t),
\end{aligned}$$

and a is a positive real constant.

Figure 3.3: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t - \tau)$. The functions associated with the product in the convolution integral for (e) $t < 0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, (h) $2 \le t < 3$, and (i) $t \ge 3$. (j) The convolution result $x(t) * h(t)$.

Solution. Since $x(t)$ and $h(t)$ are relatively simple functions, we will solve this problem without the aid of graphs. Using the definition of the convolution operation, we can write:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau. \tag{3.15}$$

The integrand is zero for $\tau < 0$ or $\tau > t$. Conversely, the integrand can only be nonzero for $\tau \geq 0$ and $\tau \leq t$. So, if $t < 0$, the integrand will be zero, and $y(t) = 0$. Now, let us consider the case of $t > 0$. From (3.15), we can write:

$$y(t) = \int_0^t e^{-a\tau}d\tau$$

$$= [-\tfrac{1}{a}e^{-a\tau}]_0^t$$

$$= \tfrac{1}{a}(1 - e^{-at}).$$

Thus, we have

$$y(t) = \begin{cases} \tfrac{1}{a}(1 - e^{-at}) & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \tfrac{1}{a}(1 - e^{-at})u(t).$$

(If some steps in the above solution are unclear, it would probably be helpful to sketch the corresponding graphs. This will yield the graphs shown in Figure 3.4.) $\qquad\square$

3.3 Properties of Convolution

Since convolution is frequently employed in the study of LTI systems, it is important for us to know some of its basic properties. In what follows, we examine some of these properties. We will later use these properties in the context of LTI system.

3.3.1 Commutative Property

The convolution operation is commutative. That is, for any two signals $x(t)$ and $h(t)$, we have

$$x(t) * h(t) = h(t) * x(t). \tag{3.16}$$

Thus, the result of the convolution operation is not affected by the order of the operands.

We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (3.16) as follows:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$x(t) * h(t) = \int_{\infty}^{-\infty} x(t-v)h(v)(-dv)$$

$$= \int_{-\infty}^{\infty} x(t-v)h(v)dv$$

$$= \int_{-\infty}^{\infty} h(v)x(t-v)dv$$

$$= h(t) * x(t).$$

Thus, we have proven that convolution is commutative.
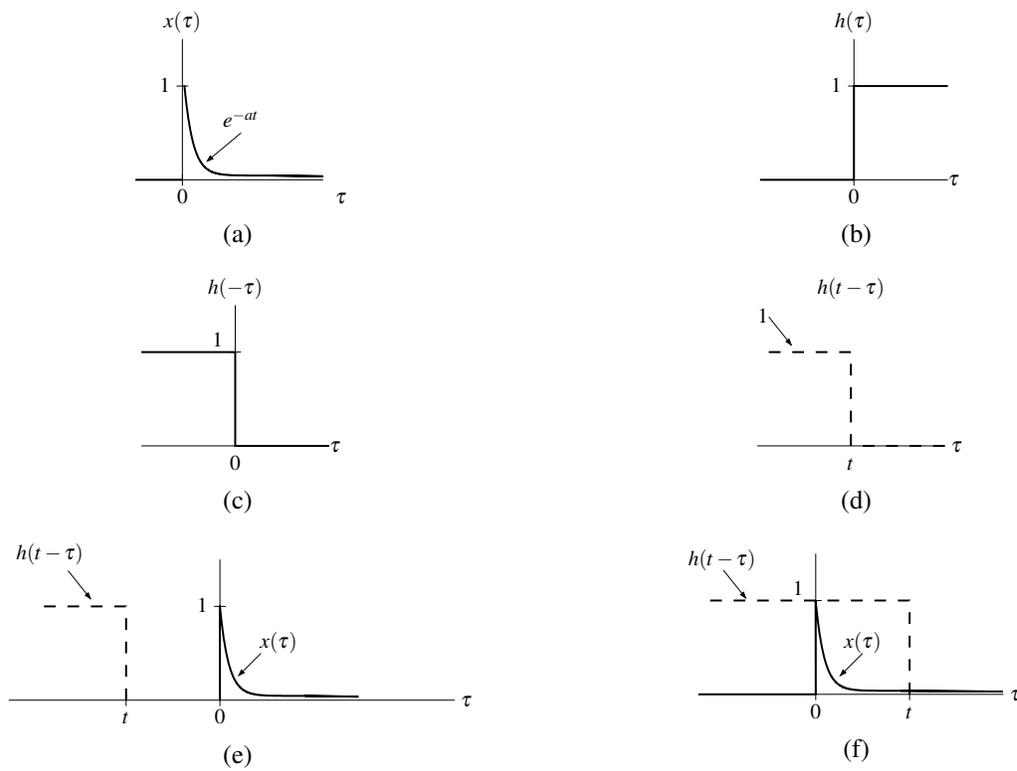
Figure 3.4: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < 0$ and (f) $t > 0$.

3.3.2 Associative Property

The convolution operation is associative. That is, for any signals $x(t)$, $h_1(t)$, and $h_2(t)$, we have

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]. \tag{3.17}$$

In other words, the final convolution result does not depend on how the intermediate operations are grouped.

The proof of this property is relatively straightforward, although somewhat confusing notationally. To begin, we use the definition of the convolution operation to expand the left-hand side of (3.17) as follows:

$$\begin{aligned}
[x(t) * h_1(t)] * h_2(t) &= \int_{-\infty}^{\infty} [x(v) * h_1(v)] h_2(t-v) dv \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h_1(v-\tau) d\tau \right) h_2(t-v) dv.
\end{aligned}$$

Now, we change the order of integration to obtain

$$[x(t) * h_1(t)] * h_2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h_1(v-\tau) h_2(t-v) dv d\tau.$$

Pulling the factor of $x(\tau)$ out of the inner integral yields

$$[x(t) * h_1(t)] * h_2(t) = \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(v-\tau) h_2(t-v) dv d\tau.$$

Next, we perform a change of variable. Let $\lambda = v - \tau$ which implies that $v = \lambda + \tau$ and $d\lambda = dv$. Using this change of variable, we can write

$$\begin{aligned}
[x(t) * h_1(t)] * h_2(t) &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\lambda-\tau) d\lambda d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h_1(\lambda) h_2([t-\tau]-\lambda) d\lambda \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left. [h_1(\rho) * h_2(\rho)] \right|_{\rho = t-\tau} d\tau \\
&= x(t) * [h_1(t) * h_2(t)].
\end{aligned}$$

Thus, we have proven that convolution is associative.

3.3.3 Distributive Property

The convolution operation is distributive. That is, for any signals $x(t)$, $h_1(t)$, and $h_2(t)$, we have

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t). \tag{3.18}$$

The proof of this property is relatively simple. Expanding the left-hand side of (3.18), we have:

$$\begin{aligned}
x(t) * [h_1(t) + h_2(t)] &= \int_{-\infty}^{\infty} x(\tau) [h_1(t-\tau) + h_2(t-\tau)] d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau) h_2(t-\tau) d\tau \\
&= x(t) * h_1(t) + x(t) * h_2(t).
\end{aligned}$$

Thus, we have shown that convolution is distributive.

Figure 3.5: System.

3.4 Representation of Continuous-Time Signals Using Impulses

The unit-impulse function is of fundamental importance in many aspects of system theory. For this reason, we sometimes have the need to express an arbitrary signal in terms of impulse functions. In what follows, we develop a means for accomplishing just this.

Suppose that we have an arbitrary signal $x(t)$. From the equivalence property of the impulse function given in (2.30), we can write

$$x(t)\delta(t-\tau) = x(\tau)\delta(t-\tau).$$

Now, let us integrate both sides of the preceding equation with respect to τ to obtain

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau. \tag{3.19}$$

The left-hand side of this equation can be further simplified as follows:

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau$$

$$= x(t)\int_{\infty}^{-\infty} \delta(\lambda)(-d\lambda)$$

$$= x(t)\int_{-\infty}^{\infty} \delta(\lambda)d\lambda$$

$$= x(t). \tag{3.20}$$

(In the above simplification, we used the change of variable $\lambda = t - \tau$, $d\lambda = -d\tau$.) Combining (3.19) and (3.20), we obtain

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)*\delta(t). \tag{3.21}$$

Thus, we can represent any signal $x(t)$ using an expression containing the impulse function. Furthermore, we have also just shown that the unit impulse function is the convolutional identity. That is, we have that, for any $x(t)$,

$$x(t)*\delta(t) = x(t)$$

(i.e., convolving a function $x(t)$ with the unit-impulse function $\delta(t)$ simply yields $x(t)$).

3.5 Continuous-Time Unit-Impulse Response and Convolution Integral Representation of LTI Systems

Suppose that we have an arbitrary system with input $x(t)$ and output $y(t)$. Let us represent the processing performed by the system with the operator $\mathscr{H}$, so that $y(t) = \mathscr{H}\{x(t)\}$. In other words, we have the system shown in Figure 3.5. As a matter of terminology, the system response to the unit-impulse function input is referred to as the **impulse response**. Let us denote the impulse response of the system as $h(t)$. Mathematically, we can state the definition of the impulse response as

$$h(t) = \mathscr{H}\{\delta(t)\}. \tag{3.22}$$

Now, let us assume that the system is LTI (i.e., the operator $\mathscr{H}$ is both linear and time invariant). As we shall demonstrate below, the behavior of a LTI system is completely characterized by its impulse response. That is, if the impulse response of a system is known, we can determine the response of the system to *any* input.

From the earlier result, we know that we can represent any signal in the form of (3.21). So, let us express the input to the system in this form as follows:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau. \tag{3.23}$$

Now, let us consider the form of the output $y(t)$. To begin, we know that

$$y(t) = \mathscr{H}\{x(t)\}.$$

From (3.23), we can rewrite this equation as

$$y(t) = \mathscr{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}.$$

Since $\mathscr{H}$ is a linear operator, we can move $\mathscr{H}$ inside the integral, and simplify the result to obtain

$$y(t) = \int_{-\infty}^{\infty} \mathscr{H}\{x(\tau)\delta(t-\tau)\}d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)\mathscr{H}\{\delta(t-\tau)\}d\tau. \tag{3.24}$$

Since the system is time invariant (by assumption), we know that

$$h(t-\tau) = \mathscr{H}\{\delta(t-\tau)\}. \tag{3.25}$$

Thus, we can rewrite (3.24) as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$= x(t) * h(t).$$

In other words, the output $y(t)$ is simply the convolution of the input $x(t)$ and the impulse response $h(t)$.

Clearly, the impulse response provides a very powerful tool for the study of LTI systems. If we know the impulse response of a system, we can determine the response of the system to *any* input.

Example 3.5. Suppose that we are given a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$ where

$$h(t) = u(t). \tag{3.26}$$

Show that this system is the integrator characterized by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau. \tag{3.27}$$

Solution. Since the system is LTI, we have that

$$y(t) = x(t) * h(t).$$

Substituting (3.26) into the preceding equation, and simplifying we obtain

$$y(t) = x(t) * h(t)$$
$$= x(t) * u(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$$
$$= \int_{-\infty}^{t} x(\tau)u(t-\tau)d\tau + \int_{t}^{\infty} x(\tau)u(t-\tau)d\tau$$
$$= \int_{-\infty}^{t} x(\tau)d\tau.$$

Therefore, the system with the impulse response $h(t)$ given by (3.26) is, in fact, the integrator given by (3.27). □

Example 3.6. Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$ where

$$h(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and plot the response of the system to the particular input $x(t)$ given by

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Plots of $x(t)$ and $h(t)$ are given in Figures 3.6(a) and (b), respectively.

Solution. Since the system is LTI, we know that

$$y(t) = x(t) * h(t).$$

Thus, in order to find the response of the system to the input $x(t)$, we simply need to compute the convolution $x(t) * h(t)$.

   We begin by plotting the signals $x(\tau)$ and $h(\tau)$ as shown in Figures 3.6(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of $h(\tau)$. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 3.6(c). Second, we time-shift the resulting signal by t to obtain $h(t - \tau)$ as shown in Figure 3.6(d).

   At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t - \tau)$ and integrate the resulting product with respect to τ. Due to the form of $x(\tau)$ and $h(\tau)$, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 3.6(e) to (h).

   First, we consider the case of $t < 0$. From Figure 3.6(e), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \tag{3.28}$$
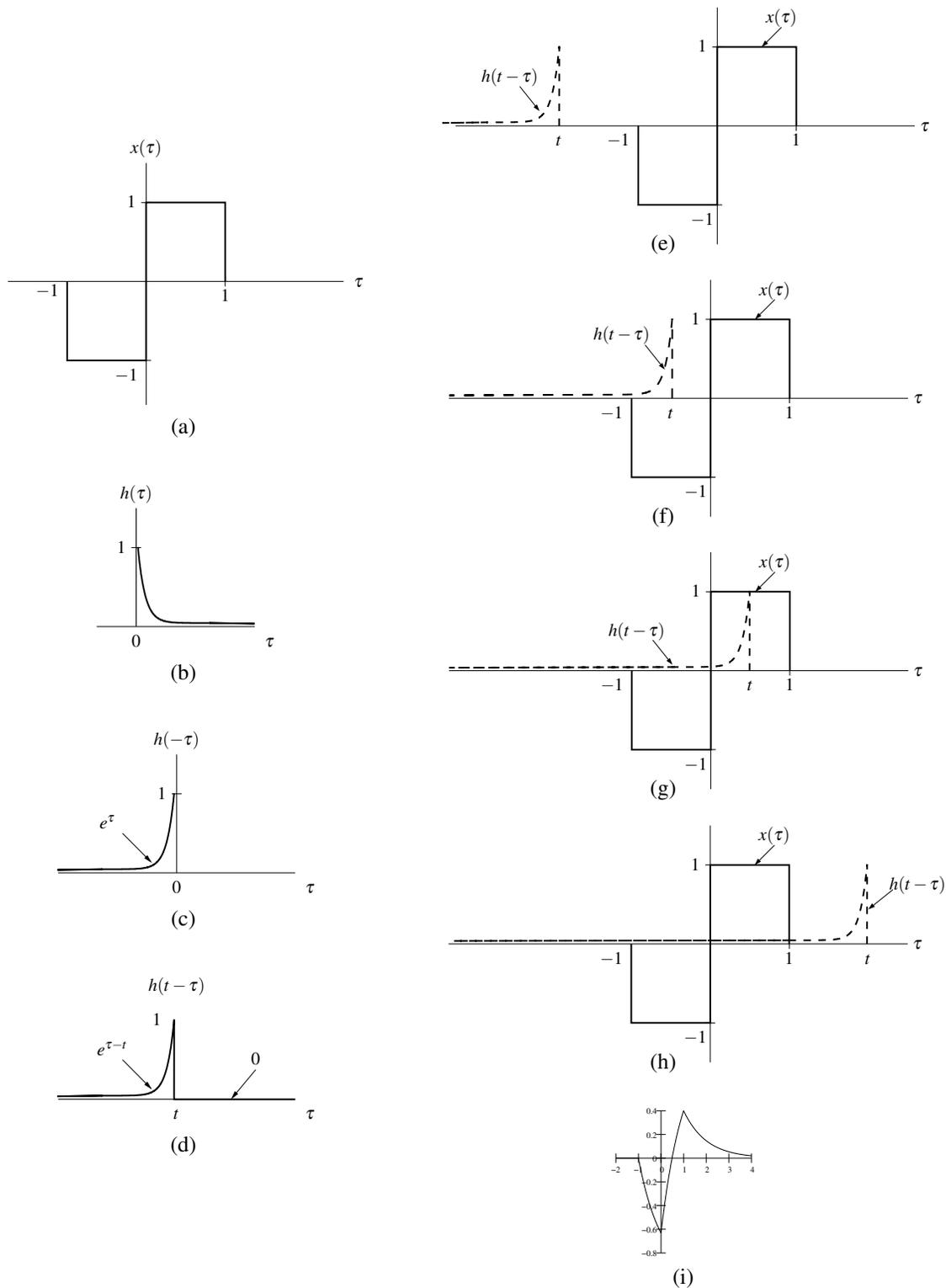
Second, we consider the case of $0 \le t < 1$. From Figure 3.6(f), we can see that

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau &= \int_0^t d\tau \\ &= [\tau]|_0^t \\ &= t. \end{aligned} \tag{3.29}$$

Third, we consider the case of $1 \le t < 2$. From Figure 3.6(g), we can see that

$$\begin{aligned} \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau &= \int_{t-1}^1 d\tau \\ &= [\tau]|_{t-1}^1 \\ &= 1 - (t - 1) \\ &= 2 - t. \end{aligned} \tag{3.30}$$

Fourth, we consider the case of $t \ge 2$. From Figure 3.6(h), we can see that

$$\int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = 0. \tag{3.31}$$

Combining the results of (3.28), (3.29), (3.30), and (3.31), we have that

$$x(t) * h(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \le t < 1 \\ 2 - t & \text{for } 1 \le t < 2 \\ 0 & \text{for } t \ge 2. \end{cases}$$

The convolution result $x(t) * h(t)$ is plotted in Figure 3.6(i). The response of the system to the specified input is simply the result of the convolution (i.e., $x(t) * h(t)$). □

3.6 Unit-Step Response of LTI Systems

Suppose that we have a system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Consider the response $s(t)$ of the system to the unit-step function input $u(t)$. We call $s(t)$ the **step response** of the system. This response $s(t)$ is given by

$$
\begin{aligned}
s(t) &= u(t) * h(t) \\
&= h(t) * u(t) \\
&= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{t} h(\tau)d\tau.
\end{aligned}
$$

Taking the derivative of $s(t)$ with respect to t, we obtain

$$
\begin{aligned}
\frac{ds(t)}{dt} &= \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{-\infty}^{t+\Delta t} h(\tau)d\tau - \int_{-\infty}^{t} h(\tau)d\tau \right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} h(\tau)d\tau \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} (h(t)\Delta t) \\
&= h(t).
\end{aligned}
$$

Thus, we have shown that

$$
\frac{ds(t)}{dt} = h(t).
$$

That is, the impulse response $h(t)$ of a system is equal to the derivative of its step response $s(t)$. Therefore, the impulse response of a system can be determined from its step response simply through differentiation.

The step response is often of great practical interest, since it can be used to determine the impulse response of a LTI system. From a practical point of view, the step response is more useful for characterizing a system based on experimental measurements. Obviously, we cannot directly measure the impulse response of a system because we cannot (in the real world) produce a unit-impulse signal. We can, however, produce a reasonably good approximation of the unit-step function in the real world. Thus, we can measure the step response and from it determine the impulse response.

3.7 Block Diagram Representation of Continuous-Time LTI Systems

Frequently, it is convenient to represent continuous-time LTI systems in block diagram form. Since such systems are completely characterized by their impulse response, we often label the system with its impulse response. That is, we represent a system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, as shown in Figure 3.7.
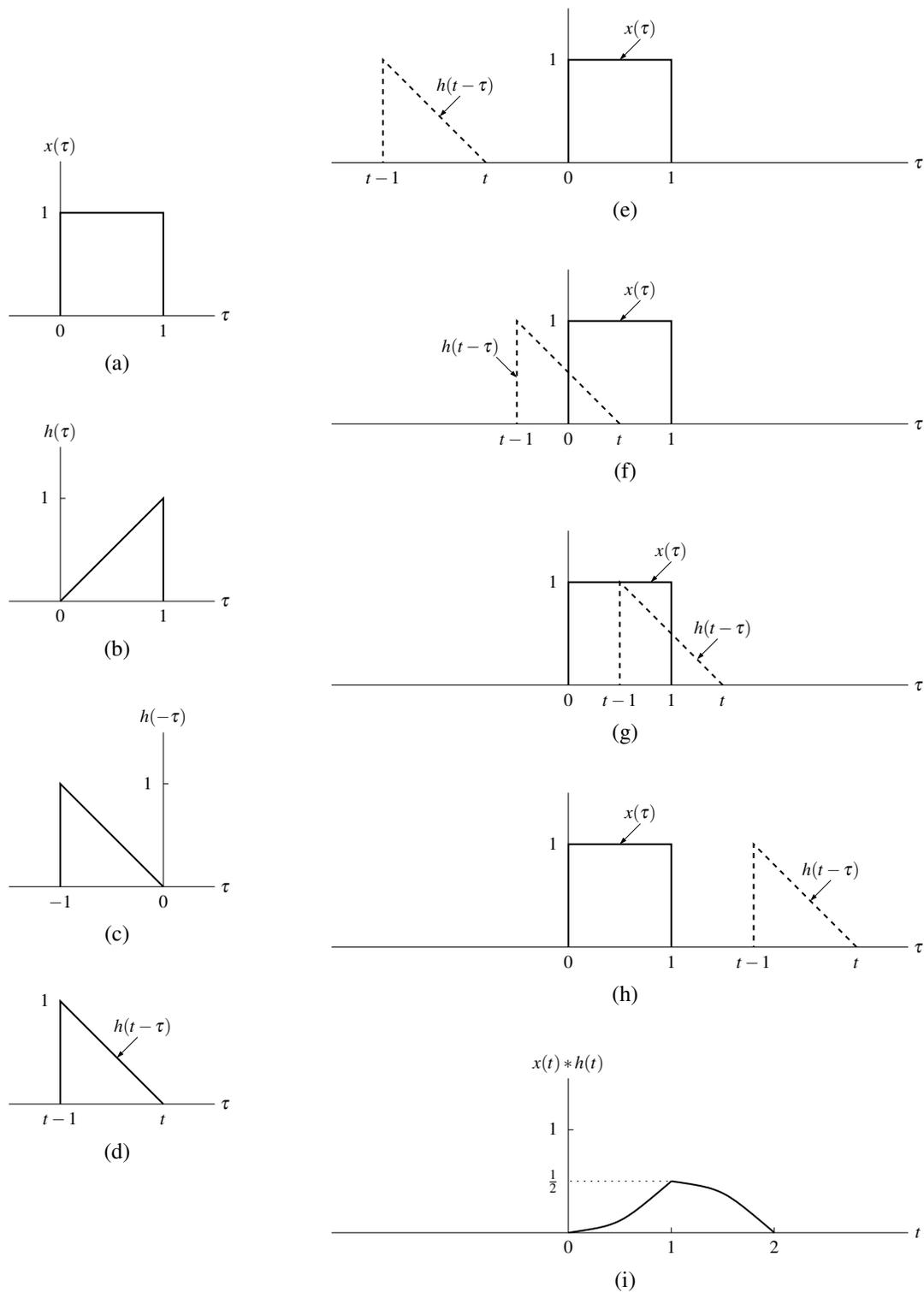
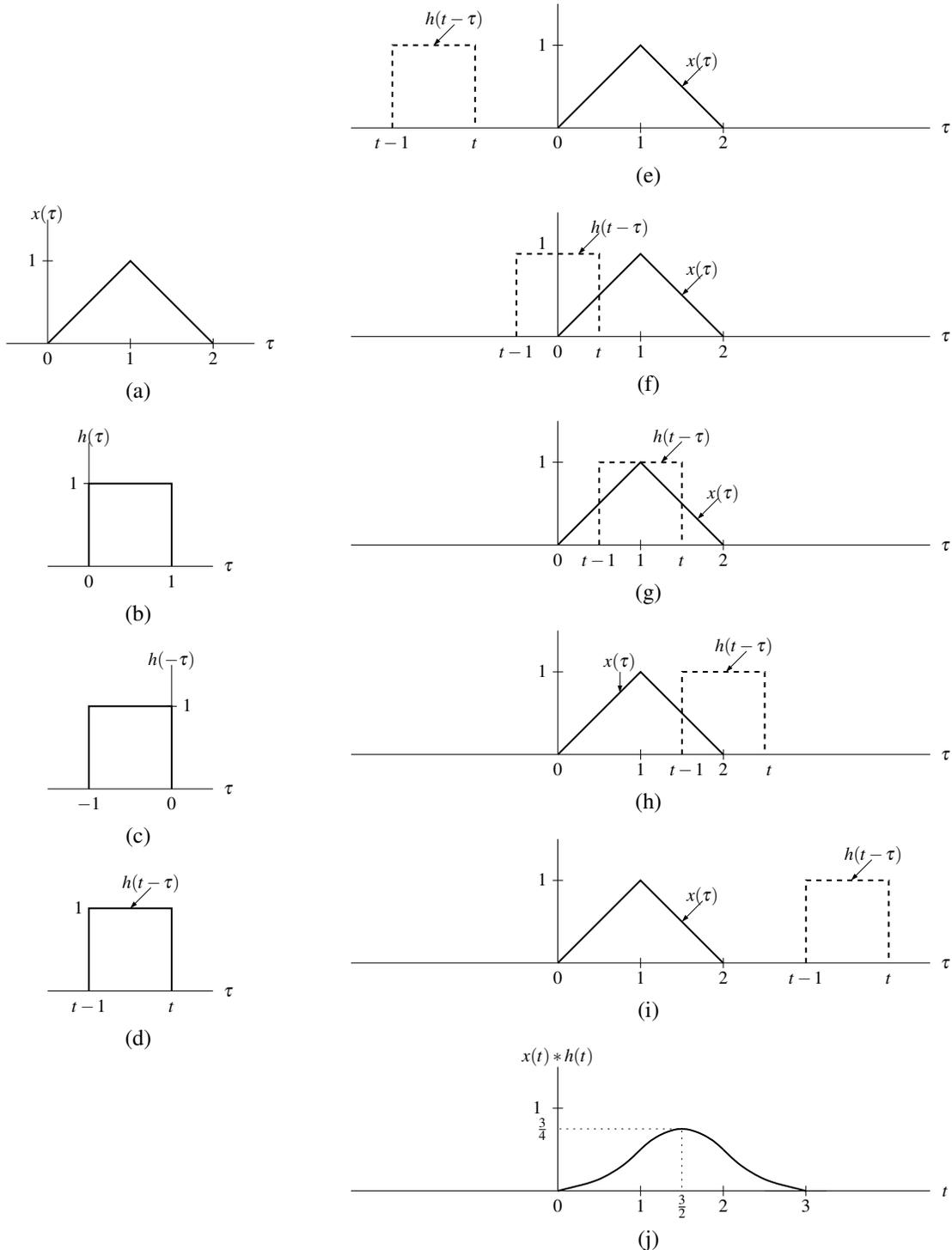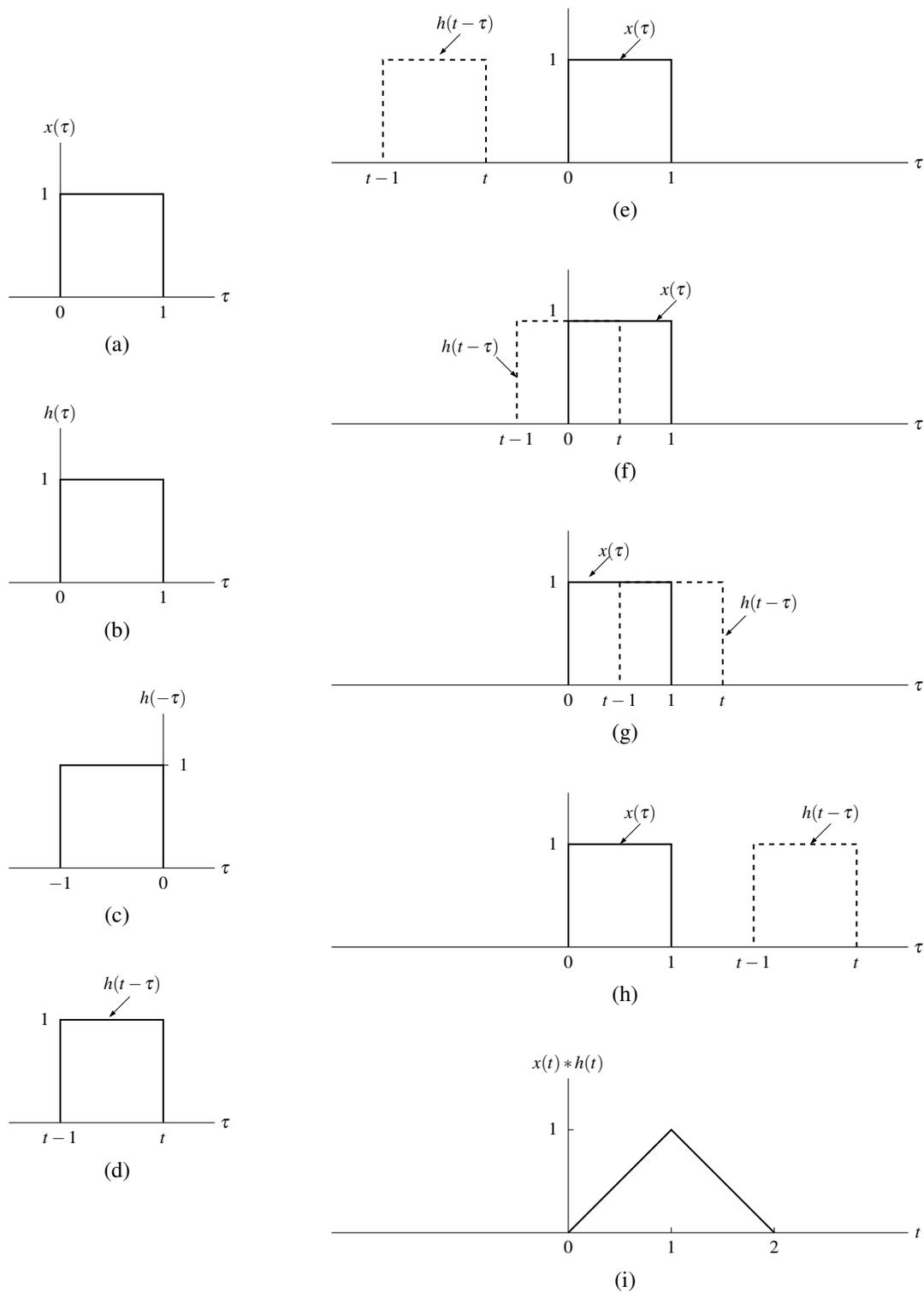Figure 3.6: Evaluation of the convolution integral. The (a) input signal $x(\tau)$, (b) impulse response $h(\tau)$, (c) time-reversed impulse response $h(-\tau)$, and (d) impulse response after time-reversal and time-shifting $h(t-\tau)$. The functions associated with the product in the convolution integral for (e) $t < 0$, (f) $0 \le t < 1$, (g) $1 \le t < 2$, and (h) $t \ge 2$, (i) The convolution result $x(t) * h(t)$.
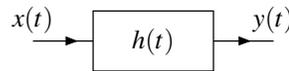
Figure 3.7: Block diagram representation of continuous-time LTI system.

(a)

(b)
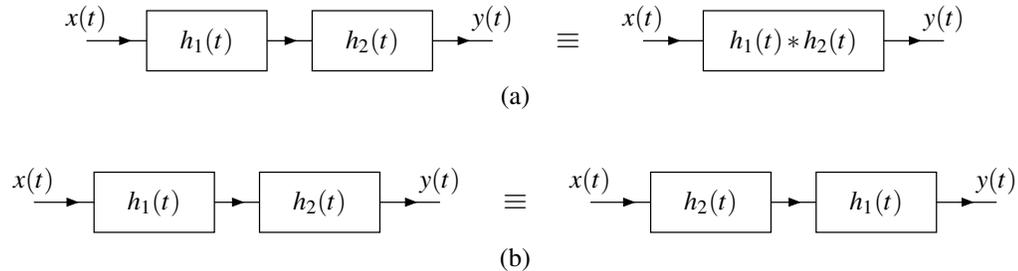
Figure 3.8: Series interconnection of continuous-time LTI systems.

3.8 Interconnection of Continuous-Time LTI Systems

Suppose that we have an LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. We know that $x(t)$ and $y(t)$ are related as $y(t) = x(t) * h(t)$. In other words, the system can be viewed as performing a convolution operation. From the properties of convolution introduced earlier, we can derive a number of equivalences involving the impulse responses of series- and parallel-interconnected systems.

Suppose that we have two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ that are connected in a series configuration, as shown on the left-side of Figure 3.8(a). For convenience, let us define $v(t) = x(t) * h_1(t)$. Using the associative property of convolution, we can simplify the expression for the output $y(t)$ as follows:

$$
\begin{aligned}
y(t) &= v(t) * h_2(t) \\
&= [x(t) * h_1(t)] * h_2(t) \\
&= x(t) * [h_1(t) * h_2(t)].
\end{aligned}
$$

In other words, we have the equivalence shown in Figure 3.8(a).

Suppose that we have two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ that are connected in a series configuration, as shown on the left-side of Figure 3.8(b). Using the commutative property of convolution, we can simplify the expression for the output $y(t)$ as follows:

$$
\begin{aligned}
y(t) &= x(t) * h_1(t) * h_2(t) \\
&= x(t) * h_2(t) * h_1(t).
\end{aligned}
$$

In other words, we have the equivalence shown in Figure 3.8(b).

Suppose that we have two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$ that are connected in a parallel configuration, as shown on the left-side of Figure 3.9. Using the distributive property of convolution, we can simplify the expression for the output $y(t)$ as follows:

$$
\begin{aligned}
y(t) &= x(t) * h_1(t) + x(t) * h_2(t) \\
&= x(t) * [h_1(t) + h_2(t)].
\end{aligned}
$$

Thus, we have the equivalence shown in Figure 3.9.

Example 3.7. Consider the system shown in Figure 3.10 with input $x(t)$, output $y(t)$. Find the impulse response $h(t)$ of the system. Use the properties of convolution in order to accomplish this.
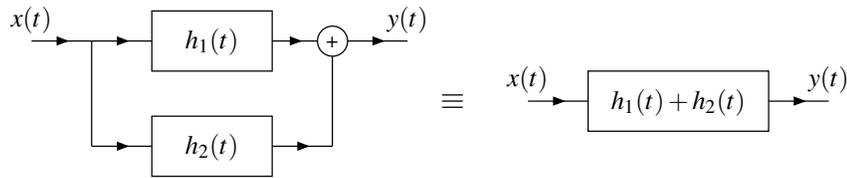
Figure 3.9: Parallel interconnection of continuous-time LTI systems.
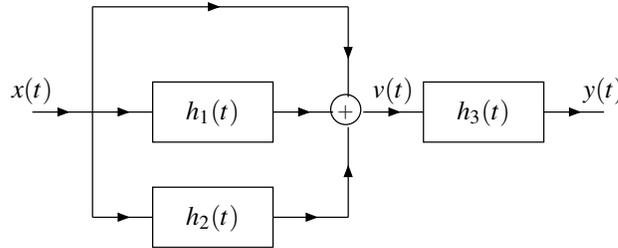
Figure 3.10: System interconnection example.

Solution. From the diagram, we can write:

$$
\begin{aligned}
v(t) &= x(t) + x(t) * h_1(t) + x(t) * h_2(t) \\
&= x(t) * \delta(t) + x(t) * h_1(t) + x(t) * h_2(t) \\
&= x(t) * [\delta(t) + h_1(t) + h_2(t)].
\end{aligned}
$$

Similarly, we can write:

$$
y(t) = v(t) * h_3(t).
$$

Substituting the expression for $v(t)$ into the preceding equation we obtain:

$$
\begin{aligned}
y(t) &= v(t) * h_3(t) \\
&= x(t) * [\delta(t) + h_1(t) + h_2(t)] * h_3(t) \\
&= x(t) * [h_3(t) + h_1(t) * h_3(t) + h_2(t) * h_3(t)].
\end{aligned}
$$

Therefore, the impulse response $h(t)$ of the overall system is

$$
h(t) = h_3(t) + h_1(t) * h_3(t) + h_2(t) * h_3(t).
$$

$\square$

3.9 Properties of Continuous-Time LTI Systems

In the previous chapter, we introduced a number of properties that might be possessed by a system (e.g., memory, causality, stability, invertibility). Since a LTI system is completely characterized by its impulse response, one might wonder if there is a relationship between some of the properties introduced previously and the impulse response. In what follows, we explore some of these relationships.

3.9.1 Memory

Recall that a system is memoryless if its output $y(t)$ at any arbitrary time t_0 depends only on the value of its input $x(t)$ at that same time. Suppose now that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. The output $y(t)$ at some arbitrary time $t = t_0$ is given by

$$
\begin{aligned}
y(t_0) &= x(t) * h(t)|_{t=t_0} \\
&= h(t) * x(t)|_{t=t_0} \\
&= \left. \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|_{t=t_0} \\
&= \int_{-\infty}^{\infty} h(\tau)x(t_0-\tau)d\tau.
\end{aligned}
$$

Consider the integral in the above equation. In order for the system to be memoryless, the result of the integration must depend only on $x(t)$ at $t = t_0$. This, however, is only possible if

$$h(t) = 0 \quad \text{for all } t \neq 0. \tag{3.32}$$

Consequently, $h(t)$ must be of the form

$$h(t) = K\delta(t) \tag{3.33}$$

where K is a complex constant. Thus, we have that a LTI system is memoryless if and only if its impulse response satisfies (3.32). This, in turn, implies that the impulse response is of the form of (3.33). As a consequence of this fact, we also have that all memoryless LTI systems must have an input-output relation of the form

$$
\begin{aligned}
y(t) &= x(t) * K\delta(t) \\
&= Kx(t).
\end{aligned}
$$

Example 3.8. Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$h(t) = e^{-at}u(t)$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). $\square$

Example 3.9. Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

Solution. Clearly, $h(t)$ is only nonzero at $t = 0$. This follows immediately from the definition of the unit-impulse function $\delta(t)$. Therefore, the system is memoryless (i.e., does not have memory). $\square$

3.9.2 Causality

Recall that a system is causal if its output $y(t)$ at any arbitrary time t_0 depends only on its input $x(t)$ for $t \leq t_0$. Suppose that we have the LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. The value of the output $y(t)$ for $t = t_0$ is given by

$$
\begin{aligned}
y(t_0) &= [x(t) * h(t)]|_{t=t_0} \\
&= \left. \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right|_{t=t_0} \\
&= \int_{-\infty}^{\infty} x(\tau)h(t_0-\tau)d\tau \\
&= \int_{-\infty}^{t_0} x(\tau)h(t_0-\tau)d\tau + \int_{t_0^+}^{\infty} x(\tau)h(t_0-\tau)d\tau. \tag{3.34}
\end{aligned}
$$
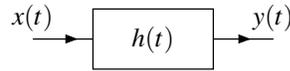
Figure 3.11: System.

In order for the expression for $y(t_0)$ in (3.34) not to depend on $x(t)$ for $t > t_0$, we must have that

$$h(t) = 0 \quad \text{for } t < 0 \tag{3.35}$$

(i.e., $h(t)$ is a causal signal). In this case, (3.34) simplifies to

$$y(t_0) = \int_{-\infty}^{t_0} x(\tau)h(t_0 - \tau)d\tau.$$

Clearly, the result of this integration does not depend on $x(t)$ for $t > t_0$ (since τ varies from $-\infty$ to t_0). Therefore, a LTI system is causal if its impulse response $h(t)$ satisfies (3.35).

Example 3.10. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$). Therefore, the system is causal. $\square$

Example 3.11. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = \delta(t + t_0),$$

where t_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of $\delta(t)$, we can easily deduce that $h(t) = 0$ except at $t = -t_0$. Since $-t_0 < 0$, the system is not causal. $\square$

3.9.3 Invertibility

Recall that a system is invertible if we can always uniquely determine its input $x(t)$ from its output $y(t)$. An equivalent way of stating this is that an inverse system must exist.

Suppose now that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Such a system is illustrated in Figure 3.11.

One can readily show that the inverse of a LTI system, if it exists, must also be LTI. This follows from the fact that the inverse of a linear system must be linear, and the inverse of a time-invariant system must be time invariant. Since the inverse system, if it exists, must be LTI, the system can be completely characterized by its impulse response. Let us denote the impulse response of the inverse system as $h^{\text{inv}}(t)$. Now, we want to find the relationship between $h(t)$ and $h^{inv}(t)$.

If the inverse system exists, then by definition, it must be such that

$$x(t) * h(t) * h^{\text{inv}}(t) = x(t). \tag{3.36}$$

This relationship is expressed diagrammatically in Figure 3.12. Since the unit-impulse function is the convolutional identity, we can equivalently rewrite (3.36) as

$$x(t) * h(t) * h^{\text{inv}}(t) = x(t) * \delta(t).$$
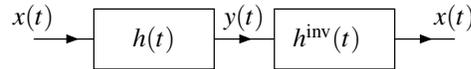
Figure 3.12: System in cascade with its inverse.

By comparing the left- and right-hand sides of the preceding equation, we have

$$h(t) * h^{\text{inv}}(t) = \delta(t). \tag{3.37}$$

Therefore, a system will have an inverse if and only if a solution for $h^{inv}(t)$ exists in (3.37). Thus, a system is invertible if and only if a solution to this equation exists.

Example 3.12. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$h(t) = A\delta(t - t_0) \tag{3.38}$$

where A is a nonzero real constant and t_0 is a real constant. Determine whether this system is invertible. If it is invertible, find the impulse response $h^{\text{inv}}(t)$ of the inverse system.

Solution. The inverse system, if it exists, is given by the solution to the equation

$$h(t) * h^{\text{inv}}(t) = \delta(t).$$

So, let us attempt to solve this equation for $h^{\text{inv}}(t)$. Substituting (3.38) into (3.37) and using straightforward algebraic manipulation, we can write:

$$h(t) * h^{\text{inv}}(t) = \delta(t)$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} h(\tau) h^{\text{inv}}(t - \tau) d\tau = \delta(t)$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} A\delta(\tau - t_0) h^{\text{inv}}(t - \tau) d\tau = \delta(t)$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} \delta(\tau - t_0) h^{\text{inv}}(t - \tau) d\tau = \tfrac{1}{A}\delta(t).$$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$$h^{\text{inv}}(t - t_0) = \tfrac{1}{A}\delta(t). \tag{3.39}$$

Now, we perform a change of variable. Let $\tau = t - t_0$ so that $t = \tau + t_0$. Using this change of variable, we can rewrite (3.39) as

$$h^{\text{inv}}(\tau) = \tfrac{1}{A}\delta(\tau + t_0).$$

Since $A \neq 0$, the function $h^{\text{inv}}(t)$ is always well defined. Therefore, the inverse system exists and has the impulse response $h^{\text{inv}}(t)$ given by

$$h^{\text{inv}}(t) = \tfrac{1}{A}\delta(t + t_0).$$

$\square$

Example 3.13. Consider the system shown in Figure 3.13 with the input $x(t)$ and output $y(t)$. Use the notion of an inverse system in order to express $y(t)$ in terms of $x(t)$.
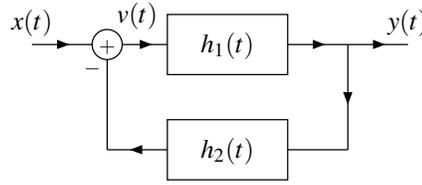
Figure 3.13: Feedback system.

Solution. From Figure 3.13, we can write:

$$v(t) = x(t) - y(t) * h_2(t) \quad \text{and} \tag{3.40}$$
$$y(t) = v(t) * h_1(t). \tag{3.41}$$

Substituting (3.40) into (3.41), and simplifying we obtain:

$$
\begin{aligned}
& y(t) = [x(t) - y(t) * h_2(t)] * h_1(t) \\
\Rightarrow \quad & y(t) = x(t) * h_1(t) - y(t) * h_2(t) * h_1(t) \\
\Rightarrow \quad & y(t) + y(t) * h_2(t) * h_1(t) = x(t) * h_1(t) \\
\Rightarrow \quad & y(t) * \delta(t) + y(t) * h_2(t) * h_1(t) = x(t) * h_1(t) \\
\Rightarrow \quad & y(t) * [\delta(t) + h_2(t) * h_1(t)] = x(t) * h_1(t). \tag{3.42}
\end{aligned}
$$

For convenience, we now define the function $g(t)$ as

$$g(t) = \delta(t) + h_2(t) * h_1(t). \tag{3.43}$$

So, we can rewrite (3.42) as

$$y(t) * g(t) = x(t) * h_1(t). \tag{3.44}$$

Thus, we have almost solved for $y(t)$ in terms of $x(t)$. To complete the solution, we need to eliminate $g(t)$ from the left-hand side of the equation. To do this, we use the notion of an inverse system. Consider the inverse of the system with impulse response $g(t)$. This inverse system has an impulse response $g^{\text{inv}}(t)$ given by

$$g(t) * g^{\text{inv}}(t) = \delta(t). \tag{3.45}$$

This relationship follows from the definition of an inverse system (associated with (3.37)). Now, we use $g^{\text{inv}}(t)$ in order to simplify (3.44) as follows:

$$
\begin{aligned}
& y(t) * g(t) = x(t) * h_1(t) \\
\Rightarrow \quad & y(t) * g(t) * g^{\text{inv}}(t) = x(t) * h_1(t) * g^{\text{inv}}(t) \\
\Rightarrow \quad & y(t) * \delta(t) = x(t) * h_1(t) * g^{\text{inv}}(t) \\
\Rightarrow \quad & y(t) = x(t) * h_1(t) * g^{\text{inv}}(t).
\end{aligned}
$$

Thus, we can express the output $y(t)$ in terms of the input $x(t)$ as

$$y(t) = x(t) * h_1(t) * g^{\text{inv}}(t),$$

where $g^{\text{inv}}(t)$ is given by (3.45) and $g(t)$ is given by (3.43). $\qquad \square$

3.9.4 Stability

Recall that a system is BIBO stable if any arbitrary bounded input produces a bounded output. Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Further, suppose that $|x(t)| \leq A < \infty$ for all t (i.e., $x(t)$ is bounded). We can write

$$y(t) = x(t) * h(t)$$
$$= h(t) * x(t)$$
$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

So, we have (by taking the magnitude of both sides of the preceding equation)

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right|. \tag{3.46}$$

One can show, for any two functions $f_1(t)$ and $f_2(t)$, that

$$\left| \int_{-\infty}^{\infty} f_1(t)f_2(t)dt \right| \leq \int_{-\infty}^{\infty} |f_1(t)f_2(t)|\, dt.$$

Using this inequality, we can rewrite (3.46) as

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau = \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\, d\tau.$$

We know (by assumption) that $|x(t)| \leq A$, so we can replace $|x(t)|$ by its bound A in the above inequality to obtain

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\, d\tau \leq \int_{-\infty}^{\infty} A\,|h(\tau)|\, d\tau = A \int_{-\infty}^{\infty} |h(\tau)|\, d\tau. \tag{3.47}$$

Thus, we have

$$|y(t)| \leq A \int_{-\infty}^{\infty} |h(\tau)|\, d\tau. \tag{3.48}$$

Since A is finite, we can deduce from (3.48) that $y(t)$ is bounded if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty. \tag{3.49}$$

(i.e., $h(t)$ is absolutely integrable). Therefore, a LTI system is BIBO stable if its impulse response $h(t)$ is absolutely integrable (as in (3.49)). In other words, the absolute integrability of the impulse response $h(t)$ is a sufficient condition for BIBO stability.

As it turns out, condition (3.49) is also necessary for BIBO stability. Suppose that $h(t)$ is not absolutely integrable. That is, suppose that

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \infty.$$

If such is the case, we can show that the system is not BIBO stable. To begin, consider the particular input $x(t)$ given by the following:

$$x(t) = \operatorname{sgn}(h(-t)),$$

where

$$\operatorname{sgn}\alpha = \begin{cases} -1 & \text{for } \alpha < 0 \\ 0 & \text{for } \alpha = 0 \\ 1 & \text{for } \alpha > 0. \end{cases}$$

The signal $x(t)$ can only assume the values -1, 0, and 1. So, $x(t)$ is obviously bounded (i.e., $|x(t)| \leq 1$). The output $y(t)$ is given by

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\
&= \int_{-\infty}^{\infty} (\operatorname{sgn} h(-\tau)) h(t - \tau) d\tau.
\end{aligned}
\tag{3.50}
$$

Now, let us consider the output $y(t)$ at $t = 0$. From (3.50), we have

$$
y(0) = \int_{-\infty}^{\infty} (\operatorname{sgn} h(-\tau)) h(-\tau) d\tau.
\tag{3.51}
$$

Then, we observe that

$$
\alpha \operatorname{sgn} \alpha = |\alpha| \quad \text{for any real } \alpha.
$$

So $(\operatorname{sgn} h(-\tau)) h(-\tau) = |h(-\tau)|$, and we can simplify (3.51) to obtain

$$
\begin{aligned}
y(0) &= \int_{-\infty}^{\infty} |h(-\tau)| \, d\tau \\
&= \int_{-\infty}^{\infty} |h(\tau)| \, d\tau \\
&= \infty.
\end{aligned}
$$

Thus, we have shown that the bounded input $x(t)$ will result in an unbounded output $y(t)$ (where $y(t)$ is unbounded at $t = 0$). Consequently, if the impulse response $h(t)$ is not absolutely integrable (i.e., does not satisfy (3.49)), the system is not BIBO stable. Consequently, the condition (3.49) is not only sufficient but also necessary for a system to be BIBO stable. In other words, we have that a LTI system is BIBO stable if and only if its impulse response $h(t)$ satisfies

$$
\int_{-\infty}^{\infty} |h(t)| \, dt < \infty
$$

(i.e., $h(t)$ is absolutely integrable).

Example 3.14. Suppose that we have the LTI system with impulse response $h(t)$ given by

$$
h(t) = e^{at} u(t),
$$

where a is a real constant. Determine for what values of the constant a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response $h(t)$ is absolutely integrable. Suppose that $a \neq 0$. We can write

$$
\begin{aligned}
\int_{-\infty}^{\infty} |h(t)| \, dt &= \int_{-\infty}^{\infty} \left| e^{at} u(t) \right| \, dt \\
&= \int_{0}^{\infty} \left| e^{at} \right| \, dt \\
&= \int_{0}^{\infty} e^{at} \, dt \\
&= \left[\tfrac{1}{a} e^{at} \right] \big|_{0}^{\infty} \\
&= \tfrac{1}{a} \left(e^{a\infty} - 1 \right).
\end{aligned}
$$

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\int_{-\infty}^{\infty} |h(t)| \, dt = 0 - \frac{1}{a}$$
$$= -\frac{1}{a}.$$

Let us now consider the case of $a = 0$. In this case, we have

$$\int_{-\infty}^{\infty} |h(t)| \, dt = \int_{-\infty}^{\infty} |u(t)| \, dt$$
$$= \int_{0}^{\infty} dt$$
$$= [t]|_{0}^{\infty}$$
$$= \infty.$$

Thus, we have shown that

$$\int_{-\infty}^{\infty} |h(t)| \, dt = \begin{cases} -\frac{1}{a} & \text{for } a < 0 \\ \infty & \text{for } a \geq 0. \end{cases}$$

In other words, the impulse response $h(t)$ is absolutely integrable if $a < 0$. Consequently, the system is BIBO stable if $a < 0$. $\square$

Example 3.15. Suppose that we have the LTI system with input $x(t)$ and output $y(t)$ defined by

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response $h(t)$ of the system.

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$
$$= \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$
$$= u(t).$$

Using this expression for $h(t)$, we now check to see if $h(t)$ is absolutely integrable. We have

$$\int_{-\infty}^{\infty} |h(t)| \, dt = \int_{-\infty}^{\infty} |u(t)| \, dt$$
$$= \int_{0}^{\infty} dt$$
$$= \infty.$$

So, we have shown that $h(t)$ is not absolutely integrable. Therefore, the system is not BIBO stable. $\square$

3.10 Response of Continuous-Time LTI Systems to Complex Exponential Signals

Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Consider now the response of the system to the complex exponential $x(t) = e^{st}$. We can write:

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \\
&= H(s) e^{st}
\end{aligned}
$$

where $H(s)$ is the complex function given by

$$
H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau. \tag{3.52}
$$

(The function $H(s)$ depends only on the complex quantity s.) From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$. Thus, we have that e^{st} and $H(s)$ satisfy an eigenvalue problem of the form

$$
\mathcal{H}\{\Phi(t)\} = \lambda \Phi(t)
$$

where $\Phi(t) = e^{st}$ and $\lambda = H(s)$. As a matter of terminology, $\Phi(t)$ is said to be an **eigenfunction** and λ its corresponding **eigenvalue**. Therefore, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.

Suppose now that we can express some arbitrary input signal $x(t)$ as a sum of complex exponentials as follows:

$$
x(t) = \sum_k a_k e^{s_k t}.
$$

From the eigenvalue property, the response to the input $a_k e^{s_k t}$ is $a_k H(s_k) e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$
\begin{aligned}
y(t) &= \mathcal{H}\{x(t)\} \\
&= \mathcal{H}\left\{ \sum_k a_k e^{s_k t} \right\} \\
&= \sum_k a_k \mathcal{H}\{e^{s_k t}\} \\
&= \sum_k a_k H(s_k) e^{s_k t}.
\end{aligned}
$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. As a matter of terminology, we refer to $H(s)$ as the **system function**.

Example 3.16. Suppose that we have the LTI system with the impulse response $h(t)$ given by

$$
h(t) = \delta(t - 1). \tag{3.53}
$$

Find the system function $H(s)$.

Solution. Substituting (3.53) into (3.52), we obtain

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$
$$= \int_{-\infty}^{\infty} \delta(t-1)e^{-st}\,dt$$
$$= e^{-s}.$$

Therefore, the system function $H(s)$ is given by $H(s) = e^{-s}$. □

Example 3.17. Suppose that we have the LTI system from Example 3.16. Use the system function to determine the response $y(t)$ of the system to the particular input $x(t)$ given by

$$x(t) = e^t \cos(\pi t) = \tfrac{1}{2} e^{(1+j\pi)t} + \tfrac{1}{2} e^{(1-j\pi)t}.$$

Solution. The input $x(t)$ has already been expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t}$$

where $a_0 = a_1 = \tfrac{1}{2}$, $s_0 = 1 + j\pi$, and $s_1 = s_0^* = 1 - j\pi$. In Example 3.16, we found the system function $H(s)$ to be $H(s) = e^{-s}$. So we can calculate $y(t)$ by using the system function as follows:

$$y(t) = \sum_{k} a_k H(s_k) e^{s_k t}$$
$$= a_0 H(s_0) e^{s_0 t} + a_1 H(s_1) e^{s_1 t}$$
$$= \tfrac{1}{2} H(1+j\pi) e^{(1+j\pi)t} + \tfrac{1}{2} H(1-j\pi) e^{(1-j\pi)t}$$
$$= \tfrac{1}{2} e^{-(1+j\pi)} e^{(1+j\pi)t} + \tfrac{1}{2} e^{-(1-j\pi)} e^{(1-j\pi)t}$$
$$= \tfrac{1}{2} e^{t-1+j\pi t - j\pi} + \tfrac{1}{2} e^{t-1-j\pi t + j\pi}$$
$$= \tfrac{1}{2} e^{t-1} e^{j\pi(t-1)} + \tfrac{1}{2} e^{t-1} e^{-j\pi(t-1)}$$
$$= e^{t-1} \left[\tfrac{1}{2} \left(e^{j\pi(t-1)} + e^{-j\pi(t-1)} \right) \right]$$
$$= e^{t-1} \cos \pi(t-1).$$

In passing, we note that we should expect the above result for $y(t)$, since the output of the system is nothing more than the input time-shifted by one. □

3.11 Problems

3.1 Using graphical methods, for each pair of signals $x(t)$ and $h(t)$ given in the figures below, compute the convolution $y(t) = x(t) * h(t)$.

(a)

(b)

(c)

(d)
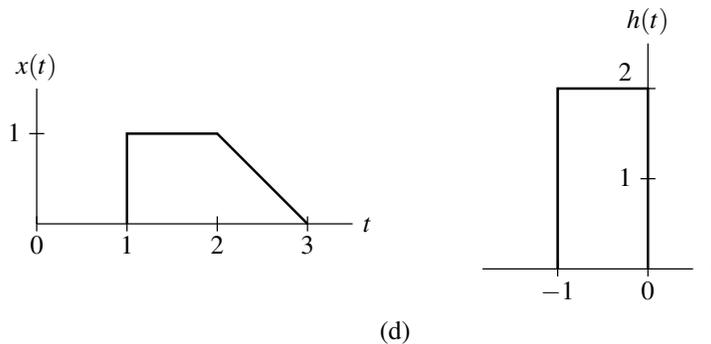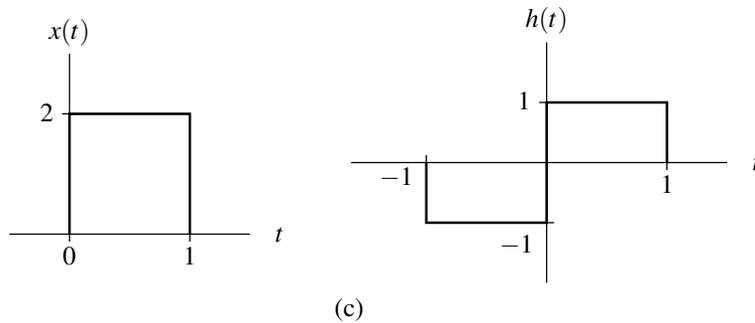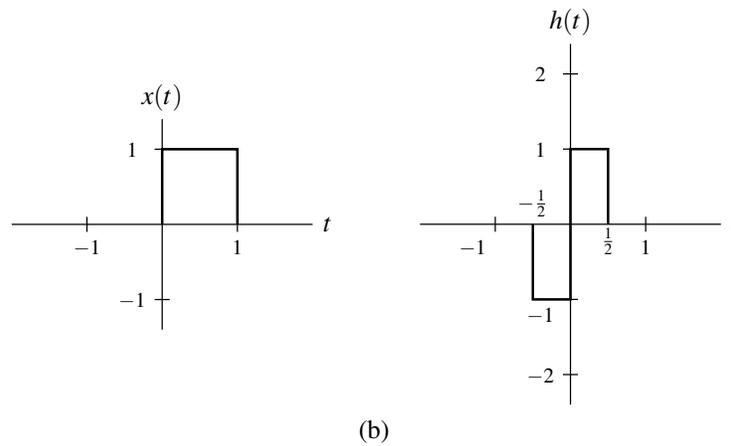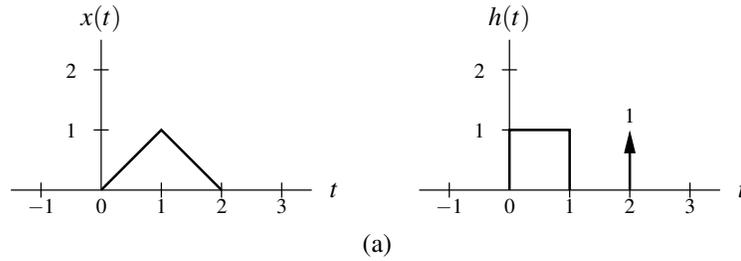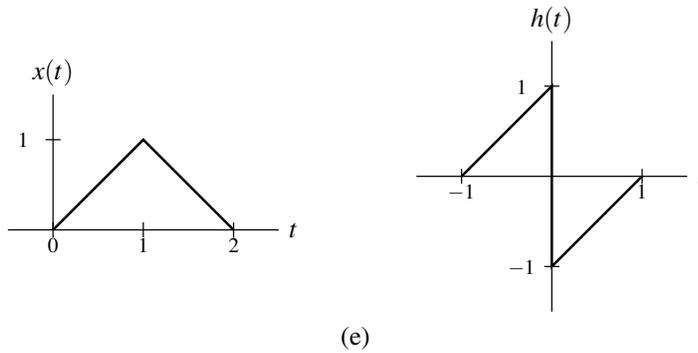
(e)
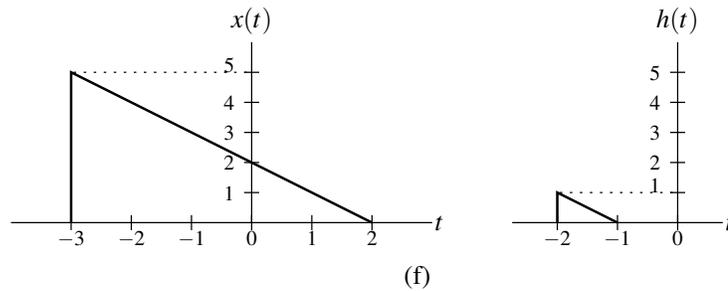
(f)

3.2 For each pair of signals $x(t)$ and $h(t)$ given below, compute the convolution $y(t) = x(t) * h(t)$.
(a) $x(t) = e^{at}u(t)$ and $h(t) = e^{-at}u(t)$ where a is a nonzero real constant;
(b) $x(t) = e^{-j\omega_0 t}u(t)$ and $h(t) = e^{j\omega_0 t}u(t)$ where ω_0 is a strictly positive real constant;
(c) $x(t) = u(t-2)$ and $h(t) = u(t+3)$;
(d) $x(t) = u(t)$ and $h(t) = e^{-2t}u(t-1)$;
(e) $x(t) = u(t-1) - u(t-2)$ and $h(t) = e^t u(-t)$.

3.3 Let $y(t) = x(t) * h(t)$. Given that

$$v(t) = \int_{-\infty}^{\infty} x(-\tau - b)h(\tau + at)d\tau,$$

where a and b are constants, express $v(t)$ in terms of $y(t)$.

3.4 Consider the convolution $y(t) = x(t) * h(t)$. Assuming that the convolution $y(t)$ exists, prove that each of the following assertions is true:
(a) If $x(t)$ is periodic then $y(t)$ is periodic.
(b) If $x(t)$ is even and $h(t)$ is odd, then $y(t)$ is odd.

3.5 From the definition of the convolution operation, show that if $y(t) = x(t) * h(t)$, then $\frac{d}{dt}y(t) = x(t) * [\frac{d}{dt}h(t)]$.

3.6 Let $x(t)$ and $h(t)$ be signals satisfying

$$x(t) = 0 \quad \text{for } t < A_1 \text{ or } t > A_2, \quad \text{and}$$
$$h(t) = 0 \quad \text{for } t < B_1 \text{ or } t > B_2$$

(i.e., $x(t)$ and $h(t)$ are finite duration). Determine for which values of t the convolution $y(t) = x(t) * h(t)$ must be zero.

3.7 Find the impulse response of the LTI system characterized by each of the equations below. In each case, the input and output of the system are denoted as $x(t)$ and $y(t)$, respectively.
(a) $y(t) = \int_{-\infty}^{t+1} x(\tau)d\tau$;

(b) $y(t) = \int_{-\infty}^{\infty} x(\tau + 5)e^{\tau - t + 1}u(t - \tau - 2)d\tau$;

(c) $y(t) = \int_{-\infty}^{t} x(\tau)v(t - \tau)d\tau$ and

(d) $y(t) = \int_{t-1}^{t} x(\tau)d\tau$.

3.8 Consider the system with input $x(t)$ and output $y(t)$ as shown in the figure below. Suppose that the systems $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are LTI systems with impulse responses $h_1(t)$, $h_2(t)$, and $h_3(t)$, respectively.
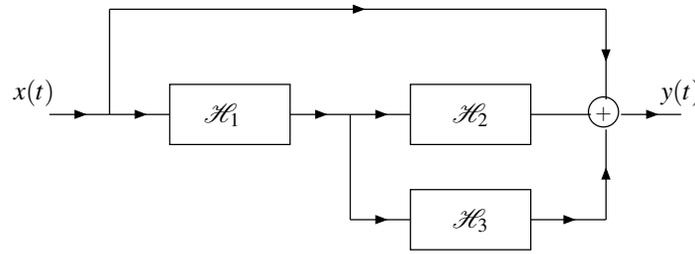
(a) Find the impulse response $h(t)$ of the overall system in terms of $h_1(t)$, $h_2(t)$, and $h_3(t)$.
(b) Determine the impulse response $h(t)$ in the specific case that

$$h_1(t) = \delta(t+1), \quad h_2(t) = \delta(t), \quad \text{and} \quad h_3(t) = \delta(t).$$

3.9 Consider a LTI system whose response to the signal $x_1(t) = u(t) - u(t-1)$ is the signal $y_1(t)$. Determine the response $y_2(t)$ of the system to the input $x_2(t)$ shown in the figure below in terms of $y_1(t)$.
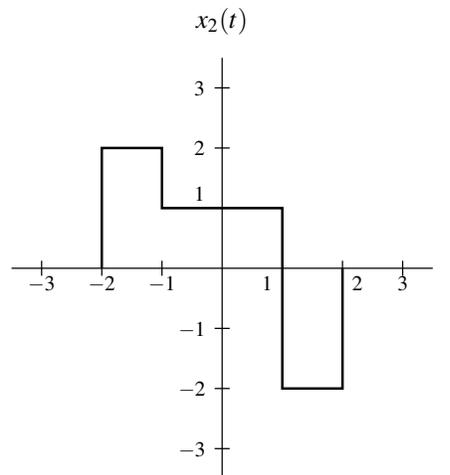
3.10 Suppose that we have the system shown in the figure below with input $x(t)$ and output $y(t)$. This system is formed by the interconnection of two LTI systems with the impulse responses $h_1(t)$ and $h_2(t)$.
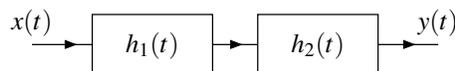
For each pair of $h_1(t)$ and $h_2(t)$ given below, find the output $y(t)$ if the input $x(t) = u(t)$.
(a) $h_1(t) = \delta(t)$ and $h_2(t) = \delta(t)$;
(b) $h_1(t) = \delta(t+1)$ and $h_2(t) = \delta(t+1)$;
(c) $h_1(t) = e^{-3t}u(t)$ and $h_2(t) = \delta(t)$.

3.11 Show that a linear system is invertible if and only if the only input $x(t)$ that produces the output $y(t) = 0$ for all t is $x(t) = 0$ for all t.

3.12 Consider the LTI systems with the impulse responses given below. Determine whether each of these systems is causal and/or memoryless.
(a) $h(t) = (t+1)u(t-1)$;
(b) $h(t) = 2\delta(t+1)$;
(c) $h(t) = \frac{\omega_c}{\pi} \operatorname{sinc} \omega_c t$;
(d) $h(t) = e^{-4t}u(t-1)$;
(e) $h(t) = e^t u(-1-t)$;
(f) $h(t) = e^{-3|t|}$; and
(g) $h(t) = 3\delta(t)$.

3.13 Consider the LTI systems with the impulse responses given below. Determine whether each of these systems is BIBO stable.
(a) $h(t) = e^{at}u(-t)$ where a is a strictly positive real constant;
(b) $h(t) = (1/t)u(t-1)$;
(c) $h(t) = e^t u(t)$;
(d) $h(t) = \delta(t-10)$;
(e) $h(t) = \operatorname{rect} t$; and
(f) $h(t) = e^{-|t|}$.

3.14 Suppose that we have two LTI systems with impulse responses

$$h_1(t) = \tfrac{1}{2}\delta(t-1) \quad \text{and} \quad h_2(t) = 2\delta(t+1).$$

Determine whether these systems are inverses of one another.

3.15 Consider the system shown in the figure below, where $\mathscr{H}$ is a LTI system and $\mathscr{G}$ is known to be the inverse system of $\mathscr{H}$. Let $y_1(t)$ and $y_2(t)$ denote the responses of the system $\mathscr{H}$ to the inputs $x_1(t)$ and $x_2(t)$, respectively.
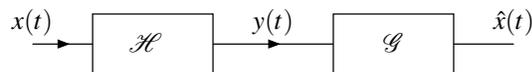
(a) Determine the response of the system $\mathscr{G}$ to the input $a_1 y_1(t) + a_2 y_2(t)$ where a_1 and a_2 are complex constants.
(b) Determine the response of the system $\mathscr{G}$ to the input $y_1(t - t_0)$ where t_0 is a real constant.
(c) Using the results of the previous parts of this question, determine whether the system $\mathscr{G}$ is linear and/or time invariant.

3.16 Suppose that we have the systems $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_3$, and $\mathscr{H}_4$, whose responses to a complex exponential input e^{j2t} are given by

$$e^{j2t} \xrightarrow{\mathscr{H}_1} 2e^{j2t},$$

$$e^{j2t} \xrightarrow{\mathscr{H}_2} te^{j2t},$$

$$e^{j2t} \xrightarrow{\mathscr{H}_3} e^{j2t+\pi/3}, \quad \text{and}$$

$$e^{j2t} \xrightarrow{\mathscr{H}_4} \cos 2t.$$

Indicate which of these systems cannot be LTI.

Chapter 4

Continuous-Time Fourier Series

4.1 Introduction

One very important tool in the study of signals and systems is the Fourier series. A very large class of signals can be represented using Fourier series, namely most practically useful periodic signals. The Fourier series represents a signal as a linear combination of complex sinusoids. This is often desirable since complex sinusoids are easy functions with which to work. For example, complex sinusoids are easy to integrate and differentiate. Also, complex sinusoids have important properties in relation to LTI systems. In particular, complex sinusoids are eigenfunctions of LTI systems. Therefore, the response of a LTI system to a complex sinusoid is the same complex sinusoid multiplied by a complex scaling factor.

4.2 Definition of Continuous-Time Fourier Series

Suppose that we have a set of **harmonically-related** complex sinusoids of the form

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \quad k = 0, \pm 1, \pm 2, \ldots.$$

The fundamental frequency of the kth complex sinusoid $\phi_k(t)$ is $k\omega_0$, an integer multiple of ω_0. Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic. More specifically, a linear combination of these complex sinusoids is periodic with period $T = 2\pi/\omega_0$.

Suppose that we can represent a periodic complex signal $x(t)$ as a linear combination of harmonically-related complex sinusoids:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{4.1}$$

Such a representation is known as a **Fourier series**. More specifically, this is the **complex exponential form** of the Fourier series. The terms in the summation for $k = 1$ and $k = -1$ are known as the fundamental frequency components or **first harmonic components**, and have the fundamental frequency ω_0. More generally, the terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$. Since the complex sinusoids are harmonically related, the signal $x(t)$ is periodic with period $T = 2\pi/\omega_0$ (and frequency ω_0).

Since we often work with Fourier series, it is sometimes convenient to have an abbreviated notation to indicate that a signal is associated with particular Fourier series coefficients. If a signal $x(t)$ has the Fourier series coefficient sequence c_k, we sometimes indicate this using the notation

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} c_k.$$

Consider the Fourier series representation of the periodic signal $x(t)$ given by (4.1). In the most general case, $x(t)$ is a complex signal, but let us now suppose that $x(t)$ is a real signal. In the case of real signals, an important

relationship exists between the Fourier series coefficients c_k and c_{-k}. To show this, we proceed as follows. Suppose that we can represent $x(t)$ in the form of (4.1). So, we have

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{4.2}$$

Taking the complex conjugate of both sides of the preceding equation, we obtain

$$
\begin{aligned}
x^*(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right)^* \\
&= \sum_{k=-\infty}^{\infty} \left(c_k e^{jk\omega_0 t} \right)^* \\
&= \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}.
\end{aligned}
\tag{4.3}
$$

Since $x(t)$ is real, we know that $x(t) = x^*(t)$, and we can rewrite (4.3) as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^* e^{-jk\omega_0 t}.$$

Replacing k by $-k$ in the summation, we obtain

$$x(t) = \sum_{k=-\infty}^{\infty} c_{-k}^* e^{jk\omega_0 t}. \tag{4.4}$$

Comparing (4.2) and (4.4), we can see that

$$c_k = c_{-k}^*. \tag{4.5}$$

Consequently, if $x(t)$ is a real signal, we have that c_k and c_{-k} are complex conjugates of each other.

Using the relationship in (4.5), we can derive two alternative forms of the Fourier series for the case of real signals. We begin by rewriting (4.1) in a slightly different form. In particular, we rearrange the summation to obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t} \right].$$

Substituting $c_k = c_{-k}^*$ from (4.5), we obtain

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[c_k e^{jk\omega_0 t} + c_k^* e^{-jk\omega_0 t} \right].$$

Now, we observe that the two terms inside the summation are complex conjugates of each other. So, we can rewrite the equation as

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\{ c_k e^{jk\omega_0 t} \}. \tag{4.6}$$

Let us now rewrite c_k in polar form as

$$c_k = |c_k| e^{j\theta_k},$$

where θ_k is real (i.e., $\theta_k = \arg c_k$). Substituting this expression for c_k into (4.6) yields

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{|c_k|\,e^{j(k\omega_0 t + \theta_k)}\right\}$$

$$= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{|c_k|\left[\cos(k\omega_0 t + \theta_k) + j\sin(k\omega_0 t + \theta_k)\right]\right\}$$

$$= c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{|c_k|\cos(k\omega_0 t + \theta_k) + j|c_k|\sin(k\omega_0 t + \theta_k)\right\}.$$

Finally, further simplification yields

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k|\cos(k\omega_0 t + \theta_k)$$

(where $\theta_k = \arg c_k$). This is known as the **combined trigonometric form** of a Fourier series.

A second alternative form of the Fourier series can be obtained by expressing c_k in Cartesian form as

$$c_k = \tfrac{1}{2}(a_k - jb_k).$$

where a_k and b_k are real. Substituting this expression for c_k into (4.6) from earlier yields

$$x(t) = c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left\{(\tfrac{1}{2}(a_k - jb_k)e^{jk\omega_0 t}\right\}$$

$$= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\left\{(a_k - jb_k)(\cos k\omega_0 t + j\sin k\omega_0 t)\right\}$$

$$= c_0 + \sum_{k=1}^{\infty} \operatorname{Re}\left\{a_k\cos k\omega_0 t + ja_k\sin k\omega_0 t - jb_k\cos k\omega_0 t + b_k\sin k\omega_0 t\right\}.$$

Further simplification yields

$$x(t) = c_0 + \sum_{k=1}^{\infty} [a_k\cos k\omega_0 t + b_k\sin k\omega_0 t]$$

(where $a_k = \operatorname{Re}2c_k$ and $b_k = -\operatorname{Im}2c_k$). This is known as the **trigonometric form** of a Fourier series.

By comparing the various forms of the Fourier series introduced above, we can see that the quantities c_k, a_k, b_k, and θ_k are related as follows:

$$2c_k = a_k - jb_k \quad \text{and} \quad c_k = |c_k|\,e^{j\theta_k}.$$

(Recall that a_k, b_k, and θ_k are real and c_k is complex.)

4.3 Determining the Fourier Series Representation of a Continuous-Time Periodic Signal

Given an arbitrary periodic signal $x(t)$, we need some means for finding its corresponding Fourier series representation. In other words, we need a method for calculating the Fourier series coefficients c_k. In what follows, we derive a formula for the calculation of the c_k. We begin with the definition of the Fourier series in (4.1). Multiplying both sides of this equation by $e^{-jn\omega_0 t}$ yields

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t}.$$

As a matter of notation, we use $\int_T$ to denote the integral over an arbitrary interval of length T (i.e., the interval $(t_0, t_0 + T)$ for arbitrary t_0). Integrating both sides of this equation over one period T of $x(t)$, we obtain

$$\int_T x(t) e^{-jn\omega_0 t} dt = \int_T \sum_{k=-\infty}^{\infty} c_k e^{j(k-n)\omega_0 t} dt.$$

Reversing the order of integration and summation yields

$$\int_T x(t) e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{\infty} c_k \left(\int_T e^{j(k-n)\omega_0 t} dt \right). \tag{4.7}$$

Consider now the integral on the right-hand side of this equation. In order to evaluate this integral, we employ Euler's formula to write

$$\begin{aligned} \int_T e^{j(k-n)\omega_0 t} dt &= \int_T [\cos((k-n)\omega_0 t) + j\sin((k-n)\omega_0 t)] \, dt \\ &= \int_T \cos((k-n)\omega_0 t) \, dt + j \int_T \sin((k-n)\omega_0 t) \, dt. \end{aligned} \tag{4.8}$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are both sinusoids of frequency $(k-n)\omega_0$ and period $T/|k-n|$. Since we are integrating over an interval of length T, both sinusoids will be integrated over an integral number of periods, resulting in a value of zero. For $k = n$, the integrand on the left-hand side of (4.8) is simply $e^{j0} = 1$, and the result of integration is T. To summarize, we have the following:

$$\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Substituting (4.9) into (4.7), we obtain

$$\int_T x(t) e^{-jn\omega_0 t} dt = c_n T. \tag{4.10}$$

Rearranging, we obtain the following expression for c_n:

$$c_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt.$$

Thus, we can calculate the Fourier series coefficient sequence c_k for an arbitrary periodic signal $x(t)$ using the formula:

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt. \tag{4.11}$$

As a matter of terminology, we refer to (4.11) as the **Fourier series analysis equation** and (4.1) as the **Fourier series synthesis equation**.

Suppose that we have a complex periodic function $x(t)$ with period T and Fourier series coefficient sequence c_k. One can easily show that the coefficient c_0 is the average value of $x(t)$ over a single period T. The proof is trivial. Consider the Fourier series analysis equation given by (4.11). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left[\frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \right]\Bigg|_{k=0} \\ &= \frac{1}{T} \int_T x(t) e^0 dt \\ &= \frac{1}{T} \int_T x(t) dt. \end{aligned}$$

Thus, c_0 is simply the average value of $x(t)$ over a single period.

Example 4.1 (Fourier series a of periodic square wave). Find the Fourier series representation of the periodic square wave $x(t)$ shown in Figure 4.1.

Solution. Let us consider the single period of $x(t)$ for $0 \leq t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & \text{for } 0 \leq t < \frac{T}{2} \\ -A & \text{for } \frac{T}{2} \leq t < T. \end{cases}$$

We use the Fourier series analysis equation (4.11) to write (and subsequently assume that $k \neq 0$):

$$\begin{aligned}
c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \left(\int_0^{T/2} A e^{-jk\omega_0 t} dt + \int_{T/2}^T (-A) e^{-jk\omega_0 t} dt \right) \\
&= \frac{1}{T} \left(\left[\frac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\frac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^T \right) \\
&= \frac{-A}{j2\pi k} \left(\left[e^{-jk\omega_0 t} \right] \Big|_0^{T/2} - \left[e^{-jk\omega_0 t} \right] \Big|_{T/2}^T \right) \\
&= \frac{jA}{2\pi k} \left(\left[e^{-j\pi k} - 1 \right] - \left[e^{-j2\pi k} - e^{-j\pi k} \right] \right) \\
&= \frac{jA}{2\pi k} \left[2e^{-j\pi k} - e^{-j2\pi k} - 1 \right] \\
&= \frac{jA}{2\pi k} \left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1 \right].
\end{aligned}$$

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

$$\begin{aligned}
c_k &= \frac{jA}{2\pi k} [2(-1)^k - 1^k - 1] \\
&= \frac{jA}{2\pi k} [2(-1)^k - 2] \\
&= \frac{jA}{\pi k} [(-1)^k - 1] \\
&= \begin{cases} \dfrac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even}, k \neq 0. \end{cases}
\end{aligned}$$

Now, we consider the case of c_0. We have

$$\begin{aligned}
c_0 &= \frac{1}{T} \int_T x(t) dt \\
&= \frac{1}{T} \left[\int_0^{T/2} A dt - \int_{T/2}^T A dt \right] \\
&= \frac{1}{T} \left[\frac{AT}{2} - \frac{AT}{2} \right] \\
&= 0.
\end{aligned}$$

Thus, the Fourier series of $x(t)$ is given by (4.1) where the coefficient sequence c_k is

$$c_k = \begin{cases} \dfrac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even}. \end{cases}$$

$\square$
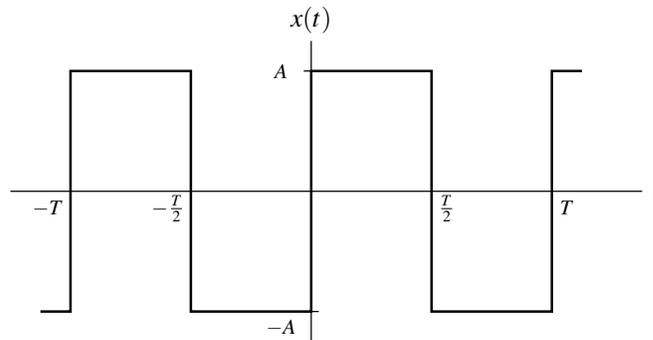
Figure 4.1: Periodic square wave.
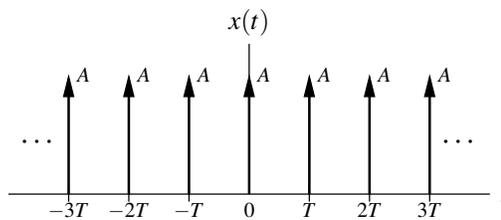
Figure 4.2: Periodic impulse train.

Example 4.2 (Fourier series of periodic impulse train). Suppose that we have the periodic impulse train $x(t)$ shown in Figure 4.2. Find the Fourier series representation of $x(t)$.

Solution. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$. We use the Fourier series analysis equation (4.11) to write:

$$
\begin{aligned}
c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \int_{-T/2}^{T/2} A\delta(t) e^{-jk\omega_0 t} dt \\
&= \frac{A}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt.
\end{aligned}
$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$
c_k = \frac{A}{T}.
$$

Thus, the Fourier series for $x(t)$ is given by (4.1) where the coefficient sequence c_k is $c_k = \frac{A}{T}$. □

Example 4.3. Consider the periodic function $x(t)$ with fundamental period $T = 3$ as shown in Figure 4.3. Find the Fourier series representation of $x(t)$.

Solution. The signal $x(t)$ has the fundamental frequency $\omega_0 = 2\pi/T = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$
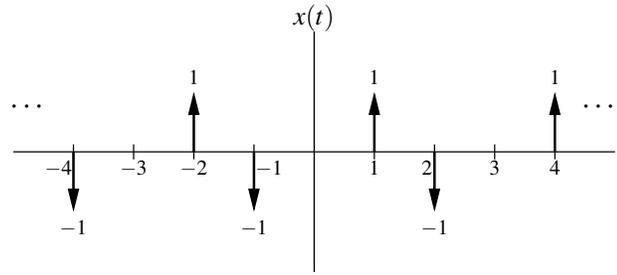
Figure 4.3: Periodic impulse train.

for $-\frac{T}{2} \le t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \le t < \frac{3}{2}$). Computing the Fourier series coefficients c_k, we have

$$
\begin{aligned}
c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \tfrac{1}{T} \int_{-3/2}^{3/2} x(t) e^{-jk\omega_0 t} dt \\
&= \tfrac{1}{T} \left[\int_{-3/2}^{0} -\delta(t+1) e^{-jk\omega_0 t} dt + \int_{0}^{3/2} \delta(t-1) e^{-jk\omega_0 t} dt \right] \\
&= \tfrac{1}{T} \left[-e^{-jk\omega_0(-1)} + e^{-jk\omega_0(1)} \right] \\
&= \tfrac{1}{T} \left[e^{-jk\omega_0} - e^{jk\omega_0} \right] \\
&= \tfrac{1}{T} \left[2j\sin(-k\omega_0) \right] \\
&= \tfrac{2j}{T} \sin(-k\omega_0) \\
&= -\tfrac{2j}{T} \sin k\omega_0 \\
&= -\tfrac{2j}{3} \sin \tfrac{2\pi k}{3}.
\end{aligned}
$$

Thus, $x(t)$ has the Fourier series representation

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3} (\sin \tfrac{2\pi k}{3}) e^{j2\pi kt/3}.
\end{aligned}
$$

$\square$

Example 4.4 (Fourier series of an even real function). Let $x(t)$ be an arbitrary periodic real function that is even. Let c_k denote the Fourier series coefficient sequence for $x(t)$. Show that $\text{Im}\{c_k\} = 0$, $c_k = c_{-k}$ and $c_0 = \frac{1}{T} \int_0^T x(t) dt$.

Proof. From the Fourier series analysis equation (4.11) and using Euler's relation, we can write

$$
\begin{aligned}
c_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \int_T (x(t) [\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)]) dt.
\end{aligned}
$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$
\begin{aligned}
c_k &= \frac{1}{T} \int_T [x(t) (\cos k\omega_0 t - j\sin k\omega_0 t)] dt \\
&= \frac{1}{T} \left[\int_T x(t) \cos k\omega_0 t \, dt - j \int_T x(t) \sin k\omega_0 t \, dt \right].
\end{aligned}
$$

Consider the first integral above (i.e., the one involving the cos function). Since $x(t)$ is even and $\cos k\omega_0 t$ is even, we have that $x(t)\cos k\omega_0 t$ is even. Thus, $\int_T x(t)\cos k\omega_0 t dt = 2\int_0^{T/2} x(t)\cos k\omega_0 t dt$. Consider the second integral above (i.e., the one involving the sin function). Since $x(t)$ is even and $\sin k\omega_0 t$ is odd, we have that $x(t)\sin k\omega_0 t$ is odd. If we integrate an odd periodic function over one period (or an integer multiple thereof), the result is zero. Therefore, the second integral is zero. Combining these results, we can write

$$c_k = \frac{1}{T}\left[2\int_0^{T/2} x(t)\cos k\omega_0 t dt\right]$$
$$= \frac{2}{T}\int_0^{T/2} x(t)\cos k\omega_0 t dt. \tag{4.12}$$

Since $x(t)$ is real, the quantity c_k must also be real. Thus, we have that $\text{Im}\{c_k\} = 0$.

Consider now the expression for c_{-k}. We substitute $-k$ for k in (4.12) to obtain

$$c_{-k} = \frac{2}{T}\int_0^{T/2} x(t)\cos(-k\omega_0 t)dt.$$

Since cos is an even function, we can simplify this expression to obtain

$$c_{-k} = \frac{2}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)dt$$
$$= c_k.$$

Thus, $c_k = c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ into (4.12), we can write

$$c_0 = \frac{2}{T}\int_0^{T/2} x(t)\cos(0)dt$$
$$= \frac{2}{T}\int_0^{T/2} x(t)dt$$
$$= \frac{2}{T}\left[\frac{1}{2}\int_0^T x(t)dt\right]$$
$$= \frac{1}{T}\int_0^T x(t)dt.$$

Thus, we have shown that $c_0 = \frac{1}{T}\int_0^T x(t)dt$. Therefore, in summary, we have shown that $\text{Im}\{c_k\} = 0$, $c_k = c_{-k}$, and $c_0 = \frac{1}{T}\int_0^T x(t)dt$. □

Example 4.5 (Fourier series of an odd real function). Let $x(t)$ be a periodic real function that is odd. Let c_k denote the Fourier series coefficient sequence for $x(t)$. Show that $\text{Re}\{c_k\} = 0$, $c_k = -c_{-k}$, and $c_0 = 0$.

Proof. From the Fourier series analysis equation (4.11) and Euler's formula, we can write

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$
$$= \frac{1}{T}\left[\int_T x(t)\left[\cos(-k\omega_0 t) + j\sin(-k\omega_0 t)\right]dt\right].$$

Since cos and sin are even and odd functions, respectively, we can rewrite the above equation as

$$c_k = \frac{1}{T}\left[\int_T x(t)\left[\cos k\omega_0 t - j\sin k\omega_0 t\right]dt\right]$$
$$= \frac{1}{T}\left[\int_T x(t)\cos k\omega_0 t dt - j\int_T x(t)\sin k\omega_0 t dt\right].$$

Consider the first integral above (i.e., the one involving the cos function). Since $x(t)$ is odd and $\cos k\omega_0 t$ is even, we have that $x(t)\cos k\omega_0 t$ is odd. If we integrate an odd periodic function over a single period (or an integer multiple thereof), the result is zero. Therefore, the first integral is zero. Consider the second integral above (i.e., the one involving the sin function). Since $x(t)$ is odd and $\sin k\omega_0 t$ is odd, we have that $x(t)\sin k\omega_0 t$ is even. Thus, $\int_T x(t)\sin k\omega_0 t\,dt = 2\int_0^{T/2} x(t)\sin k\omega_0 t\,dt$. Combining these results, we can write

$$
\begin{aligned}
c_k &= \frac{-j}{T}\int_T x(t)\sin k\omega_0 t\,dt \\
&= \frac{-j2}{T}\int_0^{T/2} x(t)\sin k\omega_0 t\,dt.
\end{aligned}
\tag{4.13}
$$

Since $x(t)$ is real, the result of the integration, and c_k is purely imaginary. Thus, $\mathrm{Re}\{c_k\} = 0$.

Consider the quantity c_{-k}. Substituting $-k$ for k in (4.13), we obtain

$$
\begin{aligned}
c_{-k} &= \frac{-j2}{T}\int_0^{T/2} x(t)\sin(-k\omega_0 t)\,dt \\
&= \frac{-j2}{T}\int_0^{T/2} x(t)[-\sin(k\omega_0 t)]\,dt \\
&= \frac{j2}{T}\int_0^{T/2} x(t)\sin k\omega_0 t\,dt \\
&= -c_k.
\end{aligned}
$$

Thus, $c_k = -c_{-k}$.

Consider now the quantity c_0. Substituting $k = 0$ in the expression (4.13), we have

$$
\begin{aligned}
c_0 &= \frac{-j2}{T}\int_0^{T/2} x(t)\sin(0)\,dt \\
&= 0.
\end{aligned}
$$

Thus, $c_0 = 0$. Therefore, in summary, we have shown that $\mathrm{Re}\{c_k\} = 0$, $c_k = -c_{-k}$, and $c_0 = 0$. $\qquad\square$

4.4 Convergence of Continuous-Time Fourier Series

So far we have assumed that a given periodic signal $x(t)$ can be represented by a Fourier series. Since a Fourier series consists of an infinite number of terms, we need to more carefully consider the issue of convergence. That is, we want to know under what circumstances the Fourier series of $x(t)$ converges (in some sense) to $x(t)$.

Suppose that we have an arbitrary periodic signal $x(t)$. This signal has the Fourier series representation given by (4.1) and (4.11). Let $x_N(t)$ denote the finite series

$$
x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.
$$

(i.e., $x_N(t)$ is a Fourier series truncated after the Nth harmonic components). The approximation error is given by

$$
e_N(t) = x(t) - x_N(t).
$$

Let us also define the mean-squared error (MSE) as

$$
E_N = \frac{1}{T}\int_T |e_N(t)|^2\,dt.
$$

Before we can proceed further, we need to more precisely specify what we mean by convergence. This is necessary because convergence can be defined in more than one way. For example, two common types of convergence are:

pointwise and MSE. In the case of pointwise convergence, the error goes to zero at every point. If convergence is pointwise and the rate of convergence is the same everywhere, we call this uniform convergence. In the case of MSE convergence, the MSE goes to zero, which does not necessarily imply that the error goes to zero at every point.

Now, we introduce a few important results regarding the convergence of Fourier series for various types of periodic signals. The first result that we consider is for the case of continuous signals as given below.

Theorem 4.1 (Convergence of Fourier series (continuous case)). *If the periodic signal $x(t)$ is a continuous function of t, then its Fourier series converges uniformly (i.e., converges pointwise and at the same rate everywhere).*

In other words, in the above theorem, we have that if $x(t)$ is continuous, then as $N \to \infty$, $e_N(t) \to 0$ for all t. Often, however, we must work with signals that are not continuous. For example, many useful periodic signals are not continuous (e.g., the square wave). Consequently, we must consider the matter of convergence for signals with discontinuities.

Another important result regarding convergence applies to signals that have finite energy over a single period. Mathematically, a signal $x(t)$ has finite energy over a single period if it satisfies

$$\int_T |x(t)|^2 \, dt < \infty.$$

In the case of such a signal, we have the following important result.

Theorem 4.2 (Convergence of Fourier series (finite-energy case)). *If the periodic signal $x(t)$ has finite energy in a single period (i.e., $\int_T |x(t)|^2 \, dt < \infty$), the Fourier series converges in the MSE sense.*

In other words, in the above theorem, we have that if $x(t)$ is of finite energy, then as $N \to \infty$, $E_N \to 0$.

The last important result regarding convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet[1] conditions for the periodic signal $x(t)$ are as follows:

1. Over a single period, $x(t)$ is absolutely integrable (i.e., $\int_T |x(t)| \, dt < \infty$).

2. In any finite interval of time, $x(t)$ is of bounded variation. In other words, there must be a finite number of maxima and minima in a single period of $x(t)$.

3. In any finite interval of time, $x(t)$ has a finite number of discontinuities, each of which is finite.

Theorem 4.3 (Convergence of Fourier series (Dirichlet case)). *If $x(t)$ is a periodic signal satisfying the Dirichlet conditions, then:*

1. *The Fourier series converges pointwise everywhere to $x(t)$, except at the points of discontinuity of $x(t)$.*

2. *At each point $t = t_a$ of discontinuity of $x(t)$, the Fourier series converges to $\frac{1}{2}(x(t_a^-) + x(t_a^+))$ where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal on the left- and right-hand sides of the discontinuity, respectively.*

In other words, if the Dirichlet conditions are satisfied, then as $N \to \infty$, $e_N(t) \to 0$ for all t except at discontinuities. Furthermore, at each discontinuity, the Fourier series converges to the average of the signal values on the left- and right-hand side of the discontinuity.

Example 4.6. Consider the periodic function $x(t)$ with period $T = 2$ as shown in Figure 4.4. Let $\hat{x}(t)$ denote the Fourier series representation of $x(t)$ (i.e., $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where $\omega_0 = \pi$). Determine the values $\hat{x}(0)$ and $\hat{x}(1)$.

Solution. We begin by observing that $x(t)$ satisfies the Dirichlet conditions. Consequently, Theorem 4.3 applies. Thus, we have that

$$\hat{x}(0) = \tfrac{1}{2}\left[x(0^-) + x(0^+)\right] = \tfrac{1}{2}(0+1) = \tfrac{1}{2} \quad \text{and}$$
$$\hat{x}(1) = \tfrac{1}{2}\left[x(1^-) + x(1^+)\right] = \tfrac{1}{2}(1+0) = \tfrac{1}{2}.$$

$\square$

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[1] Pronounced Dee-ree-klay.
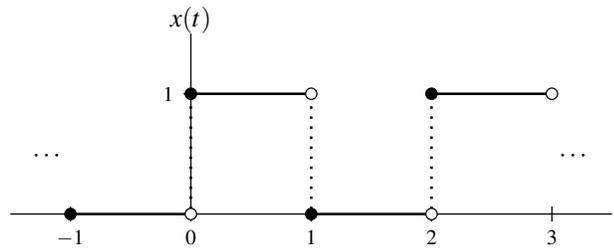
Figure 4.4: Periodic signal $x(t)$.

Although many signals of practical interest satisfy the Dirichlet conditions, not all signals satisfy these conditions. For example, consider the periodic signal $x(t)$ defined by

$$x(t) = 1/t \quad \text{for } 0 < t \le 1 \quad \text{and} \quad x(t) = x(t+1).$$

This signal is plotted in Figure 4.5(a). This signal violates the first condition, since the signal is not absolutely integrable over a single period.

Consider the periodic signal $x(t)$ defined by

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad \text{for } 0 < t \le 1 \quad \text{and} \quad x(t) = x(t+1).$$

This signal is plotted in Figure 4.5(b). Since this signal has an infinite number of minima and maxima in a single period, the second condition is violated.

The third condition is violated by the periodic signal shown in Figure 4.5(c), which has an infinite number of discontinuities in a single period.

One might wonder how the Fourier series converges for periodic signals with discontinuities. Let us consider the periodic square wave from Example 4.1. In Figure 4.6, we have plotted the truncated Fourier series $x_N(t)$ for the square wave (with period $T = 1$ and amplitude $A = 1$) for several values of N. At the discontinuities of $x(t)$, we can see that the series appears to converge to the average of the signal values on either side of the discontinuity. In the vicinity of a discontinuity, however, the truncated series $x_N(t)$ exhibits ripples and the peak amplitude of the ripples does not seem to decrease with increasing N. As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains constant. This behavior is known as **Gibbs phenomenon**.

4.5 Properties of Continuous-Time Fourier Series

Fourier series representations possess a number of important properties. In the sections that follow, we introduce a few of these properties. For convenience, these properties are also summarized later in Table 4.1 (on page 89).

4.5.1 Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T and frequency $\omega_0 = 2\pi/T$. If

$$x(t) \xleftrightarrow{\mathscr{FS}} a_k \quad \text{and}$$
$$y(t) \xleftrightarrow{\mathscr{FS}} b_k,$$

then

$$Ax(t) + By(t) \xleftrightarrow{\mathscr{FS}} Aa_k + Bb_k,$$
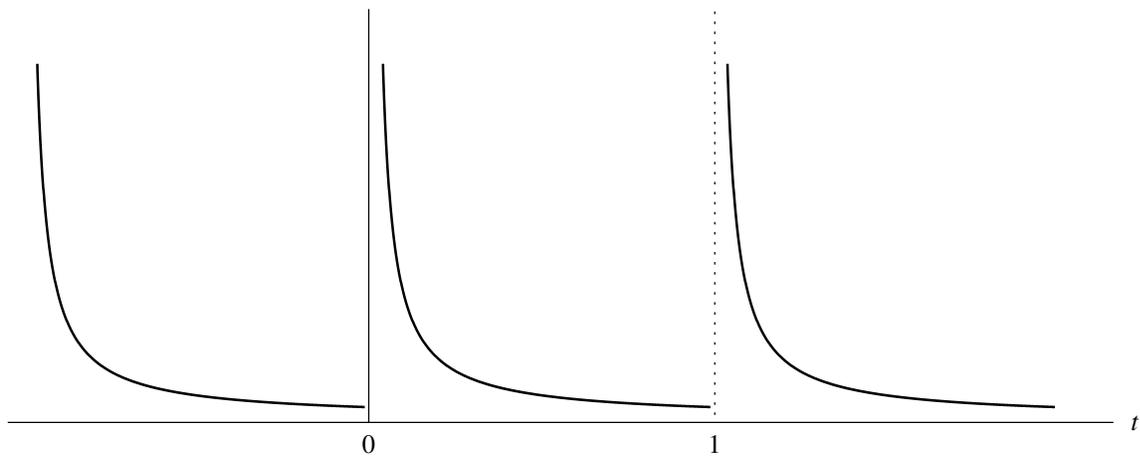
(a)
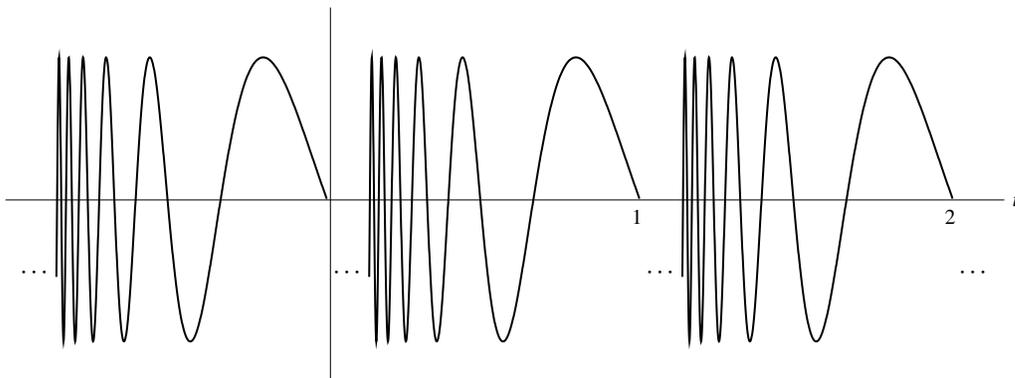
(b)
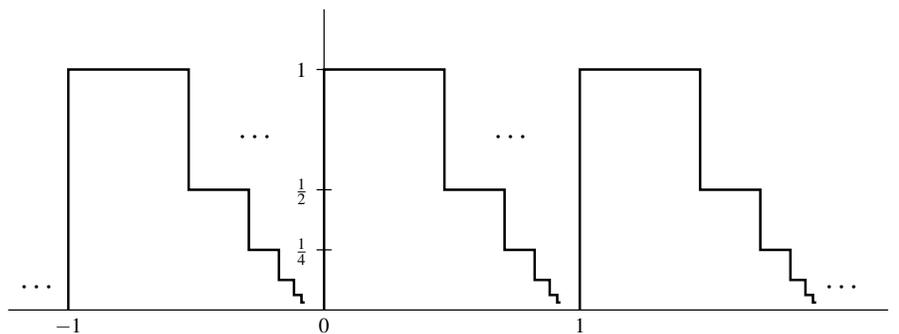
(c)

Figure 4.5: Examples of signals that violate the Dirichlet conditions. (a) A signal that is not absolutely integrable over a single period. (b) A signal that has an infinite number of maxima and minima over a single period. (c) A signal that has an infinite number of discontinuities over a single period.
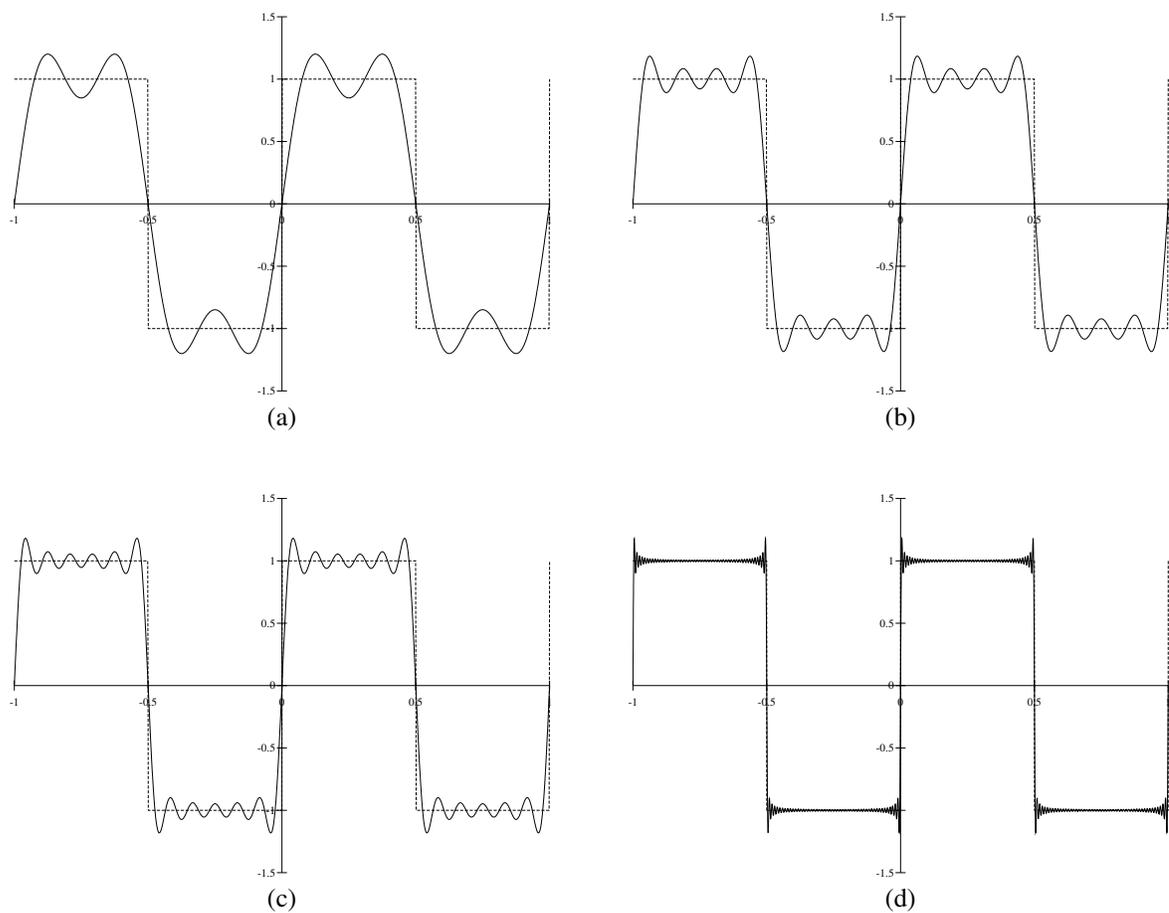
Figure 4.6: Gibbs phenomenon. The Fourier series for the periodic square wave truncated after the Nth harmonic components for (a) $N = 3$, (b) $N = 7$, (c) $N = 11$, and (d) $N = 101$.

where A and B are complex constants. In other words, a linear combination of signals produces the same linear combination of their Fourier series coefficients.

To prove the above property, we proceed as follows. First, we express $x(t)$ and $y(t)$ in terms of their corresponding Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{and}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}.$$

Now, we determine the Fourier series of $Ax(t) + By(t)$. We have

$$Ax(t) + By(t) = A \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + B \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} A a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} B b_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} (A a_k + B b_k) e^{jk\omega_0 t}.$$

Therefore, we have that $Ax(t) + By(t) \overset{\mathscr{FS}}{\longleftrightarrow} A a_k + B b_k$.

4.5.2   Time Shifting

Let $x(t)$ denote a periodic signal with period T and frequency $\omega_0 = 2\pi/T$. If

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} a_k,$$

then

$$x(t - t_0) \overset{\mathscr{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k,$$

where t_0 is a real constant. From this, we can see that time shifting a periodic signal does not change the magnitude of its Fourier series coefficients (since $\left| e^{j\theta} \right| = 1$ for any real θ).

To prove the time-shifting property, we proceed as follows. The Fourier series of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{4.14}$$

We express $x(t - t_0)$ in terms of its Fourier series, and then use algebraic manipulation to obtain:

$$x(t - t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t - t_0)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0}$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_0} \right) e^{jk\omega_0 t}. \tag{4.15}$$

Comparing (4.14) and (4.15), we have that $x(t - t_0) \overset{\mathscr{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k$.

Table 4.1: Fourier Series Properties

Property	Time Domain	Fourier Domain
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time-Domain Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} a_k$
Time Reversal	$x(-t)$	a_{-k}

4.5.3 Time Reversal

Let $x(t)$ denote a periodic signal with period T and frequency $\omega_0 = 2\pi/T$. If

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} a_k,$$

then

$$x(-t) \overset{\mathscr{FS}}{\longleftrightarrow} a_{-k}.$$

In other words, the time reversal of a signal results in the time reversal of the corresponding sequence of Fourier series coefficients. Furthermore, if $x(t)$ is a real signal, we have from (4.5) that $a_{-k} = a_k^*$ and obtain $x(-t) \overset{\mathscr{FS}}{\longleftrightarrow} a_k^*$.

To prove the time-reversal property, we proceed in the following manner. The Fourier series of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{4.16}$$

Now, we consider the Fourier series expansion of $x(-t)$. The Fourier series in this case is given by

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)}. \tag{4.17}$$

Now, we employ a change of variable. Let $l = -k$ so that $k = -l$. Performing this change of variable, we can rewrite (4.17) to obtain

$$x(-t) = \sum_{l=-\infty}^{\infty} a_{-l} e^{j(-l)\omega_0(-t)}$$

$$= \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\omega_0 t}. \tag{4.18}$$

Comparing (4.16) and (4.18), we have that $x(-t) \overset{\mathscr{FS}}{\longleftrightarrow} a_{-k}$.

4.6 Fourier Series and Frequency Spectra

The Fourier series represents a signal in terms of harmonically-related complex sinusoids. In this sense, the Fourier series captures information about the frequency content of a signal. Each complex sinusoid is associated with a particular frequency (which is some integer multiple of the fundamental frequency). So, these coefficients indicate at which frequencies the information/energy in a signal is concentrated. For example, if only the Fourier series coefficients for the low order harmonics have large magnitudes, then the signal is mostly associated with low frequencies. On the other hand, if a signal has many large magnitude coefficients for high order harmonics, then the signal has a considerable amount of information/energy associated with high frequencies. In this way, the Fourier series representation provides a means for measuring the frequency content of a signal. The distribution of the energy/information in a signal over different frequencies is referred to as the **frequency spectrum** of the signal.

To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the signal $x(t)$, it is helpful to write the Fourier series with the c_k expressed in polar form as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} |c_k| e^{j\arg c_k} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}.$$

Clearly (from the last line of the above equation), the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has had its amplitude scaled by a factor of $|c_k|$ and has been time-shifted by an amount that depends on $\arg c_k$. For a given k, the larger $|c_k|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the larger the contribution the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation. In this way, we can use $|c_k|$ as a measure of how much information a signal $x(t)$ has at the frequency $k\omega_0$.

Various ways exist to illustrate the frequency spectrum of a signal. Typically, we plot the Fourier series coefficients as a function of frequency. Since, in general, the Fourier series coefficients are complex valued, we usually display this information using two plots. One plot shows the magnitude of the coefficients as a function of frequency. This is called the **magnitude spectrum**. The other plot shows the arguments of the coefficients as a function of frequency. In this context, the argument is referred to as the phase, and the plot is called the **phase spectrum** of the signal.

Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, we only have values to plot for these particular frequencies. In other words, the frequency spectrum is discrete in the independent variable (i.e., frequency). For this reason, we use a stem graph to plot such functions. Due to the general appearance of the graph (i.e., a number of vertical lines at various frequencies) we refer to such spectra as **line spectra**.

Recall that, for a real signal $x(t)$, the Fourier series coefficient sequence c_k satisfies $c_k = c_{-k}^*$. This, however, implies that $|c_k| = |c_{-k}|$ and $\arg c_k = -\arg c_{-k}$. Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real signal is always even. Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real signal is always odd.

Example 4.7. The periodic square wave $x(t)$ in Example 4.1 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even,} \end{cases}$$

where A is a positive constant. Find and plot the magnitude and phase spectra of this signal. Determine at what frequency (or frequencies) this signal has the most information.

Solution. First, we compute the magnitude spectrum of $x(t)$, which is given by $|c_k|$. We have

$$|c_k| = \begin{cases} \left| \frac{-j2A}{\pi k} \right| & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

$$= \begin{cases} \frac{2A}{\pi |k|} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even.} \end{cases}$$

Next, we compute the phase spectrum of $x(t)$, which is given by $\arg c_k$. Using the fact that $\arg 0 = 0$ and $\arg \frac{-j2A}{\pi k} =$
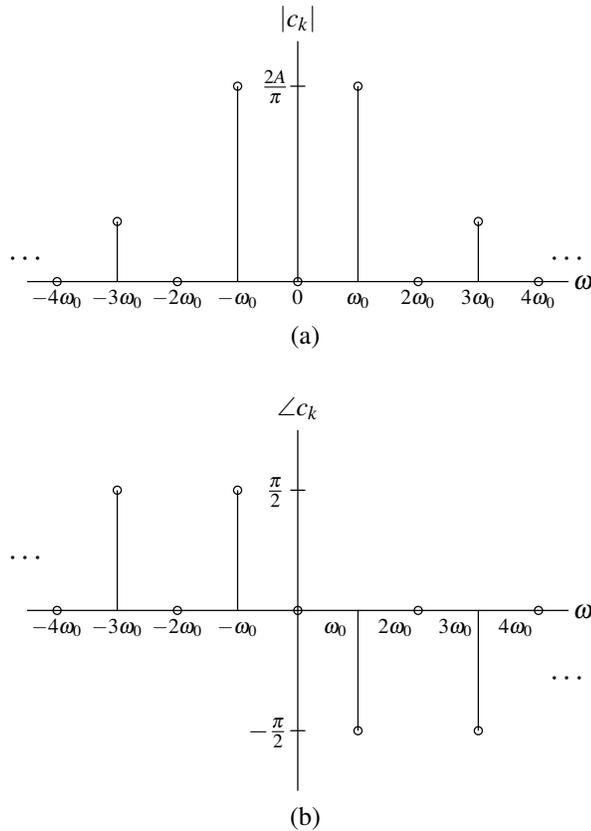
Figure 4.7: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

$-\frac{\pi}{2} \operatorname{sgn} k$, we have

$$
\arg c_k = \begin{cases} \arg \frac{-j2A}{\pi k} & \text{for } k \text{ odd} \\ \arg 0 & \text{for } k \text{ even} \end{cases}
$$

$$
= \begin{cases} \frac{\pi}{2} & \text{for } k \text{ odd, } k < 0 \\ -\frac{\pi}{2} & \text{for } k \text{ odd, } k > 0 \\ 0 & \text{for } k \text{ even} \end{cases}
$$

$$
= \begin{cases} -\frac{\pi}{2} \operatorname{sgn} k & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even.} \end{cases}
$$

The magnitude and phase spectra of $x(t)$ are plotted in Figures 4.7(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since $x(t)$ is real. Since c_k is largest in magnitude for $k = -1$ and $k = 1$, the signal $x(t)$ has the most information at frequencies $-\omega_0$ and ω_0. □

Example 4.8. The periodic impulse train $x(t)$ in Example 4.2 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$
c_k = \frac{A}{T},
$$

where A is a positive constant. Find and plot the magnitude and phase spectra of this signal.
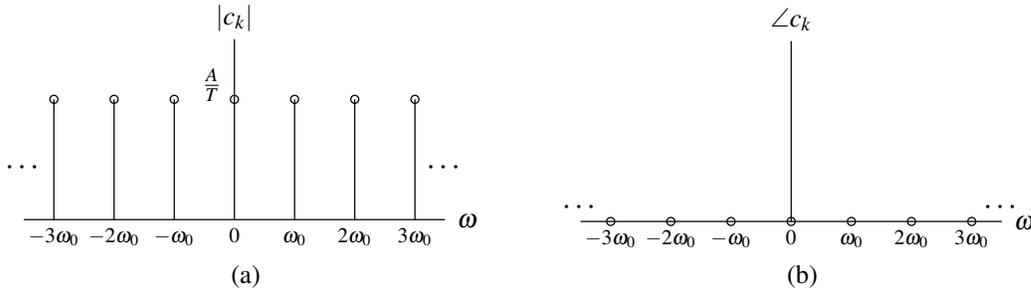
Figure 4.8: Frequency spectrum for the periodic impulse train. (a) Magnitude spectrum and (b) phase spectrum.

Solution. We have $|c_k| = \frac{A}{T}$ and $\arg c_k = 0$. The magnitude and phase spectra of $x(t)$ are plotted in Figures 4.8(a) and (b), respectively. □

4.7 Fourier Series and LTI Systems

In Section 3.10, we showed that complex exponentials are eigenfunctions of LTI systems. More specifically, for a LTI system defined by the operator $\mathscr{H}$ and having impulse response $h(t)$, we showed that

$$e^{st} \xrightarrow{\mathscr{H}} H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \tag{4.19}$$

(i.e., $H(s)$ is the system function).

Often, we are interested in the case of $H(s)$ when $\mathrm{Re}\{s\} = 0$ (i.e., s is purely imaginary). Let $s = j\omega$ where ω is real. Substituting $s = j\omega$ for s in (4.19), we obtain

$$H(j\omega) = H(s)|_{s=j\omega} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

We call $H(j\omega)$ the **frequency response** of the system. From above, it follows that an LTI system must be such that

$$e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t}. \tag{4.20}$$

Suppose now that we have a periodic signal $x(t)$ represented in terms of a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using (4.20) and the superposition property, we can determine the system response $y(t)$ to the input $x(t)$ as follows:

$$
\begin{aligned}
y(t) &= \mathscr{H}\{x(t)\} \\
&= \mathscr{H}\left\{ \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right\} \\
&= \sum_{k=-\infty}^{\infty} \mathscr{H}\left\{ c_k e^{jk\omega_0 t} \right\} \\
&= \sum_{k=-\infty}^{\infty} c_k \mathscr{H}\left\{ e^{jk\omega_0 t} \right\} \\
&= \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0)e^{jk\omega_0 t}.
\end{aligned}
$$

Therefore, we can view a LTI system as an entity that operates on the individual coefficients of a Fourier series. In particular, the system forms its output by multiplying each Fourier series coefficient by the value of the frequency response function at the frequency to which the Fourier series coefficient corresponds. In other words, if

$$x(t) \overset{\mathscr{FS}}{\longleftrightarrow} c_k$$

then

$$y(t) \overset{\mathscr{FS}}{\longleftrightarrow} H(jk\omega_0)c_k.$$

Example 4.9. Suppose that we have a LTI system with the frequency response

$$H(j\omega) = e^{-j\omega/4}.$$

Find the response $y(t)$ of the system to the input $x(t)$ where

$$x(t) = \tfrac{1}{2}\cos(2\pi t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Solution. The Fourier series for $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $c_0 = 0$, $c_1 = \tfrac{1}{4}$, $c_{-1} = \tfrac{1}{4}$, and $\omega_0 = 2\pi$. Thus, we can write

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t} \\
&= c_{-1}H(-j\omega_0)e^{-j\omega_0 t} + c_1 H(j\omega_0)e^{j\omega_0 t} \\
&= \tfrac{1}{4}H(-j2\pi)e^{-j2\pi t} + \tfrac{1}{4}H(j2\pi)e^{j2\pi t} \\
&= \tfrac{1}{4}e^{j\pi/2}e^{-j2\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j2\pi t} \\
&= \tfrac{1}{4}[e^{-j(2\pi t - \pi/2)} + e^{j(2\pi t - \pi/2)}] \\
&= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\
&= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\
&= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right).
\end{aligned}
$$

In other words, the output $y(t)$ is just a shifted version of the input $x(t)$, namely $x(t - \tfrac{1}{4})$. As it turns out, the frequency response $H(j\omega)$ corresponds to an ideal delay of $\tfrac{1}{4}$. $\square$

4.8 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a signal or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a signal is referred to as **filtering**. Various types of filters exist. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An ideal lowpass filter eliminates all frequency components with a frequency greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

(a)

(b)

(c)
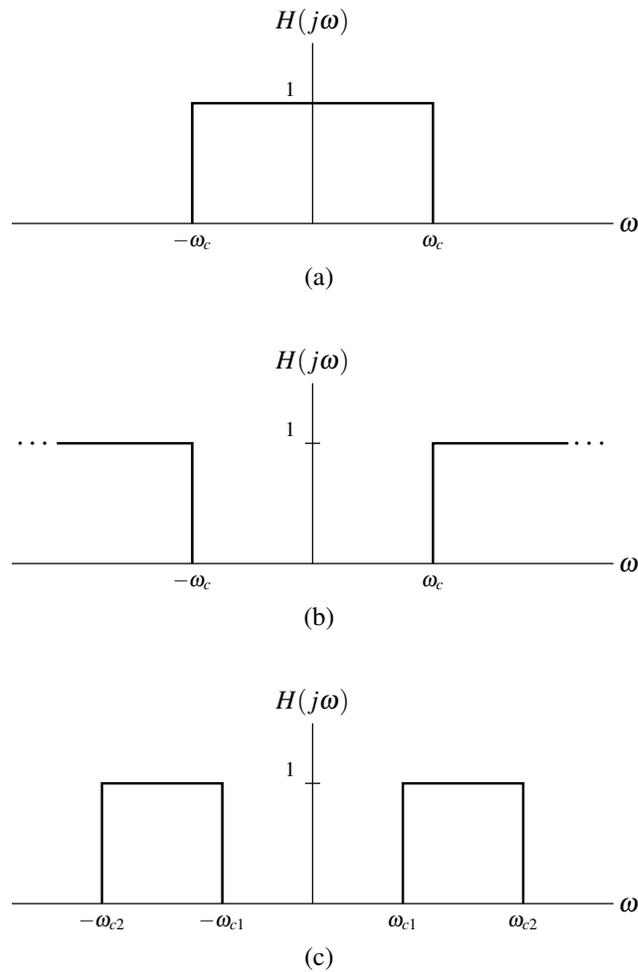
Figure 4.9: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 4.9(a).

The ideal highpass filter eliminates all frequency components with a frequency less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 4.9(b).

An ideal bandpass filter eliminates all frequency components that do not lie in its passband, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A plot of this frequency response is given in Figure 4.9(c).

Example 4.10 (Lowpass filtering). Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and frequency

response $H(j\omega)$, where

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \le 3\pi \\ 0 & \text{otherwise.} \end{cases}$$

Further, suppose that the input $x(t)$ is the periodic signal

$$x(t) = 1 + 2\cos 2\pi t + \cos 4\pi t + \tfrac{1}{2}\cos 6\pi t.$$

Find the Fourier series representation of $x(t)$. Use this representation in order to find the response $y(t)$ of the system to the input $x(t)$. Plot the frequency spectra of $x(t)$ and $y(t)$.

Solution. We begin by finding the Fourier series representation of $x(t)$. Using Euler's formula, we can re-express $x(t)$ as

$$\begin{aligned} x(t) &= 1 + 2\cos 2\pi t + \cos 4\pi t + \tfrac{1}{2}\cos 6\pi t \\ &= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\ &= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\ &= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\ &= \tfrac{1}{4}e^{j(-3)(2\pi)t} + \tfrac{1}{2}e^{j(-2)(2\pi)t} + e^{j(-1)(2\pi)t} + e^{j(0)(2\pi)t} + e^{j(1)(2\pi)t} + \tfrac{1}{2}e^{j(2)(2\pi)t} + \tfrac{1}{4}e^{j(3)(2\pi)t}. \end{aligned}$$

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & \text{for } k = 0 \\ 1 & \text{for } k = \pm 1 \\ \tfrac{1}{2} & \text{for } k = \pm 2 \\ \tfrac{1}{4} & \text{for } k = \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Since the system is LTI, we know that the output $y(t)$ has the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(jk\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$$\begin{aligned} b_0 &= a_0 H(j[0][2\pi]) = 1(1) = 1, \\ b_1 &= a_1 H(j[1][2\pi]) = 1(1) = 1, \\ b_{-1} &= a_{-1} H(j[-1][2\pi]) = 1(1) = 1, \\ b_2 &= a_2 H(j[2][2\pi]) = \tfrac{1}{2}(0) = 0, \\ b_{-2} &= a_{-2} H(j[-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\ b_3 &= a_3 H(j[3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\ b_{-3} &= a_{-3} H(j[-3][2\pi]) = \tfrac{1}{4}(0) = 0. \end{aligned}$$

(a)

(b)
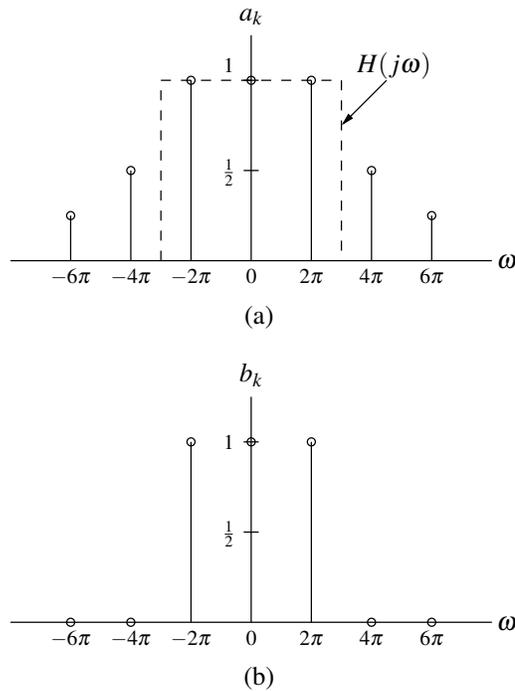
Figure 4.10: Frequency spectra of the input and output signals.

Thus, we have

$$
b_k = \begin{cases} 1 & \text{for } k = 0 \\ 1 & \text{for } k = \pm 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Lastly, we plot the frequency spectra of $x(t)$ and $y(t)$ in Figures 4.10(a) and (b), respectively. The frequency response $H(j\omega)$ is superimposed on the plot for $x(t)$ for illustrative purposes.

$\square$

4.9 Problems

4.1 Find the Fourier series representation (in complex exponential form) of each of the signals given below. In each case, explicitly identify the fundamental period and Fourier series coefficient sequence c_k.
(a) $x(t) = 1 + \cos \pi t + \sin^2 \pi t$;
(b) $x(t) = [\cos 4t][\sin t]$; and
(c) $x(t) = |\sin 2\pi t|$. [Hint: $\int e^{ax} \sin bx \, dx = \frac{e^{ax}[a \sin bx - b \cos bx]}{a^2 + b^2} + C$, where a and b are arbitrary complex and nonzero real constants, respectively.]

4.2 For each of the signals shown in the figure below, find the corresponding Fourier series coefficient sequence.

(a)

(b)

(c)

(d)
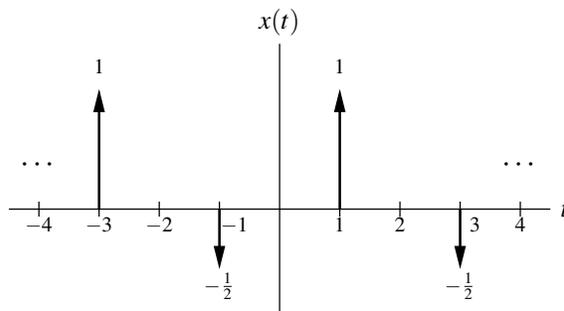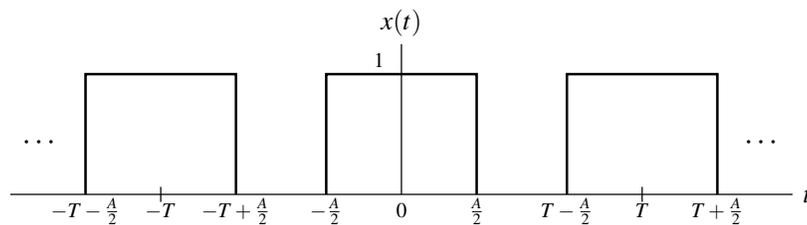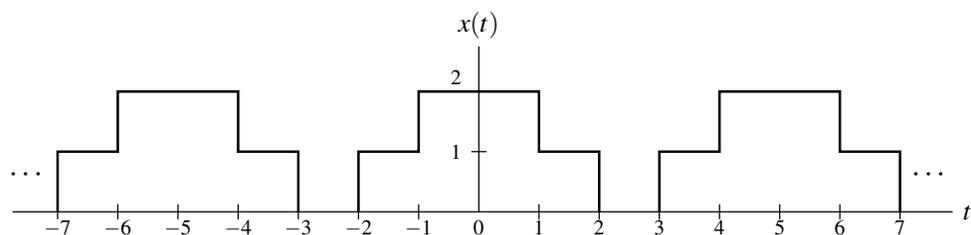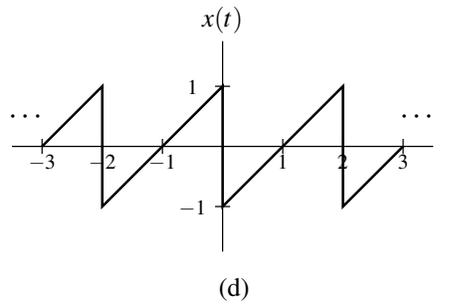
4.3 Let $x(t)$ be a periodic signal with the Fourier series coefficient sequence c_k given by

$$c_k = \begin{cases} 1 & \text{for } k = 0 \\ j \left(\frac{1}{2}\right)^{|k|} & \text{otherwise.} \end{cases}$$

Use the properties of the Fourier series to answer the following questions:
(a) Is $x(t)$ real?
(b) Is $x(t)$ even?
(c) Is $\frac{d}{dt} x(t)$ even? [Hint: Try Problems 4.4 and 4.5 first.]

4.4 Show that, if a complex periodic signal $x(t)$ is even, then its Fourier series coefficient sequence c_k satisfies $c_k = c_{-k}$.

4.5 Suppose that the periodic signal $x(t)$ has the Fourier series coefficient sequence c_k. Determine the Fourier series coefficient sequence d_k of the signal $\frac{d}{dt} x(t)$.

4.6 A periodic signal $x(t)$ with period T and Fourier series coefficient sequence c_k is said to be odd harmonic if $c_k = 0$ for all even k.
(a) Show that if $x(t)$ is odd harmonic, then $x(t) = -x(t - \frac{T}{2})$ for all t.
(b) Show that if $x(t) = -x(t - \frac{T}{2})$ for all t, then $x(t)$ is odd harmonic.

4.7 Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficient sequence c_k. Find the Fourier series coefficient sequence of each of the following signals in terms of c_k:
(a) $\text{Even}\{x(t)\}$
(b) $\text{Re}\{x(t)\}$.

4.8 Find the Fourier series coefficient sequence c_k of the periodic signal $x(t)$ shown in the figure below. Plot the frequency spectrum of this signal including the first five harmonics.
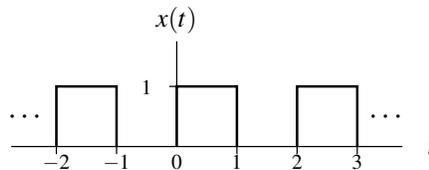
4.9 Suppose that we have a LTI system with frequency response

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Using frequency-domain methods, find the output $y(t)$ of the system if the input $x(t)$ is given by

$$x(t) = 1 + 2\cos 2t + 2\cos 4t + \frac{1}{2}\cos 6t.$$

4.10 MATLAB Problems

4.101 Consider the periodic signal $x(t)$ shown in Figure B of Problem 4.2 where $T = 1$ and $A = \frac{1}{2}$. We can show that this signal $x(t)$ has the Fourier series representation

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $c_k = \frac{1}{2} \operatorname{sinc} \frac{\pi k}{2}$ and $\omega_0 = 2\pi$. Let $\hat{x}_N(t)$ denote the above infinite series truncated after the Nth harmonic component. That is,

$$\hat{x}_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

(a) Use MATLAB to plot $\hat{x}_N(t)$ for $N = 1, 5, 10, 50, 100$. You should see that as N increases, $\hat{x}_N(t)$ converges to $x(t)$. [**HINT:** You may find the `sym`, `symsum`, `subs`, and `ezplot` functions useful for this problem. Please note that the MATLAB `sinc` function is NOT defined in the same way as in the lecture notes. The MATLAB `sinc` function is defined as $\operatorname{sinc} x = (\sin(\pi x))/(\pi x)$. So, it might be wise to avoid the use of this MATLAB function altogether.]

(b) By examining the graphs obtained in part (a), answer the following: As $N \to \infty$, does $\hat{x}_N(t)$ converge to $x(t)$ uniformly (i.e., equally fast) everywhere? If not, where is the rate of convergence slower?

(c) The signal $x(t)$ is not continuous everywhere. For example, the signal has a discontinuity at $t = \frac{1}{4}$. As $N \to \infty$, to what value does $\hat{x}_N(t)$ appear to converge at this point? Again, deduce your answer from the graphs obtained in part (a).

Chapter 5

Continuous-Time Fourier Transform

5.1 Introduction

The Fourier series provides an extremely useful representation for periodic signals. Often, however, we need to deal with signals that are not periodic. A more general tool than the Fourier series is needed in this case. In this chapter, we will introduce a tool for representing arbitrary (i.e., possibly aperiodic) signals, known as the Fourier transform.

5.2 Development of the Continuous-Time Fourier Transform

As demonstrated earlier, the Fourier series is an extremely useful signal representation. Unfortunately, this signal representation can only be used for periodic signals, since a Fourier series is inherently periodic. Many signals, however, are not periodic. Therefore, one might wonder if we can somehow use the Fourier series to develop a representation for aperiodic signals. As it turns out, this is possible. In order to understand why, we must make the following key observation. An aperiodic signal can be viewed as a periodic signal with a period of infinity. By viewing an aperiodic signal as this limiting case of a periodic signal where the period is infinite, we can use the Fourier series to develop a more general signal representation that can be used in the aperiodic case. (In what follows, our development of the Fourier transform is not completely rigorous, as we assume that various integrals, summations, and limits converge. Such assumptions are not valid in all cases. Our development is mathematically sound, however, provided that the Fourier transform of the signal being considered exists.)

Suppose that we have an arbitrary signal $x(t)$ that is not necessarily periodic. Let us define the signal $x_T(t)$ as

$$x_T(t) = \begin{cases} x(t) & \text{for } -\frac{T}{2} \le t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

In other words, $x_T(t)$ is identical to $x(t)$ over the interval $-\frac{T}{2} \le t < \frac{T}{2}$ and zero otherwise. Let us now repeat the portion of $x_T(t)$ for $-\frac{T}{2} \le t < \frac{T}{2}$ to form a periodic signal $\tilde{x}(t)$ with period T. That is, we define $\tilde{x}(t)$ as

$$\tilde{x}(t) = x_T(t) \text{ for } -\frac{T}{2} \le t < \frac{T}{2} \quad \text{and} \quad \tilde{x}(t) = \tilde{x}(t+T).$$

In Figures 5.1 and 5.2, we provide illustrative examples of the signals $x(t)$, $x_T(t)$, and $\tilde{x}(t)$.

Before proceeding further, we make two important observations that we will use later. First, from the definition of $x_T(t)$, we have

$$\lim_{T \to \infty} x_T(t) = x(t). \tag{5.2}$$

Second, from the definition of $x_T(t)$ and $\tilde{x}(t)$, we have

$$\lim_{T \to \infty} \tilde{x}(t) = x(t). \tag{5.3}$$

Now, let us consider the signal $\tilde{x}(t)$. Since $\tilde{x}(t)$ is periodic, we can represent it using a Fourier series as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{5.4}$$

where

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt \tag{5.5}$$

and $\omega_0 = 2\pi/T$. Since $x_T(t) = \tilde{x}(t)$ for $-\frac{T}{2} \leq t < \frac{T}{2}$, we can rewrite (5.5) as

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jk\omega_0 t} dt.$$

Furthermore, since $x_T(t) = 0$ for $t < -\frac{T}{2}$ and $t \geq \frac{T}{2}$, we can rewrite the preceding expression for a_k as

$$a_k = \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt.$$

Substituting this expression for a_k into (5.4) and rearranging, we obtain the following Fourier series representation for $\tilde{x}(t)$:

$$\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} \left[\frac{1}{T} \int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \left[\frac{\omega_0}{2\pi} \int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \omega_0.
\end{aligned}$$

Substituting the above expression for $\tilde{x}(t)$ into (5.3), we obtain

$$x(t) = \lim_{T \to \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(\tau) e^{-jk\omega_0 \tau} d\tau \right] e^{jk\omega_0 t} \omega_0. \tag{5.6}$$

Now, we must evaluate the above limit. As $T \to \infty$, we have that $\omega_0 \to 0$. Thus, in the limit above, $k\omega_0$ becomes a continuous variable which we denote as ω, ω_0 becomes the infinitesimal $d\omega$, and the summation becomes an integral. This is illustrated in Figure 5.3. Also, as $T \to \infty$, we have that $x_T(t) \to x(t)$. Combining these results, we can rewrite (5.6) to obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Thus, we have found a representation of the arbitrary signal $x(t)$ in terms of complex sinusoids at all frequencies. We call this the Fourier transform representation of the signal $x(t)$.
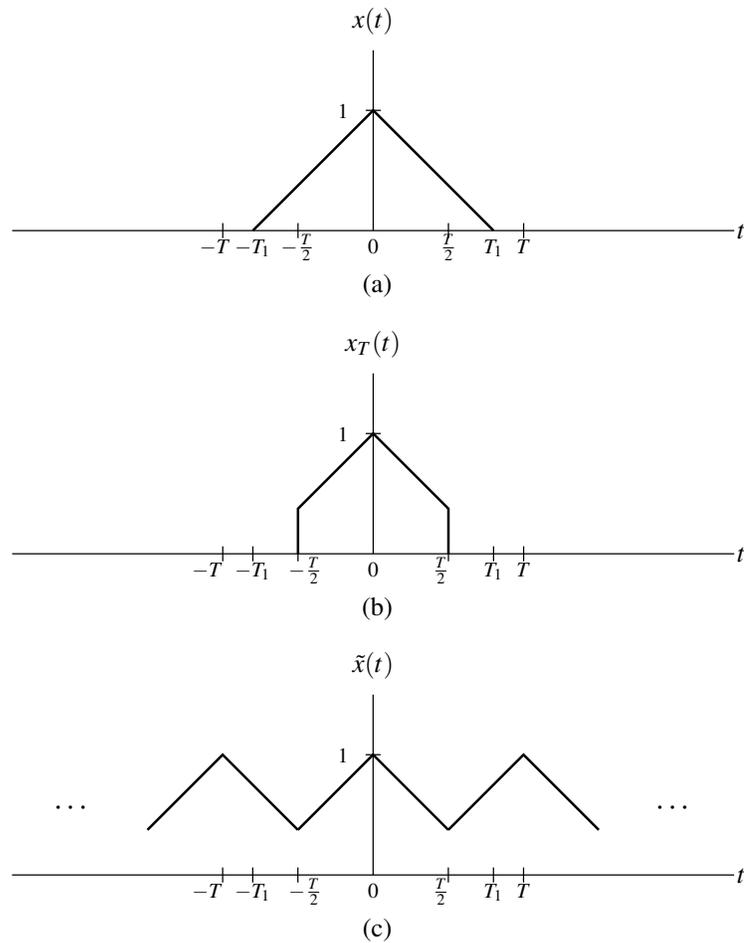
Figure 5.1: Example of signals used in derivation of Fourier transform representation (where $T_1 > \frac{T}{2}$).

(a)

(b)

(c)
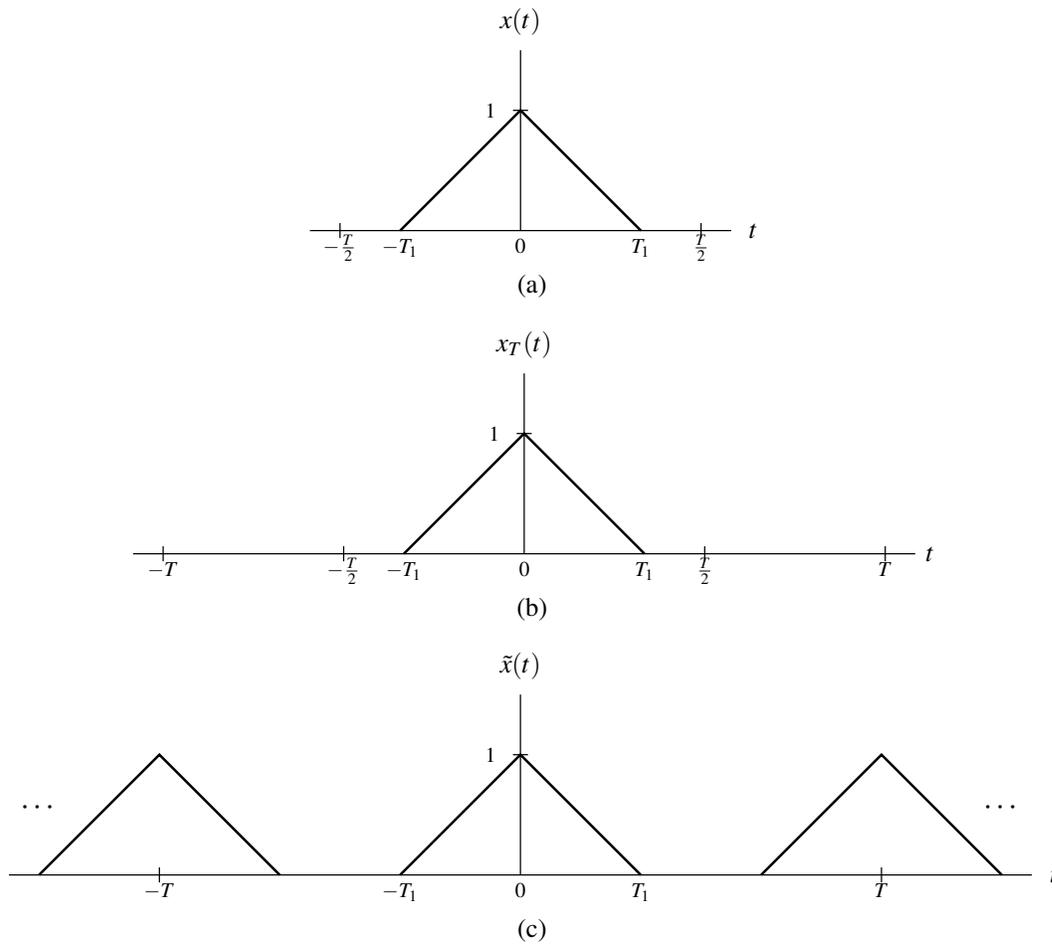
Figure 5.2: Example of signals used in derivation of Fourier transform representation (where $T_1 < \frac{T}{2}$).
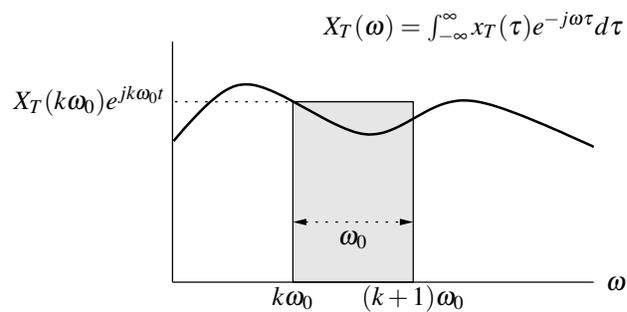
Figure 5.3: Integral obtained in the derivation of the Fourier transform representation.

5.3 Definition of the Continuous-Time Fourier Transform

In the previous section, we derived the Fourier transform representation of an arbitrary signal. This representation expresses a signal in terms of complex sinusoids at all frequencies. Given a signal $x(t)$, its Fourier transform representation is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega, \tag{5.7a}$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt. \tag{5.7b}$$

We refer to (5.7b) as the **Fourier transform analysis equation** and (5.7a) as the **Fourier transform synthesis equation**.

The quantity $X(\omega)$ is called the Fourier transform of $x(t)$. That is, the Fourier transform of the signal $x(t)$, denoted as $\mathscr{F}\{x(t)\}$ or $X(\omega)$, is defined as

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt. \tag{5.8}$$

Similarly, the inverse Fourier transform of $X(\omega)$ is given by

$$\mathscr{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega. \tag{5.9}$$

If a signal $x(t)$ has the Fourier transform $X(\omega)$ we often denote this as

$$x(t) \xleftrightarrow{\mathscr{F}} X(\omega).$$

As a matter of terminology, $x(t)$ and $X(\omega)$ are said to be a **Fourier transform pair**.

Example 5.1 (Fourier transform of the unit-impulse function). Find the Fourier transform $X(\omega)$ of the signal $x(t) = A\delta(t - t_0)$. Then, from this result, write the Fourier transform representation of $x(t)$.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{x(t)\} \\
&= \mathscr{F}\{A\delta(t - t_0)\} \\
&= \int_{-\infty}^{\infty} A\delta(t - t_0) e^{-j\omega t} \, dt \\
&= A \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} \, dt.
\end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$X(\omega) = A e^{-j\omega t_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \xleftrightarrow{\mathscr{F}} A e^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of $x(t)$ is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

where

$$X(\omega) = Ae^{-j\omega t_0}.$$

<div style="text-align: right">□</div>

Example 5.2 (Inverse Fourier transform of the unit-impulse function). Find the inverse Fourier transform of $X(\omega) = 2\pi A\delta(\omega - \omega_0)$.

Solution. From the definition of the inverse Fourier transform, we can write

$$\mathscr{F}^{-1}\{2\pi A\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A\delta(\omega - \omega_0)e^{j\omega t}d\omega$$

$$= A\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega.$$

Using the sifting property of the unit-impulse function, we can simplify the preceding equation to obtain

$$\mathscr{F}^{-1}\{2\pi A\delta(\omega - \omega_0)\} = Ae^{j\omega_0 t}.$$

Thus, we have that

$$Ae^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi A\delta(\omega - \omega_0).$$

<div style="text-align: right">□</div>

Example 5.3 (Fourier transform of the rectangular pulse). Find the Fourier transform $X(\omega)$ of the signal $x(t) = \text{rect}\, t$.

Solution. From the definition of the Fourier transform, we can write

$$X(\omega) = \mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} [\text{rect}\, t]e^{-j\omega t}dt.$$

From the definition of the rectangular pulse function, we can simplify this equation as follows:

$$X(\omega) = \int_{-1/2}^{1/2} [\text{rect}\, t]e^{-j\omega t}dt$$

$$= \int_{-1/2}^{1/2} e^{-j\omega t}dt.$$

Evaluating the integral and simplifying, we obtain

$$X(\omega) = \left[-\frac{1}{j\omega}e^{-j\omega t}\right]\Bigg|_{-1/2}^{1/2}$$

$$= \frac{1}{j\omega}[e^{j\omega/2} - e^{-j\omega/2}]$$

$$= \frac{1}{j\omega}[2j\sin\frac{\omega}{2}]$$

$$= \frac{2}{\omega}\sin\omega/2$$

$$= \text{sinc}\,\omega/2.$$

Thus, we have shown that

$$\text{rect}\, t \overset{\mathscr{F}}{\longleftrightarrow} \text{sinc}\,\omega/2.$$

<div style="text-align: right">□</div>

5.4 Convergence of the Continuous-Time Fourier Transform

When deriving the Fourier transform representation earlier, we implicitly made some assumptions about the convergence of the integrals and other expressions involved. These assumptions are not always valid. For this reason, a more careful examination of the convergence properties of the Fourier transform is in order.

Suppose that we have an arbitrary signal $x(t)$. This signal has the Fourier transform representation $\hat{x}(t)$ given by

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega,$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Now, we need to concern ourselves with the convergence properties of this representation. In other words, we want to know when $\hat{x}(t)$ is a valid representation of $x(t)$. In our earlier derivation of the Fourier transform, we relied heavily on the Fourier series. Therefore, one might expect that the convergence of the Fourier transform representation is closely related to the convergence properties of Fourier series. This is, in fact, the case. The convergence properties of the Fourier transform are very similar to the convergence properties of the Fourier series (as studied in Section 4.4).

The first important result concerning convergence relates to finite-energy signals as stated by the theorem below.

Theorem 5.1 (Convergence of Fourier transform (finite-energy case)). *If a signal $x(t)$ is of finite energy (i.e., $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$), then its Fourier transform representation converges in the MSE sense.*

In other words, if $x(t)$ is of finite energy, then

$$E = \int_{-\infty}^{\infty} |\hat{x}(t) - x(t)|^2 dt = 0.$$

Although $x(t)$ and $\hat{x}(t)$ may differ at individual points, the energy E in the difference is zero.

The other important result concerning convergence that we shall consider relates to what are known as the Dirichlet conditions. The Dirichlet conditions for the signal $x(t)$ are as follows:

1. The signal $x(t)$ is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$).

2. The signal $x(t)$ has a finite number of maxima and minima on any finite interval.

3. The signal $x(t)$ has a finite number of discontinuities on any finite interval, and each discontinuity is itself finite.

For a signal satisfying the Dirichlet conditions, we have the important convergence result stated below.

Theorem 5.2 (Convergence of Fourier transform (Dirichlet case)). *If a signal $x(t)$ satisfies the Dirichlet conditions, then its Fourier transform representation $\hat{x}(t)$ converges pointwise for all t, except at points of discontinuity. Furthermore, at each discontinuity point $t = t_a$, we have that*

$$\hat{x}(t_a) = \tfrac{1}{2}[x(t_a^+) + x(t_a^-)],$$

where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the signal $x(t)$ on the left- and right-hand sides of the discontinuity, respectively.

In other words, if a signal $x(t)$ satisfies the Dirichlet conditions, then the Fourier transform representation $\hat{x}(t)$ of $x(t)$ converges to $x(t)$ for all t, except at points of discontinuity where $\hat{x}(t)$ instead converges to the average of $x(t)$ on the two sides of the discontinuity.

The finite-energy and Dirichlet conditions mentioned above are only sufficient conditions for the convergence of the Fourier transform representation. They are not necessary. In other words, a signal may violate these conditions and still have a valid Fourier transform representation.
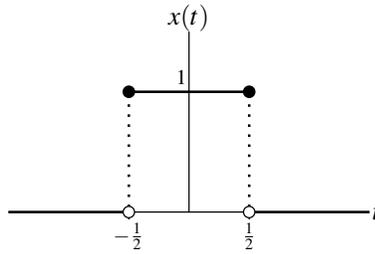
Figure 5.4: Signal $x(t)$.

Example 5.4. Consider the function $x(t)$ shown in Figure 5.4. Let $\hat{x}(t)$ denote the Fourier transform representation of $x(t)$ (i.e., $\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$, where $X(\omega)$ denotes the Fourier transform of $x(t)$). Determine the values $\hat{x}(-\frac{1}{2})$ and $\hat{x}(\frac{1}{2})$.

Solution. We begin by observing that $x(t)$ satisfies the Dirichlet conditions. Consequently, Theorem 5.2 applies. Thus, we have that

$$\hat{x}(-\tfrac{1}{2}) = \tfrac{1}{2}\left[x(-\tfrac{1}{2}^-) + x(-\tfrac{1}{2}^+)\right] = \tfrac{1}{2}(0+1) = \tfrac{1}{2} \quad \text{and}$$
$$\hat{x}(\tfrac{1}{2}) = \tfrac{1}{2}\left[x(\tfrac{1}{2}^-) + x(\tfrac{1}{2}^+)\right] = \tfrac{1}{2}(1+0) = \tfrac{1}{2}.$$

$\square$

5.5   Properties of the Continuous-Time Fourier Transform

The Fourier transform has a number of important properties. In the sections that follow, we introduce several of these properties. For convenience, these properties are also later summarized in Table 5.1 (on page 121).

5.5.1   Linearity

If $x_1(t) \xleftrightarrow{\mathscr{F}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\mathscr{F}} X_2(\omega)$, then

$$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{\mathscr{F}} a_1 X_1(\omega) + a_2 X_2(\omega),$$

where a_1 and a_2 are arbitrary complex constants. This is known as the linearity property of the Fourier transform.

To prove the above property, we proceed as follows:

$$\begin{aligned}
\mathscr{F}\{a_1 x_1(t) + a_2 x_2(t)\} &= \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} a_1 x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} a_2 x_2(t) e^{-j\omega t} dt \\
&= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt \\
&= a_1 \mathscr{F}\{x_1(t)\} + a_2 \mathscr{F}\{x_2(t)\} \\
&= a_1 X_1(\omega) + a_2 X_2(\omega).
\end{aligned}$$

Thus, we have shown that the linearity property holds.

Example 5.5 (Linearity property of the Fourier transform)**.** Find the Fourier transform $X(\omega)$ of the signal $x(t) = A\cos \omega_0 t$.

Solution. We recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Thus, we can write

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{x(t)\} \\
&= \mathscr{F}\{A\cos\omega_0 t\} \\
&= \mathscr{F}\{\tfrac{A}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]\}.
\end{aligned}$$

Then, we use the linearity property of the Fourier transform to obtain

$$X(\omega) = \tfrac{A}{2}\mathscr{F}\{e^{j\omega_0 t}\} + \tfrac{A}{2}\mathscr{F}\{e^{-j\omega_0 t}\}.$$

From Example 5.2, we know that $e^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$ and $e^{-j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega + \omega_0)$. Thus, we can further simplify the above expression for $X(\omega)$ as follows:

$$\begin{aligned}
X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega + \omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega - \omega_0)] \\
&= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].
\end{aligned}$$

Thus, we have shown that

$$A\cos\omega_0 t \overset{\mathscr{F}}{\longleftrightarrow} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$

$\square$

Example 5.6 (Fourier transform of the unit-step function). Suppose that $\operatorname{sgn} t \overset{\mathscr{F}}{\longleftrightarrow} \frac{2}{j\omega}$. Find the Fourier transform $X(\omega)$ of the signal $x(t) = u(t)$.

Solution. First, we observe that $x(t)$ can be expressed in terms of the signum function as

$$x(t) = u(t) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t.$$

Taking the Fourier transform of both sides of this equation yields

$$X(\omega) = \mathscr{F}\{x(t)\} = \mathscr{F}\{\tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn} t\}.$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$X(\omega) = \tfrac{1}{2}\mathscr{F}\{1\} + \tfrac{1}{2}\mathscr{F}\{\operatorname{sgn} t\}.$$

From Example 5.5 (with $\omega_0 = 0$), we know that $1 \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega)$. Also, we are given that $\operatorname{sgn} t \overset{\mathscr{F}}{\longleftrightarrow} \frac{2}{j\omega}$. Using these facts, we can rewrite the expression for $X(\omega)$ as

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2}[2\pi\delta(\omega)] + \tfrac{1}{2}(\tfrac{2}{j\omega}) \\
&= \pi\delta(\omega) + \tfrac{1}{j\omega}.
\end{aligned}$$

Thus, we have shown that

$$u(t) \overset{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

$\square$

5.5.2 Time-Domain Shifting

If $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$x(t - t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

where t_0 is an arbitrary real constant. This is known as the time-shifting property of the Fourier transform.

To prove the above property, we proceed as follows. To begin, we have

$$\mathscr{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt.$$

Now, we use a change of variable. Let $\lambda = t - t_0$ so that $t = \lambda + t_0$ and $dt = d\lambda$. Performing the change of variable and simplifying, we obtain

$$\begin{aligned}
\mathscr{F}\{x(t - t_0)\} &= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda + t_0)} d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} e^{-j\omega t_0} d\lambda \\
&= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\
&= e^{-j\omega t_0} \mathscr{F}\{x(t)\} \\
&= e^{-j\omega t_0} X(\omega).
\end{aligned}$$

Thus, we have proven that the time-shifting property holds.

Example 5.7 (Time-domain shifting property of the Fourier transform). Find the Fourier transform $X(\omega)$ of the signal $x(t) = A\cos(\omega_0 t + \theta)$.

Solution. Let $v(t) = A\cos\omega_0 t$ so that $x(t) = v(t + \frac{\theta}{\omega_0})$. From Example 5.5, we have that

$$\begin{aligned}
\mathscr{F}\{v(t)\} = V(\omega) &= \mathscr{F}\{A\cos\omega_0 t\} \\
&= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].
\end{aligned}$$

From the definition of $v(t)$ and the time-shifting property of the Fourier transform, we have

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{x(t)\} \\
&= e^{j\omega\theta/\omega_0} V(\omega) \\
&= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].
\end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \overset{\mathscr{F}}{\longleftrightarrow} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$

$\square$

5.5.3 Frequency-Domain Shifting

If $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$e^{j\omega_0 t} x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega - \omega_0),$$

where ω_0 is an arbitrary real constant. This is known as the frequency-domain shifting property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the Fourier transform and straightforward algebraic manipulation, we can write

$$
\begin{aligned}
\mathscr{F}\{e^{j\omega_0 t}x(t)\} &= \int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\
&= X(\omega-\omega_0).
\end{aligned}
$$

Thus, we have shown that the frequency-domain shifting property holds.

Example 5.8 (Frequency-domain shifting property of the Fourier transform). Find the Fourier transform $X(\omega)$ of the signal $x(t) = (\cos \omega_0 t)(\cos 20\pi t)$.

Solution. Recall that $\cos \alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

$$
\begin{aligned}
X(\omega) &= \mathscr{F}\{x(t)\} \\
&= \mathscr{F}\{(\cos \omega_0 t)(\tfrac{1}{2})[e^{j20\pi t} + e^{-j20\pi t}]\} \\
&= \mathscr{F}\{\tfrac{1}{2}e^{j20\pi t}\cos \omega_0 t + \tfrac{1}{2}e^{-j20\pi t}\cos \omega_0 t\} \\
&= \tfrac{1}{2}\mathscr{F}\{e^{j20\pi t}\cos \omega_0 t\} + \tfrac{1}{2}\mathscr{F}\{e^{-j20\pi t}\cos \omega_0 t\}.
\end{aligned}
$$

From Example 5.5 (where we showed $\cos \omega_0 t \overset{\mathscr{F}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$) and the frequency-domain shifting property of the Fourier transform, we have

$$
\begin{aligned}
X(\omega) &= \tfrac{1}{2}\left[\pi[\delta(v-\omega_0) + \delta(v+\omega_0)]\right]\big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi[\delta(v-\omega_0) + \delta(v+\omega_0)]\right]\big|_{v=\omega+20\pi} \\
&= \tfrac{1}{2}\left(\pi[\delta(\omega+\omega_0-20\pi) + \delta(\omega-\omega_0-20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega+\omega_0+20\pi) + \delta(\omega-\omega_0+20\pi)]\right) \\
&= \tfrac{\pi}{2}\left[\delta(\omega+\omega_0-20\pi) + \delta(\omega-\omega_0-20\pi) + \delta(\omega+\omega_0+20\pi) + \delta(\omega-\omega_0+20\pi)\right].
\end{aligned}
$$

$\square$

5.5.4 Time- and Frequency-Domain Scaling

If $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$
x(at) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right),
$$

where a is an arbitrary nonzero real constant. This is known as the time/frequency-scaling property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the Fourier transform, we can write

$$
\mathscr{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt.
$$

Now, we use a change of variable. Let $\lambda = at$ so that $t = \lambda/a$ and $dt = d\lambda/a$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$
\begin{aligned}
\mathscr{F}\{x(at)\} &= \begin{cases} \int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}(\tfrac{1}{a})d\lambda & \text{for } a > 0 \\ \int_{\infty}^{-\infty} x(\lambda)e^{-j(\omega/a)\lambda}(\tfrac{1}{a})d\lambda & \text{for } a < 0 \end{cases} \\
&= \begin{cases} \tfrac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & \text{for } a > 0 \\ -\tfrac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-j(\omega/a)\lambda}d\lambda & \text{for } a < 0. \end{cases}
\end{aligned}
$$

Combining the two cases (i.e., for $a < 0$ and $a > 0$), we obtain

$$\mathscr{F}\{x(at)\} = \frac{1}{|a|} \int_{-\infty}^{\infty} x(\lambda) e^{-j(\omega/a)\lambda} d\lambda$$
$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

Thus, we have shown that the time/frequency-scaling property holds.

Example 5.9 (Time scaling property of the Fourier transform). Find the Fourier transform $X(\omega)$ of the signal $x(t) = \text{rect}(at)$.

Solution. Let $v(t) = \text{rect}(t)$ so that $x(t) = v(at)$. From Example 5.3, we know that

$$\mathscr{F}\{v(t)\} = V(\omega) = \mathscr{F}\{\text{rect} t\} = \text{sinc } \omega/2. \tag{5.10}$$

From the definition of $v(t)$ and the time-scaling property of the Fourier transform, we have

$$X(\omega) = \mathscr{F}\{x(t)\}$$
$$= \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

Substituting the expression for $V(\omega)$ in (5.10) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|} \text{sinc } \frac{\omega}{2a}.$$

Thus, we have shown that

$$\text{rect}(at) \overset{\mathscr{F}}{\longleftrightarrow} \tfrac{1}{|a|} \text{sinc } \tfrac{\omega}{2a}.$$

$\square$

5.5.5 Conjugation

If $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$x^*(t) \overset{\mathscr{F}}{\longleftrightarrow} X^*(-\omega).$$

This is known as the conjugation property of the Fourier transform.

A proof of the above property is quite simple. From the definition of the Fourier transform, we have

$$\mathscr{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt.$$

From the properties of conjugation, we can rewrite this equation as

$$\mathscr{F}\{x^*(t)\} = \left[\left(\int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt\right)^*\right]^*$$
$$= \left[\int_{-\infty}^{\infty} [x(t)^*]^* [e^{-j\omega t}]^* dt\right]^*$$
$$= \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt\right]^*$$
$$= X^*(-\omega).$$

Thus, we have shown that the conjugation property holds.

Example 5.10 (Fourier transform of a real signal). Show that the Fourier transform $X(\omega)$ of any real signal $x(t)$ must satisfy $X(\omega) = X^*(-\omega)$, and this condition implies that $|X(\omega)| = |X(-\omega)|$ and $\arg X(\omega) = -\arg X(-\omega)$ (i.e., $|X(\omega)|$ and $\arg X(\omega)$ are even and odd functions, respectively).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathscr{F}\{x^*(t)\} = X^*(-\omega).$$

Since $x(t)$ is real, $x(t) = x^*(t)$ which implies

$$\mathscr{F}\{x(t)\} = X^*(-\omega),$$

or equivalently

$$X(\omega) = X^*(-\omega). \tag{5.11}$$

Taking the magnitude of both sides of (5.11) and observing that $|z| = |z^*|$ for any complex z, we have

$$|X(\omega)| = |X^*(-\omega)|$$
$$= |X(-\omega)|.$$

Taking the argument of both sides of (5.11) and observing that $\arg z^* = -\arg z$ for any complex z, we have

$$\arg X(\omega) = \arg X^*(-\omega)$$
$$= -\arg X(-\omega).$$

$\square$

5.5.6 Duality

If $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$X(t) \stackrel{\mathscr{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

This is known as the duality property of the Fourier transform. This property follows from the similarity/symmetry in the definition of the forward and inverse Fourier transforms. That is, the forward Fourier transform equation given by (5.8) and the inverse Fourier transform equation given by (5.9) are identical except for a factor of 2π and different sign in the parameter for the exponential function.

To prove the above property, we proceed as follows. From the Fourier transform synthesis equation, we have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

Substituting $-t$ for t, we obtain

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega.$$

Now, we multiply both sides of the equation by 2π to yield

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$
$$= \mathscr{F}\{X(t)\}.$$

Thus, we have shown that the duality property holds.

Example 5.11 (Fourier transform of the sinc function). Given that $\mathrm{rect}(t) \overset{\mathscr{F}}{\longleftrightarrow} \mathrm{sinc}\,\omega/2$, find the Fourier transform $X(\omega)$ of the signal $x(t) = \mathrm{sinc}(t/2)$.

Solution. From the given Fourier transform pair and the duality property, we have that

$$X(\omega) = \mathscr{F}\{\mathrm{sinc}\,t/2\} = 2\pi\,\mathrm{rect}(-\omega).$$

Since $\mathrm{rect}(-\omega) = \mathrm{rect}(\omega)$, we can simplify this to obtain

$$X(\omega) = 2\pi\,\mathrm{rect}(\omega).$$

Thus, we have shown that

$$\mathrm{sinc}\,t/2 \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\,\mathrm{rect}\,\omega.$$

$\square$

5.5.7   Time-Domain Convolution

If $x_1(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t) * x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

This is known as the time-domain convolution property of the Fourier transform.

The proof of this property is as follows. From the definition of the Fourier transform and convolution, we have

$$\mathscr{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) e^{-j\omega t} d\tau dt.$$

Changing the order of integration, we obtain

$$\mathscr{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) e^{-j\omega t} dt d\tau.$$

Now, we use a change of variable. Let $\lambda = t - \tau$ so that $t = \lambda + \tau$ and $d\lambda = dt$. Applying the change of variable and simplifying, we obtain

$$\mathscr{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(\lambda) e^{-j\omega(\lambda+\tau)} d\lambda d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) x_2(\lambda) e^{-j\omega\lambda} e^{-j\omega\tau} d\lambda d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} \left[\int_{-\infty}^{\infty} x_2(\lambda) e^{-j\omega\lambda} d\lambda \right] d\tau$$

$$= \left[\int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau \right] \left[\int_{-\infty}^{\infty} x_2(\lambda) e^{-j\omega\lambda} d\lambda \right]$$

$$= \mathscr{F}\{x_1(t)\} \mathscr{F}\{x_2(t)\}$$

$$= X_1(\omega)X_2(\omega).$$

Thus, we have shown that the time-domain convolution property holds.

Example 5.12 (Time-domain convolution property of the Fourier transform). With the aid of Table 5.2, find the Fourier transform $X(\omega)$ of the signal $x(t) = x_1(t) * x_2(t)$ where

$$x_1(t) = e^{-2t}u(t) \quad \text{and}$$
$$x_2(t) = u(t).$$

Solution. Let $X_1(\omega)$ and $X_2(\omega)$ denote the Fourier transforms of $x_1(t)$ and $x_2(t)$, respectively. From the time-domain convolution property of the Fourier transform, we know that

$$X(\omega) = \mathscr{F}\{x_1(t) * x_2(t)\} = X_1(\omega)X_2(\omega). \tag{5.12}$$

From Table 5.2, we know that

$$X_1(\omega) = \mathscr{F}\{x_1(t)\} = \mathscr{F}\{e^{-2t}u(t)\} = \tfrac{1}{2+j\omega} \quad \text{and}$$
$$X_2(\omega) = \mathscr{F}\{x_2(t)\} = \mathscr{F}\{u(t)\} = \pi\delta(\omega) + \tfrac{1}{j\omega}.$$

Substituting these expressions for $X_1(\omega)$ and $X_2(\omega)$ into (5.12), we obtain

$$\begin{aligned}
X(\omega) &= [\tfrac{1}{2+j\omega}](\pi\delta(\omega) + \tfrac{1}{j\omega}) \\
&= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j\omega}(\tfrac{1}{2+j\omega}) \\
&= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2} \\
&= \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}.
\end{aligned}$$

$\square$

5.5.8 Frequency-Domain Convolution

If $x_1(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\mathscr{F}}{\longleftrightarrow} \tfrac{1}{2\pi}X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\theta)X_2(\omega-\theta)d\theta.$$

This is known as the frequency-domain convolution (or time-domain multiplication) property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the inverse Fourier transform, we have

$$\begin{aligned}
\mathscr{F}^{-1}\left\{\frac{1}{2\pi}X_1(\omega) * X_2(\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}X_1(\omega) * X_2(\omega)\right]e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)d\lambda\right]e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\lambda d\omega.
\end{aligned}$$

Reversing the order of integration, we obtain

$$\mathscr{F}^{-1}\left\{\frac{1}{2\pi}X_1(\omega) * X_2(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(\omega-\lambda)e^{j\omega t}d\omega d\lambda.$$

Now, we employ a change of variable. Let $v = \omega - \lambda$ so that $\omega = v + \lambda$ and $dv = d\lambda$. Applying the change of variable

and simplifying yields

$$
\begin{aligned}
\mathscr{F}^{-1}\left\{\frac{1}{2\pi}X_1(\omega)*X_2(\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(v)e^{j(v+\lambda)t}dvd\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}X_1(\lambda)X_2(v)e^{jvt}e^{j\lambda t}dvd\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\lambda)e^{j\lambda t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(v)e^{jvt}dv\right]d\lambda \\
&= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\lambda)e^{j\lambda t}d\lambda\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(v)e^{jvt}dv\right] \\
&= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_1(\omega)e^{j\omega t}d\omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X_2(\omega)e^{j\omega t}d\omega\right] \\
&= \mathscr{F}^{-1}\{X_1(\omega)\}\mathscr{F}^{-1}\{X_2(\omega)\} \\
&= x_1(t)x_2(t).
\end{aligned}
$$

Thus, we have shown that the frequency-domain convolution property holds.

Example 5.13 (Frequency-domain convolution property). Suppose that we have the signal

$$
y(t) = x(t)\cos\omega_c t
$$

where ω_c is a nonzero real constant. Find the Fourier transform $Y(\omega)$ of the signal $y(t)$ in terms of $X(\omega) = \mathscr{F}\{x(t)\}$.

Solution. Taking the Fourier transform of both sides of the above equation for $y(t)$, we have

$$
Y(\omega) = \mathscr{F}\{x(t)\cos\omega_c t\}.
$$

Using the frequency-domain convolution property of the Fourier transform, we can write

$$
\begin{aligned}
Y(\omega) &= \mathscr{F}\{x(t)\cos\omega_c t\} \\
&= \tfrac{1}{2\pi}\mathscr{F}\{x(t)\}*\mathscr{F}\{\cos\omega_c t\} \\
&= \tfrac{1}{2\pi}X(\omega)*[\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]] \\
&= \tfrac{1}{2}X(\omega)*[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] \\
&= \tfrac{1}{2}[X(\omega)*\delta(\omega-\omega_c)+X(\omega)*\delta(\omega+\omega_c)] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty}X(\lambda)\delta(\omega-\lambda-\omega_c)d\lambda+\int_{-\infty}^{\infty}X(\lambda)\delta(\omega-\lambda+\omega_c)d\lambda\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty}X(\lambda)\delta(\lambda-\omega+\omega_c)d\lambda+\int_{-\infty}^{\infty}X(\lambda)\delta(\lambda-\omega-\omega_c)d\lambda\right] \\
&= \tfrac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)] \\
&= \tfrac{1}{2}X(\omega-\omega_c)+\tfrac{1}{2}X(\omega+\omega_c).
\end{aligned}
$$

$\square$

5.5.9   Time-Domain Differentiation

If $x(t) \xleftrightarrow{\ \mathscr{F}\ } X(\omega)$, then

$$
\frac{dx(t)}{dt} \xleftrightarrow{\ \mathscr{F}\ } j\omega X(\omega).
$$

This is known as the time-domain differentiation property of the Fourier transform.

To prove the above property, we begin by using the definition of the inverse Fourier transform to write

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

Now, we differentiate both sides of the preceding equation with respect to t and simplify to obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)(j\omega) e^{j\omega t} d\omega$$
$$= \mathscr{F}^{-1}\{j\omega X(\omega)\}.$$

Thus, we have shown that the time-differentiation property holds. In passing, we also note that by repeating the above argument, we have

$$\frac{d^n}{dt^n} x(t) \xleftrightarrow{\mathscr{F}} (j\omega)^n X(\omega).$$

Example 5.14 (Time-domain differentiation property). Find the Fourier transform $X(\omega)$ of the signal $x(t) = \frac{d}{dt}\delta(t)$.

Solution. Taking the Fourier transform of both sides of the above equation for $x(t)$ yields

$$X(\omega) = \mathscr{F}\{\tfrac{d}{dt}\delta(t)\}.$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$X(\omega) = \mathscr{F}\{\tfrac{d}{dt}\delta(t)\}$$
$$= j\omega \mathscr{F}\{\delta(t)\}$$
$$= j\omega(1)$$
$$= j\omega.$$

$\square$

5.5.10 Frequency-Domain Differentiation

If $x(t) \xleftrightarrow{\mathscr{F}} X(\omega)$, then

$$tx(t) \xleftrightarrow{\mathscr{F}} j\frac{d}{d\omega} X(\omega).$$

This is known as the frequency-domain differentiation property of the Fourier transform.

To prove the above property, we proceed as follows. From the definition of the Fourier transform, we can write

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

Now, we differentiate both sides of this equation with respect to ω and simplify to obtain

$$\frac{d}{d\omega} X(\omega) = \int_{-\infty}^{\infty} x(t)(-jt) e^{-j\omega t} dt$$
$$= -j \int_{-\infty}^{\infty} tx(t) e^{-j\omega t} dt$$
$$= -j\mathscr{F}\{tx(t)\}.$$

Multiplying both sides of the preceding equation by j yields

$$j\frac{d}{d\omega} X(\omega) = \mathscr{F}\{tx(t)\}.$$

Thus, we have shown that the frequency-domain differentiation property holds.

Example 5.15 (Frequency-domain differentiation property). Find the Fourier transform $X(\omega)$ of the signal $x(t) = t\cos\omega_0 t$ where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for $x(t)$ yields

$$X(\omega) = \mathscr{F}\{t\cos\omega_0 t\}.$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned}
X(\omega) &= \mathscr{F}\{t\cos\omega_0 t\} \\
&= j\tfrac{d}{d\omega}[\mathscr{F}\{\cos\omega_0 t\}] \\
&= j\tfrac{d}{d\omega}[\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]] \\
&= j\pi\tfrac{d}{d\omega}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \\
&= j\pi\tfrac{d}{d\omega}\delta(\omega - \omega_0) + j\pi\tfrac{d}{d\omega}\delta(\omega + \omega_0).
\end{aligned}$$

$\square$

5.5.11 Time-Domain Integration

If $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

This is known as the time-domain integration property of the Fourier transform.

The above property can be proven as follows. First, we observe that

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t).$$

Taking the Fourier transform of both sides of the preceding equation and using the time-domain convolution property of the Fourier transform, we have

$$\begin{aligned}
\mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} &= \mathscr{F}\{x(t) * u(t)\} \\
&= X(\omega)\mathscr{F}\{u(t)\}. \tag{5.13}
\end{aligned}$$

From Example 5.6, we know that $u(t) \overset{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}$. Using this fact, we can rewrite (5.13) as

$$\begin{aligned}
\mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} &= X(\omega)[\pi\delta(\omega) + \tfrac{1}{j\omega}] \\
&= \tfrac{1}{j\omega}X(\omega) + \pi X(\omega)\delta(\omega).
\end{aligned}$$

From the equivalence property of the unit-impulse function, we have

$$\mathscr{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \tfrac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

Thus, we have shown that the time-domain integration property holds.

Example 5.16 (Time-domain integration property of the Fourier transform). Use the time-domain integration property of the Fourier transform in order to find the Fourier transform $X(\omega)$ of the signal $x(t) = u(t)$.

Solution. We begin by observing that $x(t)$ can be expressed in terms of an integral as follows:

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau.$$

Now, we consider the Fourier transform of $x(t)$. We have

$$\begin{aligned} X(\omega) &= \mathscr{F}\{u(t)\} \\ &= \mathscr{F}\left\{\int_{-\infty}^{t} \delta(\tau) d\tau\right\}. \end{aligned}$$

From the time-domain integration property, we can write

$$\begin{aligned} X(\omega) &= \frac{1}{j\omega} \mathscr{F}\{\delta(t)\} + \pi \left[\mathscr{F}\{\delta(t)\}\right]|_{\omega=0} \delta(\omega) \\ &= \frac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \frac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

Thus, we have shown that $u(t) \xleftrightarrow{\mathscr{F}} \frac{1}{j\omega} + \pi\delta(\omega)$.   □

5.5.12 Parseval's Relation

Recall that the energy of a signal $f(t)$ is given by $\int_{-\infty}^{\infty} |f(t)|^2 dt$. If $x(t) \xleftrightarrow{\mathscr{F}} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \tag{5.14}$$

That is, the energy of $x(t)$ and energy of $X(\omega)$ are equal within a scaling factor of 2π. This relationship is known as Parseval's relation.

To prove the above relationship, we proceed as follows. Consider the left-hand side of (5.14) which we can write as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t) dt \\ &= \int_{-\infty}^{\infty} x(t)\mathscr{F}^{-1}\{\mathscr{F}\{x^*(t)\}\} dt. \end{aligned}$$

From the conjugation property of the Fourier transform, we have that $x^*(t) \xleftrightarrow{\mathscr{F}} X^*(-\omega)$. So, we can rewrite the above equation as

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)\mathscr{F}^{-1}\{X^*(-\omega)\} dt \\ &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(-\omega)e^{j\omega t} d\omega\right] dt. \end{aligned}$$

Now, we employ a change of variable (i.e., replace ω by $-\omega$) to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t} d\omega\right] dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)X^*(\omega)e^{-j\omega t} d\omega dt. \end{aligned}$$

Reversing the order of integration and simplifying, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 \, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) X^*(\omega) e^{-j\omega t} \, dt \, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right] d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) \, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega.
\end{aligned}
$$

Thus, Parseval's relation holds.

Example 5.17 (Energy of the sinc signal). Suppose that we have the signal $x(t) = \operatorname{sinc} t/2$ and $\mathscr{F}\{x(t)\} = 2\pi \operatorname{rect} \omega$. Compute the energy of the signal $x(t)$.

Solution. We could directly compute the energy of the signal $x(t)$ as

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\
&= \int_{-\infty}^{\infty} |\operatorname{sinc} t/2|^2 \, dt.
\end{aligned}
$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$
\begin{aligned}
E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \operatorname{rect} \omega|^2 \, d\omega \\
&= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 \, d\omega \\
&= 2\pi \int_{-1/2}^{1/2} d\omega \\
&= 2\pi [\omega] \big|_{-1/2}^{1/2} \\
&= 2\pi [\tfrac{1}{2} + \tfrac{1}{2}] \\
&= 2\pi.
\end{aligned}
$$

Thus, we have

$$
E = \int_{-\infty}^{\infty} \left| \operatorname{sinc} \tfrac{t}{2} \right|^2 \, dt = 2\pi.
$$

$\square$

5.6 Continuous-Time Fourier Transform of Periodic Signals

The Fourier transform can be applied to both aperiodic and periodic signals. In what follows, we derive a method for computing the Fourier transform of a periodic signal.

Suppose that we have a periodic signal $x(t)$ with period T. Since $x(t)$ is periodic, we can express it using a Fourier series as

$$
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{5.15}
$$

Table 5.1: Fourier Transform Properties

Property	Time Domain	Frequency Domain
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
Time-Domain Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{|a|} X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1(t) * x_2(t)$	$X_1(\omega) X_2(\omega)$
Frequency-Domain Convolution	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt} x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$t x(t)$	$j\frac{d}{d\omega} X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$

where

$$a_k = \frac{1}{T} \int_{T} x(t) e^{-jk\omega_0 t} dt. \tag{5.16}$$

Let us define the signal $x_T(t)$ to contain a single period of $x(t)$ as follows:

$$x_T(t) = \begin{cases} x(t) & \text{for } -\frac{T}{2} \le t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

Let $X_T(\omega)$ denote the Fourier transform of $x_T(t)$. Consider the expression for a_k in (5.16). Since $x_T(t) = x(t)$ for a single period of $x(t)$ and is zero otherwise, we can rewrite (5.16) as

$$a_k = \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} X_T(k\omega_0). \tag{5.18}$$

Now, let us consider the Fourier transform of $x(t)$. By taking the Fourier transform of both sides of (5.15), we obtain

$$\mathscr{F}\{x(t)\} = \mathscr{F}\left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right\}$$

$$= \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt.$$

Reversing the order of summation and integration, we have

$$\mathscr{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \mathscr{F}\{e^{jk\omega_0 t}\}. \tag{5.19}$$
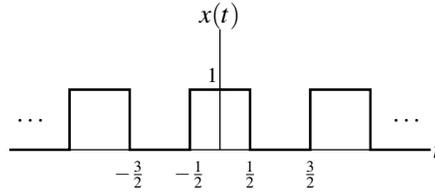
Figure 5.5: Periodic function $x(t)$.

From Example 5.2, we know that $\mathscr{F}\{e^{j\lambda t}\} = 2\pi\delta(\omega - \lambda)$. So, we can simplify (5.19) to obtain

$$\mathscr{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k[2\pi\delta(\omega - k\omega_0)]$$

$$= \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega - k\omega_0). \tag{5.20}$$

Thus, the Fourier transform of a periodic function is a series of impulse functions located at integer multiples of the fundamental frequency ω_0. The weight of each impulse is 2π times the corresponding Fourier series coefficient. Furthermore, by substituting (5.18) into (5.20), we have

$$\mathscr{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} 2\pi[\tfrac{1}{T}X_T(k\omega_0)]\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0). \tag{5.21}$$

This provides an alternative expression for the Fourier transform of $x(t)$ in terms of the Fourier transform $X_T(\omega)$ of a single period of $x(t)$.

In summary, we have shown that the periodic signal $x(t)$ with period T and frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier transform $X(\omega)$ given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega - k\omega_0). \tag{5.22}$$

(as in (5.20)), or equivalently,

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0) \tag{5.23}$$

(as in (5.21), where a_k is the Fourier series coefficient sequence of $x(t)$, $X_T(\omega)$ is the Fourier transform of $x_T(t)$, and $x_T(t)$ is a function equal to $x(t)$ over a single period and zero elsewhere (e.g., as in (5.17)). Furthermore, we have also shown that the Fourier series coefficients a_k of $x(t)$ are related to $X_T(\omega)$ by the equation

$$a_k = \frac{1}{T}X_T(k\omega_0).$$

Thus, we have that the Fourier series coefficient sequence a_k of the periodic signal $x(t)$ is produced by sampling the Fourier transform of $x_T(t)$ at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$.

Example 5.18. Consider the periodic function $x(t)$ with period $T = 2$ as shown in Figure 5.5. Using the Fourier transform, find the Fourier series representation of $x(t)$.

Table 5.2: Fourier Transform Pairs

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
8	$\operatorname{rect}(t/T)$	$T\operatorname{sinc}(\omega T/2)$
9	$\frac{B}{\pi}\operatorname{sinc}Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t),\ \operatorname{Re}\{a\} > 0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t),\ \operatorname{Re}\{a\} > 0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{T}{2}\operatorname{sinc}^2(T\omega/4)$

Solution. We have that $\omega_0 = \frac{2\pi}{T} = \pi$. Let $y(t) = \operatorname{rect}t$ (i.e., $y(t)$ corresponds to a single period of the periodic function $x(t)$). Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} y(t - 2k).$$

Taking the Fourier transform of $y(t)$, we obtain

$$Y(\omega) = \mathscr{F}\{\operatorname{rect}t\} = \operatorname{sinc}\omega/2.$$

Now, we seek to find the Fourier series representation of $x(t)$, which has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we compute c_k as follows:

$$
\begin{aligned}
c_k &= \tfrac{1}{T}Y(k\omega_0) \\
&= \tfrac{1}{2}\operatorname{sinc}(\tfrac{k\omega_0}{2}) \\
&= \tfrac{1}{2}\operatorname{sinc}(k\pi/2).
\end{aligned}
$$

$\square$

5.7 Fourier Transforms

Throughout this chapter, we have derived a number of Fourier transform pairs. Some of these and other important transform pairs are listed in Table 5.2. Using the various Fourier transform properties listed in Table 5.1 and the Fourier transform pairs listed in Table 5.2, we can determine (more easily) the Fourier transform of more complicated signals.

Example 5.19. Suppose that $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$, $x_1(t) \overset{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$, and

$$x_1(t) = \frac{d^2}{dt^2}x(t - 2).$$

Express $X_1(\omega)$ in terms of $X(\omega)$.

Solution. Let $v_1(t) = x(t-2)$. From the definition of $v_1(t)$ and the time-shifting property of the Fourier transform, we have

$$V_1(\omega) = e^{-j2\omega}X(\omega). \tag{5.24}$$

From the definition of $v_1(t)$, we have

$$x_1(t) = \frac{d^2}{dt^2}v_1(t).$$

Thus, from the time-differentiation property of the Fourier transform, we can write

$$X_1(\omega) = (j\omega)^2V_1(\omega)$$
$$= -\omega^2V_1(\omega). \tag{5.25}$$

Combining (5.24) and (5.25), we obtain

$$X_1(\omega) = -\omega^2e^{-j2\omega}X(\omega).$$

$\square$

Example 5.20. Suppose that $x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X(\omega)$, $x_1(t) \stackrel{\mathscr{F}}{\longleftrightarrow} X_1(\omega)$, and

$$x_1(t) = x(at-b),$$

where a is a nonzero real constant and b is a real constant. Express $X_1(\omega)$ in terms of $X(\omega)$.

Solution. We rewrite $x_1(t)$ as

$$x_1(t) = v_1(at)$$

where

$$v_1(t) = x(t-b).$$

We now take the Fourier transform of both sides of each of the preceding equations. Using the time-shifting property of the Fourier transform, we can write

$$V_1(\omega) = e^{-jb\omega}X(\omega). \tag{5.26}$$

Using the time-scaling property of the Fourier transform, we can write

$$X_1(\omega) = \frac{1}{|a|}V_1\left(\frac{\omega}{a}\right). \tag{5.27}$$

Substituting the expression for $V_1(\omega)$ in (5.26) into (5.27), we obtain

$$X_1(\omega) = \frac{1}{|a|}e^{-j(b/a)\omega}X\left(\frac{\omega}{a}\right).$$

$\square$

Example 5.21. Suppose that we have the periodic signal $x(t)$ given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

where a single period of $x(t)$ is given by

$$x_0(t) = A\operatorname{rect}(2t/T).$$

Find the Fourier transform $X(\omega)$ of the signal $x(t)$.

Solution. From (5.23), we know that

$$\mathscr{F}\{x(t)\} = \mathscr{F}\left\{\sum_{k=-\infty}^{\infty} x_0(t-kT)\right\}$$

$$= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0)\delta(\omega - k\omega_0).$$

So, we need to find $X_0(\omega)$. This quantity is computed (by using the linearity property of the Fourier transform and Table 5.2) as follows:

$$X_0(\omega) = \mathscr{F}\{x_0(t)\}$$
$$= \mathscr{F}\{A \operatorname{rect}(2t/T)\}$$
$$= A\mathscr{F}\{\operatorname{rect}(2t/T)\}$$
$$= \frac{AT}{2}\operatorname{sinc}(\omega T/4).$$

Thus, we have that

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 \frac{AT}{2}\operatorname{sinc}(k\omega_0 T/4)\delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} \pi A \operatorname{sinc}(\frac{\pi k}{2})\delta(\omega - k\omega_0).$$

$\square$

Example 5.22. Suppose that we have the signal $x(t)$ given by

$$x(t) = \int_{-\infty}^{t} e^{-(3+j2)\tau}u(\tau)d\tau.$$

Find the Fourier transform $X(\omega)$ of the signal $x(t)$.

Solution. We can rewrite $x(t)$ as

$$x(t) = \int_{-\infty}^{t} v_1(\tau)d\tau, \tag{5.28}$$

where

$$v_1(t) = e^{-j2t}v_2(t), \quad \text{and} \tag{5.29}$$
$$v_2(t) = e^{-3t}u(t). \tag{5.30}$$

From (5.28) and the time-domain integration property of the Fourier transform, we have

$$X(\omega) = \frac{1}{j\omega}V_1(\omega) + \pi V_1(0)\delta(\omega). \tag{5.31}$$

From (5.29) and the frequency-domain shifting property of the Fourier transform, we have

$$V_1(\omega) = V_2(\omega + 2). \tag{5.32}$$

From (5.30) and Table 5.2 (i.e., the entry for $\mathscr{F}\{e^{-at}u(t)\}$), we have

$$V_2(\omega) = \frac{1}{3 + j\omega}. \tag{5.33}$$

Combining (5.31), (5.32), and (5.33), we obtain

$$
\begin{aligned}
X(\omega) &= \frac{1}{j\omega} V_1(\omega) + \pi V_1(0)\delta(\omega) \\
&= \frac{1}{j\omega} V_2(\omega + 2) + \pi V_2(2)\delta(\omega) \\
&= \frac{1}{j\omega}\left(\frac{1}{3 + j(\omega + 2)}\right) + \pi\left(\frac{1}{3 + j2}\right)\delta(\omega).
\end{aligned}
$$

$\square$

Example 5.23. Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. Suppose that $y(t) = x(t)\cos at$, where a is a nonzero real constant. Find an expression for $Y(\omega)$ in terms of $X(\omega)$.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. Taking the Fourier transform of both sides of the given equation, we obtain

$$
\begin{aligned}
Y(\omega) &= \mathscr{F}\{x(t)\cos at\} \\
&= \tfrac{1}{2\pi}\left[X(\omega) * \mathscr{F}\{\cos at\}\right] \\
&= \tfrac{1}{2\pi}\left[X(\omega) * \left[\pi\left(\delta(\omega - a) + \delta(\omega + a)\right)\right]\right] \\
&= \tfrac{1}{2}\left[X(\omega) * \left[\delta(\omega - a) + \delta(\omega + a)\right]\right] \\
&= \tfrac{1}{2}\left[X(\omega) * \delta(\omega - a) + X(\omega) * \delta(\omega + a)\right] \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\tau)\delta(\omega - \tau - a)d\tau + \int_{-\infty}^{\infty} X(\tau)\delta(\omega - \tau + a)d\tau\right] \\
&= \tfrac{1}{2}\left[X(\omega - a) + X(\omega + a)\right] \\
&= \tfrac{1}{2}X(\omega - a) + \tfrac{1}{2}X(\omega + a).
\end{aligned}
$$

Note that the above solution is identical to the one appearing earlier in Example 5.13 on page 116.

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$
\begin{aligned}
Y(\omega) &= \mathscr{F}\{x(t)\cos at\} \\
&= \mathscr{F}\{\tfrac{1}{2}(e^{jat} + e^{-jat})x(t)\} \\
&= \tfrac{1}{2}\mathscr{F}\{e^{jat}x(t)\} + \tfrac{1}{2}\mathscr{F}\{e^{-jat}x(t)\} \\
&= \tfrac{1}{2}X(\omega - a) + \tfrac{1}{2}X(\omega + a).
\end{aligned}
$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. $\square$

Example 5.24 (Fourier transform of an even signal). Show that if a signal $x(t)$ is even, then its Fourier transform $X(\omega)$ is even.

Solution. From the definition of the Fourier transform, we have

$$
X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.
$$

Since $x(t)$ is even, we can rewrite this as

$$X(\omega) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt.$$

Now, we employ a change of variable. Let $\lambda = -t$ so that $d\lambda = -dt$. Applying the change of variable, we obtain

$$X(\omega) = \int_{\infty}^{-\infty} x(\lambda)e^{j\omega\lambda}(-1)d\lambda$$

$$= -\int_{\infty}^{-\infty} x(\lambda)e^{j\omega\lambda}d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)e^{j\omega\lambda}d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)e^{-j(-\omega)\lambda}d\lambda$$

$$= X(-\omega).$$

Therefore, $X(\omega)$ is even. □

Example 5.25 (Fourier transform of an odd signal). Show that if a signal $x(t)$ is odd, then its Fourier transform $X(\omega)$ is odd.

Solution. From the definition of the Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

Since $x(t)$ is odd, we can rewrite this as

$$X(\omega) = \int_{-\infty}^{\infty} -x(-t)e^{-j\omega t}dt.$$

Now, we employ a change of variable. Let $\lambda = -t$ so that $d\lambda = -dt$. Applying this change of variable, we obtain

$$X(\omega) = \int_{\infty}^{-\infty} -x(\lambda)e^{-j\omega(-\lambda)}(-1)d\lambda$$

$$= \int_{\infty}^{-\infty} x(\lambda)e^{j\omega\lambda}d\lambda$$

$$= -\int_{-\infty}^{\infty} x(\lambda)e^{-j(-\omega)\lambda}d\lambda$$

$$= -X(-\omega).$$

Therefore, $X(\omega)$ is odd. □

5.8 Frequency Spectra of Signals

The Fourier transform representation expresses a signal as a function of complex sinusoids at all frequencies. In this sense, the Fourier transform representation captures information about the frequency content of a signal. For example, suppose that we have a signal $x(t)$ with Fourier transform $X(\omega)$. If $X(\omega)$ is nonzero for some frequency ω_0, then the signal $x(t)$ contains some information at the frequency ω_0. On the other hand, if $X(\omega)$ is zero at the frequency ω_0, then the signal has no information at that frequency. In this way, the Fourier transform representation provides a means for measuring the frequency content of a signal. This distribution of information in a signal over different frequencies is referred to as the **frequency spectrum** of the signal. That is, $X(\omega)$ is the frequency spectrum of $x(t)$.

To gain further insight into the role played by the Fourier transform $X(\omega)$ in the context of the frequency spectrum of $x(t)$, it is helpful to write the Fourier transform representation of $x(t)$ with $X(\omega)$ expressed in polar form as follows:

$$
\begin{aligned}
x(t) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)| e^{j \arg X(\omega)} e^{j\omega t} d\omega \\
&= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)| e^{j[\omega t + \arg X(\omega)]} d\omega.
\end{aligned}
$$

In effect, the quantity $|X(\omega)|$ is a weight that determines how much the complex sinusoid at frequency ω contributes to the integration result $x(t)$. Perhaps, this can be more easily seen if we express the above integral as the limit of a sum, derived from an approximation of the integral using the area of rectangles. Expressing $x(t)$ in this way, we obtain

$$
\begin{aligned}
x(t) &= \lim_{\Delta\omega \to 0} \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \, |X(k\Delta\omega)| \, e^{j[k\Delta\omega t + \arg X(k\Delta\omega)]} \\
&= \lim_{\Delta\omega \to 0} \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega \, |X(\omega')| \, e^{j[\omega' t + \arg X(\omega')]},
\end{aligned}
$$

where $\omega' = k\Delta\omega$. From the last line of the above equation, the kth term in the summation (associated with the frequency $\omega' = k\Delta\omega$) corresponds to a complex sinusoid with fundamental frequency ω' that has had its amplitude scaled by a factor of $|X(\omega')|$ and has been time-shifted by an amount that depends on $\arg X(\omega')$. For a given $\omega' = k\Delta\omega$ (which is associated with the kth term in the summation), the larger $|X(\omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega' t}$ will be, and therefore the larger the contribution the kth term will make to the overall summation. In this way, we can use $|X(\omega')|$ as a measure of how much information a signal $x(t)$ has at the frequency ω'.

Note that, since the Fourier transform $X(\omega)$ is a function of a real variable (namely, ω), a signal can, in the most general case, have information at *any arbitrary* real frequency. This is different from the case of frequency spectra in the Fourier series context (which deals only with periodic signals), where a signal can only have information at certain specific frequencies (namely, at integer multiples of the fundamental frequency). There is no inconsistency here, however. As we saw in Section 5.6, in the case of periodic signals the Fourier transform will also be zero, except possibly at integer multiples of the fundamental frequency.

Since the frequency spectrum is complex (in the general case), it is usually represented using two plots, one showing the magnitude and one showing the argument of $X(\omega)$. We refer to $|X(\omega)|$ as the **magnitude spectrum** of the signal $x(t)$. Similarly, we refer to $\arg X(\omega)$ as the **phase spectrum** of the signal $x(t)$. In the special case that $X(\omega)$ is a real-valued (or purely imaginary) function, we usually plot the frequency spectrum directly on a single graph.

Consider the signal $x(t) = \operatorname{sgn}(t-1)$. We can show that the Fourier transform $X(\omega)$ of this signal is $X(\omega) = \frac{2}{j\omega} e^{-j\omega}$. In this case, $X(\omega)$ is neither purely real nor purely imaginary, so we use two separate graphs to represent the frequency spectrum of the signal. We plot the magnitude spectrum and phase spectrum as shown in Figures 5.6(a) and (b), respectively.

Consider the signal $x(t) = \delta(t)$. This signal has the Fourier transform $X(\omega) = 1$. Since, in this case, $X(\omega)$ is a real-valued function, we can plot the frequency spectrum $X(\omega)$ on a single graph, as shown in Figure 5.7.

Consider the signal $x(t) = \operatorname{sinc} t/2$. This signal has the Fourier transform $X(\omega) = 2\pi \operatorname{rect} \omega$. Since, in this case, $X(\omega)$ is a real-valued function, we can plot the frequency spectrum $X(\omega)$ on a single graph, as shown in Figure 5.8.

Suppose that we have a signal $x(t)$ with Fourier transform $X(\omega)$. If $x(t)$ is real, then

$$
\begin{aligned}
|X(\omega)| &= |X(-\omega)| \quad \text{and} \\
\arg X(\omega) &= -\arg X(-\omega)
\end{aligned}
$$

(i.e., $|X(\omega)|$ is an even function and $\arg X(\omega)$ is an odd function). (Earlier, these identities were shown to hold. See Example 5.10 for a detailed proof.)
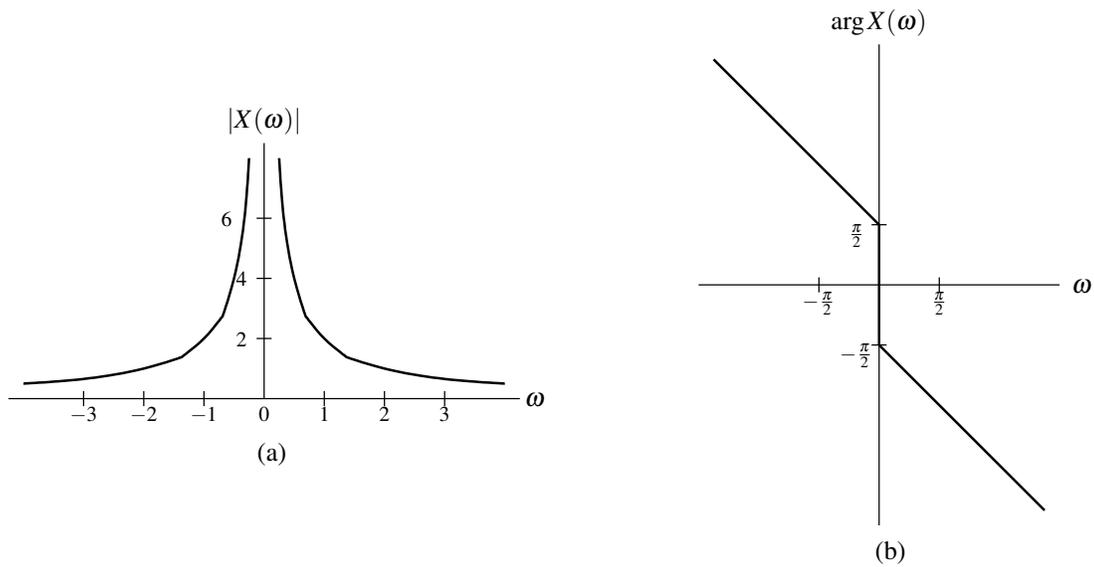
Figure 5.6: Frequency spectrum of the time-shifted signum signal. (a) Magnitude spectrum and (b) Phase spectrum.
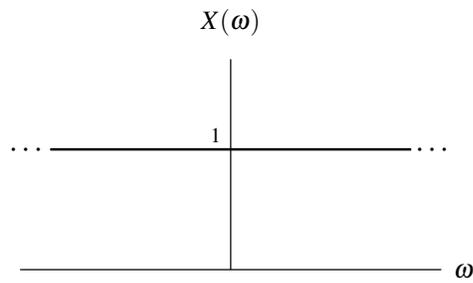
Figure 5.7: Frequency spectrum of the unit-impulse function.
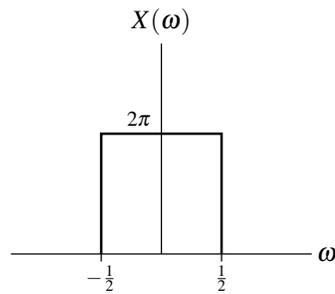
Figure 5.8: Frequency spectrum of the sinc function.

Example 5.26 (Frequency spectrum of the shifted signum signal). Suppose that we have the signal $x(t) = \text{sgn}(t - 1)$ with the Fourier transform $X(\omega) = \left(\frac{2}{j\omega}\right)e^{-j\omega}$. Find and plot the magnitude and phase spectra of this signal. Determine at what frequency (or frequencies) the signal $x(t)$ has the most information.

Solution. First, we must find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

$$
\begin{aligned}
|X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| \\
&= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| \\
&= \left|\tfrac{2}{j\omega}\right| \\
&= \tfrac{2}{|\omega|}.
\end{aligned}
$$

Next, we find the phase spectrum $\arg X(\omega)$. From the expression for $X(\omega)$, we can write

$$
\begin{aligned}
\arg X(\omega) &= \arg\left\{\tfrac{2}{j\omega}e^{-j\omega}\right\} \\
&= \arg e^{-j\omega} + \arg\tfrac{2}{j\omega} \\
&= -\omega + \arg\tfrac{2}{j\omega} \\
&= -\omega + \arg(-\tfrac{j2}{\omega}) \\
&= \begin{cases} -\tfrac{\pi}{2} - \omega & \text{for } \omega > 0 \\ \tfrac{\pi}{2} - \omega & \text{for } \omega < 0 \end{cases} \\
&= -\tfrac{\pi}{2}\text{sgn}\,\omega - \omega.
\end{aligned}
$$

Note that

$$
\arg\tfrac{2}{j\omega} = \arg(-\tfrac{j2}{\omega}) = \begin{cases} -\tfrac{\pi}{2} & \text{for } \omega > 0 \\ \tfrac{\pi}{2} & \text{for } \omega < 0. \end{cases}
$$

Finally, using numerical calculation, we can plot the graphs of the functions $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown previously in Figures 5.6(a) and (b). Since $|X(\omega)|$ is largest for $\omega = 0$, the signal $x(t)$ has the most information at the frequency 0. □

Example 5.27 (Frequency spectra of images). The human visual system is more sensitive to the phase spectrum of an image than its magnitude spectrum. This can be aptly demonstrated by separately modifying the magnitude and phase spectra of an image, and observing the effect. Below, we consider two variations on this theme.

Consider the `potatohead` and `hongkong` images shown in Figures 5.9(a) and (b), respectively. Replacing the magnitude spectrum of the `potatohead` image with the magnitude spectrum of the `hongkong` image (and leaving the phase spectrum unmodified), we obtain the new image shown in Figure 5.9(c). Although changing the magnitude spectrum has led to distortion, the basic essence of the original image has not been lost. On the other hand, replacing the phase spectrum of the `potatohead` image with the phase spectrum of the `hongkong` image (and leaving the magnitude spectrum unmodified), we obtain the image shown in Figure 5.9(d). Clearly, by changing the phase spectrum of the image, the fundamental nature of the image has been altered, with the new image more closely resembling the `hongkong` image than the original `potatohead` image.

A more extreme scenario is considered in Figure 5.10. In this case, we replace each of the magnitude and phase spectra of the `potatohead` image with random data, with this data being taken from the image consisting of random noise shown in Figure 5.10(b). When we completely replace the magnitude spectrum of the `potatohead` image with random values, we can still recognize the resulting image in Figure 5.10(c) as a very grainy version of the original `potatohead` image. On the other hand, when the phase spectrum of the `potatohead` image is replaced with random values, all visible traces of the original `potatohead` image are lost in the resulting image in Figure 5.10(d).
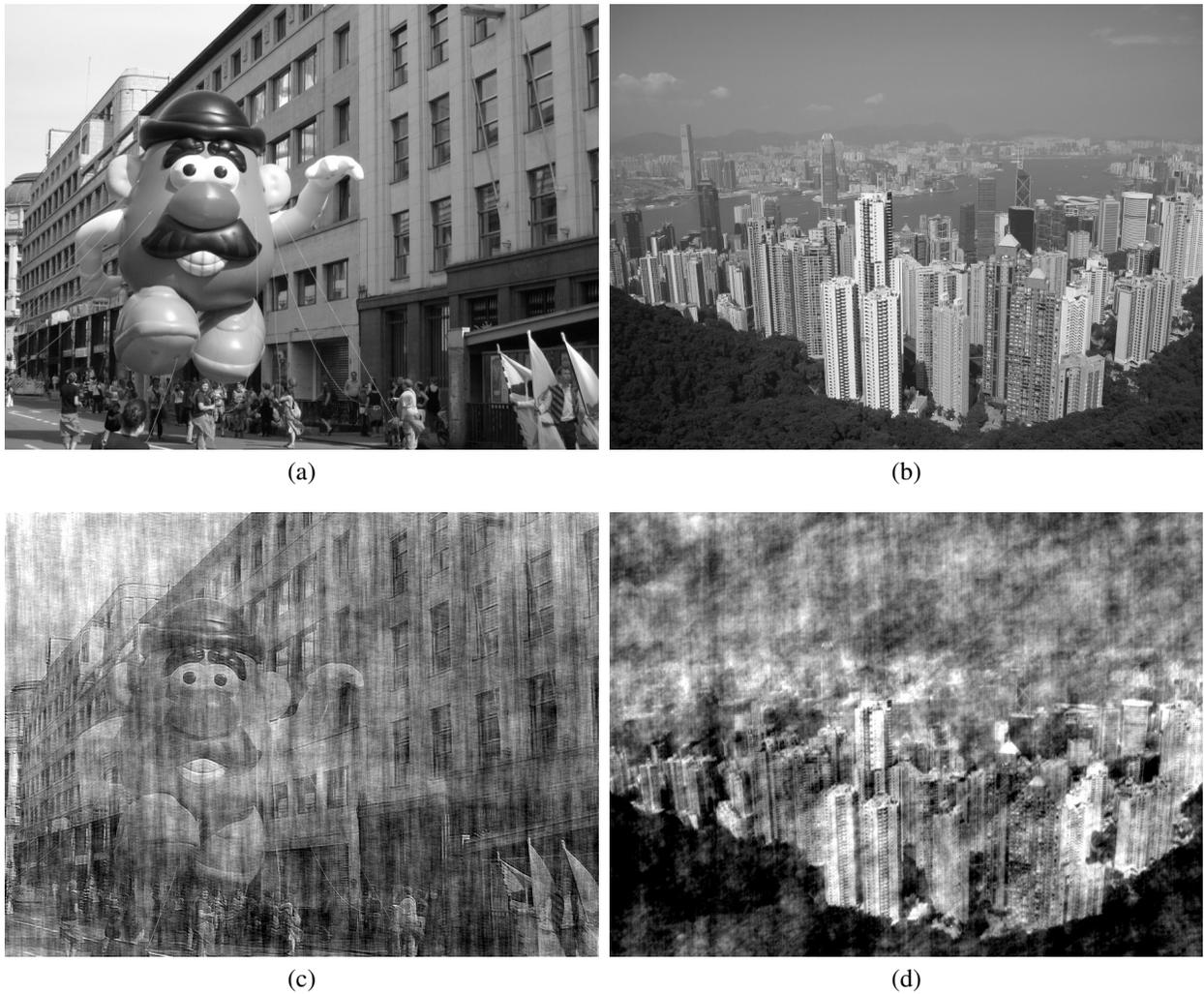
Figure 5.9: Importance of phase information in images. The (a) `potatohead` and (b) `hongkong` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `hongkong` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `hongkong` image.

(a)

(b)

(c)

(d)
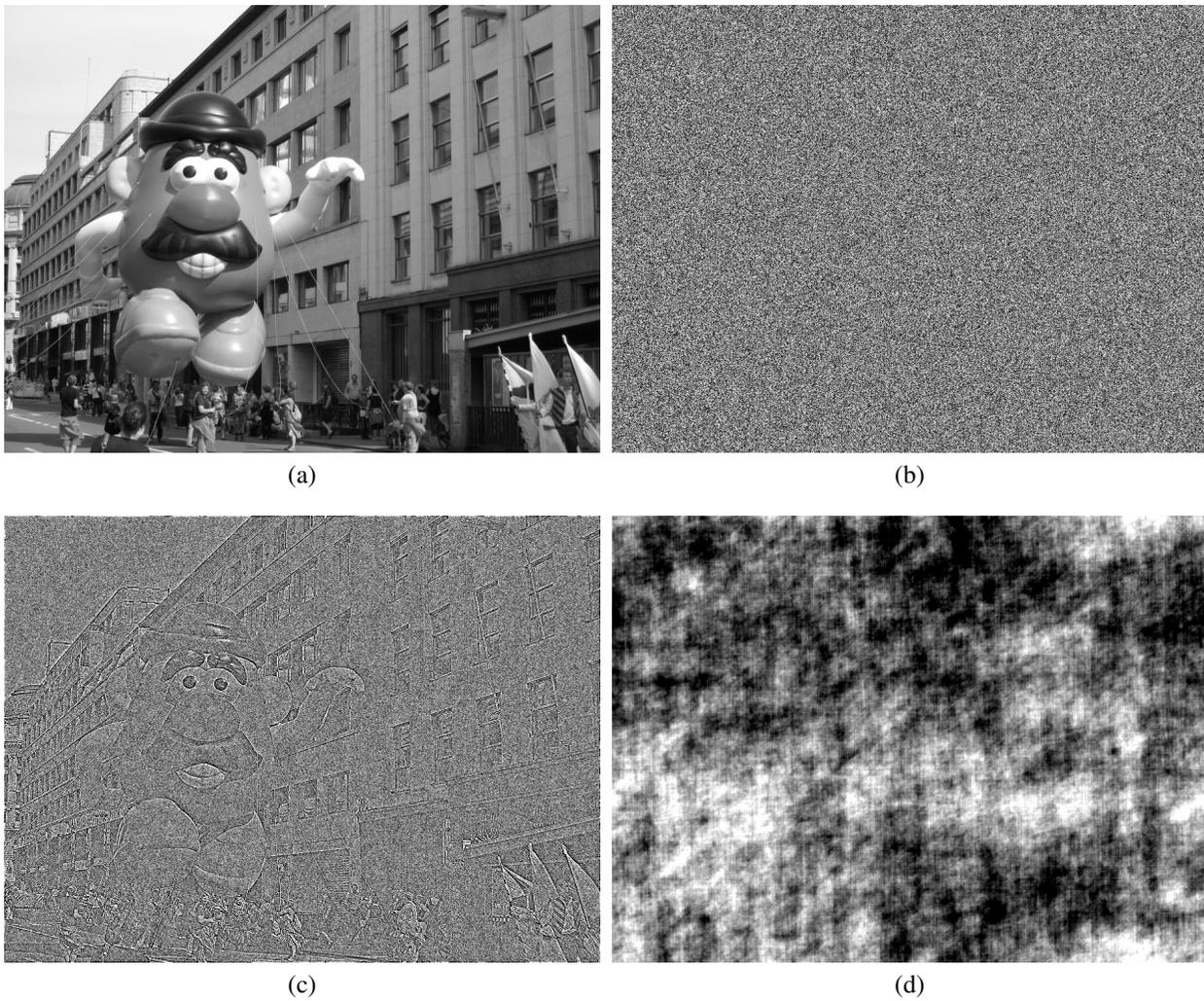
Figure 5.10: Importance of phase information in images. The (a) `potatohead` and (b) `random` images. (c) The `potatohead` image after having its magnitude spectrum replaced with the magnitude spectrum of the `random` image. (d) The `potatohead` image after having its phase spectrum replaced with the phase spectrum of the `random` image.
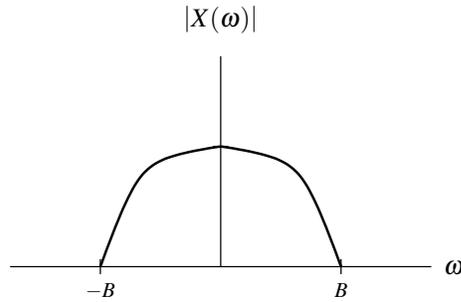
Figure 5.11: Bandwidth of a signal.

5.9 Bandwidth of Signals

A signal $x(t)$ with Fourier transform $X(\omega)$ is said to be **bandlimited** if, for some nonnegative real constant B, $X(\omega) = 0$ for all ω satisfying $|\omega| > B$. We sometimes refer to B as the **bandwidth** of the signal $x(t)$. An illustrative example is provided in Figure 5.11.

One can show that a signal cannot be both time limited and bandlimited. To help understand why this is so we recall the time/frequency scaling property of the Fourier transform. From this property, we know that as we compress a signal $x(t)$ along the t-axis, its Fourier transform $X(\omega)$ will expand along the ω-axis. Similarly, a compression of the Fourier transform $X(\omega)$ along the ω-axis corresponds to an expansion of $x(t)$ along the t-axis. Clearly, there is an inverse relationship between the time-extent and bandwidth of a signal.

5.10 Frequency Response of LTI Systems

Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Such a system is depicted in Figure 5.12. The behavior of such a system is governed by the equation

$$y(t) = x(t) * h(t). \tag{5.34}$$

Let $X(\omega)$, $Y(\omega)$, and $H(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $h(t)$, respectively. Taking the Fourier transform of both sides of (5.34) yields

$$Y(\omega) = \mathscr{F}\{x(t) * h(t)\}.$$

From the time-domain convolution property of the Fourier transform, however, we can rewrite this as

$$Y(\omega) = X(\omega)H(\omega). \tag{5.35}$$

This result provides an alternative way of viewing the behavior of an LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals. In other words, we have a system resembling that in Figure 5.13. In this case, however, the convolution operation from the time domain is replaced by multiplication in the frequency domain. The frequency spectrum (i.e., Fourier transform) of the output is the product of the frequency spectrum (i.e., Fourier transform) of the input and the frequency spectrum (i.e., Fourier transform) of the impulse response. As a matter of terminology, we refer to $H(\omega)$ as the **frequency response** of the system. The system behavior is completely characterized by the frequency response $H(\omega)$. If we know the input, we can compute its Fourier transform $X(\omega)$, and then determine the Fourier transform $Y(\omega)$ of the output. Using the inverse Fourier transform, we can then determine the output $y(t)$.

In the general case, $H(\omega)$ is a complex-valued function. Thus, we can represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$. We refer to $|H(\omega)|$ as the **magnitude response** of the system. Similarly, we call $\arg H(\omega)$ the **phase response** of the system.
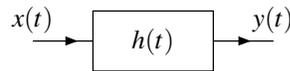
Figure 5.12: Time-domain view of LTI system.
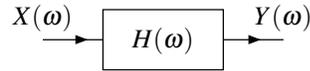
Figure 5.13: Frequency-domain view of LTI system.
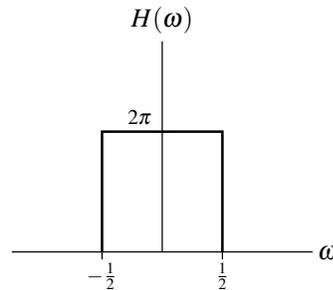
Figure 5.14: Frequency response of example system.

From (5.35), we can write

$$
\begin{aligned}
|Y(\omega)| &= |X(\omega)H(\omega)| \\
&= |X(\omega)||H(\omega)| \quad \text{and}
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
\arg Y(\omega) &= \arg(X(\omega)H(\omega)) \\
&= \arg X(\omega) + \arg H(\omega).
\end{aligned}
\tag{5.37}
$$

From (5.36), we can see that the magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude spectrum of the impulse response. From (5.37), we have that the phase spectrum of the output equals the phase spectrum of the input plus the phase spectrum of the impulse response.

Since the frequency response $H(\omega)$ is simply the frequency spectrum of the impulse response $h(t)$, for the reasons explained in Section 5.8, if $h(t)$ is real, then

$$
\begin{aligned}
|H(\omega)| &= |H(-\omega)| \quad \text{and} \\
\arg H(\omega) &= -\arg H(-\omega)
\end{aligned}
$$

(i.e., the magnitude response $|H(\omega)|$ is an even function and the phase response $\arg H(\omega)$ is an odd function).

Example 5.28. Suppose that we have a LTI system with impulse response $h(t) = \operatorname{sinc} t/2$. The frequency response of the system is $H(\omega) = 2\pi \operatorname{rect} \omega$. In this particular case, $H(\omega)$ is real. So, we can plot the frequency response $H(\omega)$ on a single graph as shown in Figure 5.14.

5.11 Frequency Response and Differential Equation Representations of LTI Systems

Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients. Suppose that we have such a system with input $x(t)$ and output $y(t)$. Then, the input-output behavior of

the system is given by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)$$

(where $M \leq N$). Let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. Taking the Fourier transform of both sides of this equation yields

$$\mathscr{F}\left\{\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t)\right\} = \mathscr{F}\left\{\sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)\right\}.$$

Using the linearity property of the Fourier transform, we can rewrite this as

$$\sum_{k=0}^{N} b_k \mathscr{F}\left\{\frac{d^k}{dt^k} y(t)\right\} = \sum_{k=0}^{M} a_k \mathscr{F}\left\{\frac{d^k}{dt^k} x(t)\right\}.$$

Using the time-differentiation property of the Fourier transform, we can re-express this as

$$\sum_{k=0}^{N} b_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} a_k (j\omega)^k X(\omega).$$

Then, factoring we have

$$Y(\omega) \sum_{k=0}^{N} b_k (j\omega)^k = X(\omega) \sum_{k=0}^{M} a_k (j\omega)^k.$$

Rearranging this equation, we find the frequency response $H(\omega)$ of the system to be

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k (j\omega)^k}{\sum_{k=0}^{N} b_k (j\omega)^k} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

Observe that, for a system of the form considered above, the frequency response is a rational function—hence, our interest in rational functions.

Example 5.29 (Resistors, inductors, and capacitors)**.** The basic building blocks of many electrical networks are resistors, inductors, and capacitors. The resistor, shown in Figure 5.15(a), is governed by the relationship

$$v(t) = Ri(t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad V(\omega) = RI(\omega)$$

where R, $v(t)$ and $i(t)$ denote the resistance of, voltage across, and current through the resistor, respectively. The inductor, shown in Figure 5.15(b), is governed by the relationship

$$v(t) = L\frac{d}{dt} i(t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad V(\omega) = j\omega L I(\omega)$$

or equivalently

$$i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad I(\omega) = \frac{1}{j\omega L} V(\omega)$$

where L, $v(t)$, and $i(t)$ denote the inductance of, voltage across, and current through the inductor, respectively. The capacitor, shown in Figure 5.15(c), is governed by the relationship

$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad V(\omega) = \frac{1}{j\omega C} I(\omega)$$

or equivalently

$$i(t) = C\frac{d}{dt} v(t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad I(\omega) = j\omega C V(\omega)$$

where C, $v(t)$, and $i(t)$ denote the capacitance of, voltage across, and current through the capacitor, respectively.
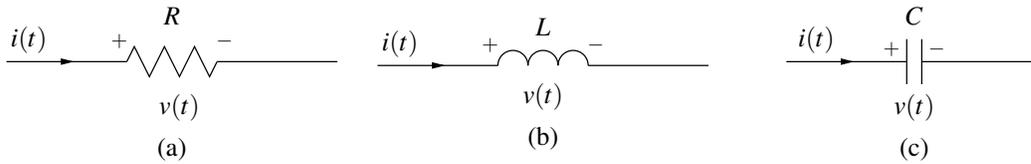
Figure 5.15: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

Example 5.30 (Simple RL network). Suppose that we have the RL network shown in Figure 5.16 with input $v_1(t)$ and output $v_2(t)$. This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response $H(\omega)$ of the system. (b) Find the response $v_2(t)$ of the system to the input $v_1(t) = \text{sgn}\, t$.

Solution. (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\tfrac{d}{dt}i(t) \quad \text{and} \tag{5.38}$$

$$v_2(t) = L\tfrac{d}{dt}i(t). \tag{5.39}$$

(Recall that the voltage $v(t)$ across an inductor L is related to the current $i(t)$ through the inductor as $v(t) = L\tfrac{d}{dt}i(t)$.) Taking the Fourier transform of (5.38) and (5.39) yields

$$V_1(\omega) = RI(\omega) + j\omega LI(\omega)$$
$$= (R + j\omega L)I(\omega) \quad \text{and} \tag{5.40}$$

$$V_2(\omega) = j\omega LI(\omega). \tag{5.41}$$

From (5.40) and (5.41), we have

$$\begin{aligned} H(\omega) &= \frac{V_2(\omega)}{V_1(\omega)} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \tag{5.42}$$

Thus, we have found the frequency response of the system.

(b) Now, suppose that $v_1(t) = \text{sgn}\, t$ (as given). Taking the Fourier transform of the input $v_1(t)$ (with the aid of Table 5.2), we have

$$V_1(\omega) = \frac{2}{j\omega}. \tag{5.43}$$

From the definition of the system, we know

$$V_2(\omega) = H(\omega)V_1(\omega). \tag{5.44}$$

Substituting (5.43) and (5.42) into (5.44), we obtain

$$\begin{aligned} V_2(\omega) &= \left(\frac{j\omega L}{R + j\omega L}\right)\left(\frac{2}{j\omega}\right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$
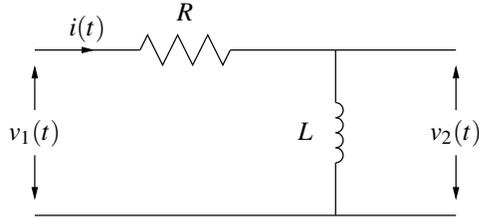
Figure 5.16: Simple RL network.

Taking the inverse Fourier transform of both sides of this equation, we obtain

$$v_2(t) = \mathscr{F}^{-1}\left\{\frac{2L}{R + j\omega L}\right\}$$

$$= \mathscr{F}^{-1}\left\{\frac{2}{R/L + j\omega}\right\}$$

$$= 2\mathscr{F}^{-1}\left\{\frac{1}{R/L + j\omega}\right\}.$$

Using Table 5.2, we can simplify to obtain

$$v_2(t) = 2e^{-(R/L)t}u(t).$$

Thus, we have found the response $v_2(t)$ to the input $v_1(t) = \operatorname{sgn} t$. □

5.12 Energy Spectral Density

Suppose that we have a signal $x(t)$ with finite energy E and Fourier transform $X(\omega)$. By definition, the energy contained in $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

We can use Parseval's relation (5.14) to express E in terms of $X(\omega)$ as

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

$$= \int_{-\infty}^{\infty} \tfrac{1}{2\pi} |X(\omega)|^2 \, d\omega.$$

Thus, the energy E is given by

$$E = \int_{-\infty}^{\infty} E_x(\omega) \, d\omega$$

where

$$E_x(\omega) = \tfrac{1}{2\pi} |X(\omega)|^2.$$

We refer to $E_x(\omega)$ as the **energy spectral density** of the signal $x(t)$. The quantity $E_x(\omega)$ indicates how the energy in the signal $x(t)$ is distributed as a function of frequency (in units of energy per rad/unit-time). For example, the energy contributed by frequency components in range $\omega_1 \leq \omega \leq \omega_2$ is simply given by

$$\int_{\omega_1}^{\omega_2} E_x(\omega) \, d\omega.$$

Example 5.31. Compute the energy spectral density $E_x(\omega)$ of the signal $x(t) = \operatorname{sinc} t/2$. Determine the amount of energy contained in the frequency components in the range $|\omega| \leq \frac{1}{4}$. Also, determine the total amount of energy in the signal.

Solution. First, we compute the Fourier transform $X(\omega)$ of $x(t)$. We obtain $X(\omega) = 2\pi \operatorname{rect} \omega$. Next, we find the energy spectral density function $E_x(\omega)$ as follows:

$$
\begin{aligned}
E_x(\omega) &= \tfrac{1}{2\pi} |X(\omega)|^2 \\
&= \tfrac{1}{2\pi} |2\pi \operatorname{rect} \omega|^2 \\
&= 2\pi \operatorname{rect}^2 \omega \\
&= 2\pi \operatorname{rect} \omega.
\end{aligned}
$$

Let E_1 denote the energy contained in the signal for frequencies $|\omega| \leq \frac{1}{4}$. Then, we have

$$
\begin{aligned}
E_1 &= \int_{-1/4}^{1/4} E_x(\omega) d\omega \\
&= \int_{-1/4}^{1/4} 2\pi \operatorname{rect} \omega \, d\omega \\
&= \int_{-1/4}^{1/4} 2\pi d\omega \\
&= \pi.
\end{aligned}
$$

Let E denote the total amount of energy in the signal. We can compute E as follows:

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} E_x(\omega) d\omega \\
&= \int_{-\infty}^{\infty} 2\pi \operatorname{rect} \omega \, d\omega \\
&= \int_{-1/2}^{1/2} 2\pi d\omega \\
&= 2\pi.
\end{aligned}
$$

$\square$

5.13   Power Spectral Density

Suppose that we have a signal $x(t)$ with finite power P and Fourier transform $X(\omega)$. Let us define $x_T(t)$ as the following truncated (or "windowed") version of $x(t)$:

$$
x_T(t) = x(t) \operatorname{rect}(t/T).
$$

Also, let $X_T(\omega)$ denote the Fourier transform of $x_T(t)$. From this definition of $x_T(t)$, we have that

$$
\lim_{T \to \infty} x_T(t) = x(t).
$$

The average power of the signal $x_T(t)$ is given by

$$
P = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt.
$$

Using Parseval's relation (5.14), we can rewrite this expression for P in terms of $X(\omega)$ to obtain

$$P = \lim_{T \to \infty} \frac{1}{T} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \right]$$

$$= \lim_{T \to \infty} \left[\frac{1}{2\pi T} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \right].$$

Interchanging the order of summation and integration, we obtain

$$P = \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{2\pi T} |X_T(\omega)|^2 d\omega].$$

Thus, we have that

$$P = \int_{-\infty}^{\infty} S_x(\omega) d\omega$$

where

$$S_x(\omega) = \lim_{T \to \infty} \frac{1}{2\pi T} |X_T(\omega)|^2.$$

We refer to $S_x(\omega)$ as the **power spectral density** of the signal $x(t)$. This quantity indicates how the power in the signal $x(t)$ is distributed as a function of frequency (in units of power per rad/unit-time). We can, therefore, determine the amount of power contained in frequency components over the range $\omega_1 \leq \omega \leq \omega_2$ as

$$\int_{\omega_1}^{\omega_2} S_x(\omega) d\omega.$$

5.14 Filtering

In some applications, we want to change the relative amplitude of the frequency components of a signal or possibly eliminate some frequency components altogether. This process of modifying the frequency components of a signal is referred to as **filtering**. Various types of filters exist. One type is frequency-selective filters. Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies. Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

An ideal lowpass filter eliminates all frequency components with a frequency greater than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 5.17(a).

The ideal highpass filter eliminates all frequency components with a frequency less than some cutoff frequency, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the cutoff frequency. A plot of this frequency response is given in Figure 5.17(b).

An ideal bandpass filter eliminates all frequency components that do not lie in its passband, while leaving the remaining frequency components unaffected. Such a filter has a frequency response of the form

$$H(\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A plot of this frequency response is given in Figure 5.17(c).

(a)

(b)

(c)
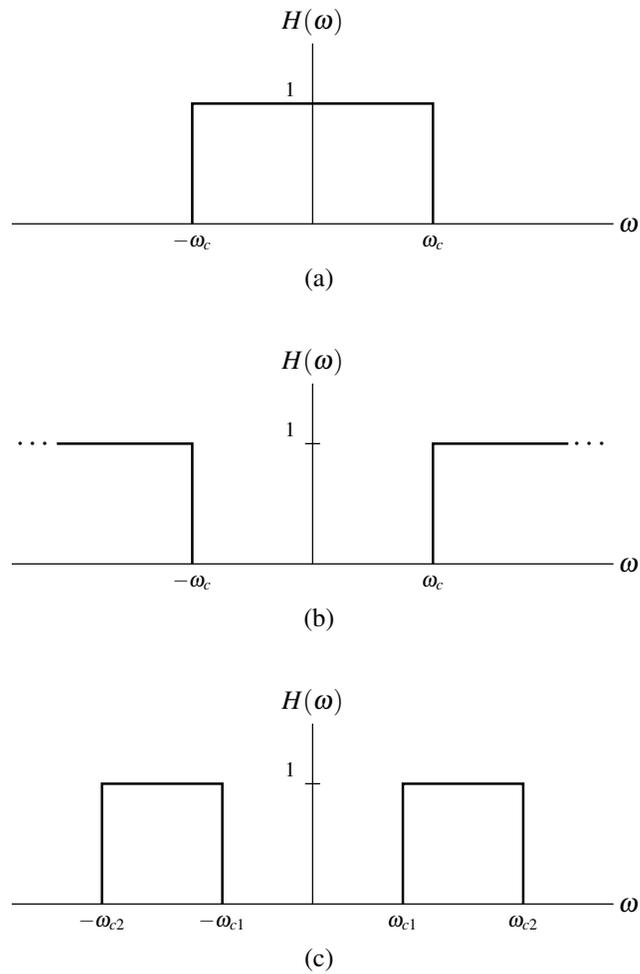
Figure 5.17: Frequency responses of (a) ideal lowpass, (b) ideal highpass, and (c) ideal bandpass filters.

Example 5.32 (Ideal filters). For each of the following impulse responses, find and plot the frequency response of the corresponding system:

$$h_{\text{LP}}(t) = \tfrac{\omega_c}{\pi} \operatorname{sinc} \omega_c t,$$
$$h_{\text{HP}}(t) = \delta(t) - \tfrac{\omega_c}{\pi} \operatorname{sinc} \omega_c t, \quad \text{and}$$
$$h_{\text{BP}}(t) = \tfrac{2\omega_b}{\pi} [\operatorname{sinc} \omega_b t] \cos \omega_a t,$$

where ω_c, ω_a, and ω_b are positive real constants. In each case, identify the type of frequency-selective filter to which the system corresponds.

Solution. In what follows, let us denote the input and output of the system as $x(t)$ and $y(t)$, respectively. Also, let $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively.

First, let us consider the system with impulse response $h_{\text{LP}}(t)$. The frequency response $H_{\text{LP}}(\omega)$ of the system is simply the Fourier transform of the impulse response $h_{\text{LP}}(t)$. Thus, we have

$$\begin{aligned}
H_{\text{LP}}(\omega) &= \mathscr{F}\{h_{\text{LP}}(t)\} \\
&= \mathscr{F}\{\tfrac{\omega_c}{\pi} \operatorname{sinc} \omega_c t\} \\
&= \tfrac{\omega_c}{\pi} \mathscr{F}\{\operatorname{sinc} \omega_c t\} \\
&= \tfrac{\omega_c}{\pi} \left[\tfrac{\pi}{\omega_c} \operatorname{rect} \tfrac{\omega}{2\omega_c} \right] \\
&= \operatorname{rect} \tfrac{\omega}{2\omega_c} \\
&= \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response $H_{\text{LP}}(\omega)$ is plotted in Figure 5.18(a). Since $Y(\omega) = H_{\text{LP}}(\omega)X(\omega)$ and $H_{\text{LP}}(\omega) = 0$ for $|\omega| > \omega_c$, $Y(\omega)$ will contain only those frequency components in $X(\omega)$ that lie in the frequency range $|\omega| \le \omega_c$. In other words, only the lower frequency components from $X(\omega)$ are kept. Thus, the system represents a lowpass filter.

Second, let us consider the system with impulse response $h_{\text{HP}}(t)$. The frequency response $H_{\text{HP}}(\omega)$ of the system is simply the Fourier transform of the impulse response $h_{\text{HP}}(t)$. Thus, we have

$$\begin{aligned}
H_{\text{HP}}(\omega) &= \mathscr{F}\{h_{\text{HP}}(t)\} \\
&= \mathscr{F}\{\delta(t) - \tfrac{\omega_c}{\pi} \operatorname{sinc} \omega_c t\} \\
&= \mathscr{F}\{\delta(t)\} - \tfrac{\omega_c}{\pi} \mathscr{F}\{\operatorname{sinc} \omega_c t\} \\
&= 1 - \tfrac{\omega_c}{\pi} \left[\tfrac{\pi}{\omega_c} \operatorname{rect} \tfrac{\omega}{2\omega_c} \right] \\
&= 1 - \operatorname{rect} \tfrac{\omega}{2\omega_c} \\
&= \begin{cases} 1 & \text{for } |\omega| \ge \omega_c \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The frequency response $H_{\text{HP}}(\omega)$ is plotted in Figure 5.18(b). Since $Y(\omega) = H_{\text{HP}}(\omega)X(\omega)$ and $H_{\text{HP}}(\omega) = 0$ for $|\omega| < \omega_c$, $Y(\omega)$ will contain only those frequency components in $X(\omega)$ that lie in the frequency range $|\omega| \ge \omega_c$. In other words, only the higher frequency components from $X(\omega)$ are kept. Thus, the system represents a highpass filter.

Third, let us consider the system with impulse response $h_{\text{BP}}(t)$. The frequency response $H_{\text{BP}}(\omega)$ of the system is

simply the Fourier transform of the impulse response $h_{BP}(t)$. Thus, we have

$$
\begin{aligned}
H_{BP}(\omega) &= \mathscr{F}\{h_{BP}(t)\} \\
&= \mathscr{F}\{\tfrac{2\omega_b}{\pi}[\operatorname{sinc}\omega_b t]\cos\omega_a t\} \\
&= \tfrac{\omega_b}{\pi}\mathscr{F}\{[\operatorname{sinc}\omega_b t](2\cos\omega_a t)\} \\
&= \tfrac{\omega_b}{\pi}\mathscr{F}\{[\operatorname{sinc}\omega_b t][e^{j\omega_a t}+e^{-j\omega_a t}]\} \\
&= \tfrac{\omega_b}{\pi}\left[\mathscr{F}\{e^{j\omega_a t}\operatorname{sinc}\omega_b t\}+\mathscr{F}\{e^{-j\omega_a t}\operatorname{sinc}\omega_b t\}\right] \\
&= \tfrac{\omega_b}{\pi}\left[\tfrac{\pi}{\omega_b}\operatorname{rect}\tfrac{\omega-\omega_a}{2\omega_b}+\tfrac{\pi}{\omega_b}\operatorname{rect}\tfrac{\omega+\omega_a}{2\omega_b}\right] \\
&= \operatorname{rect}\tfrac{\omega-\omega_a}{2\omega_b}+\operatorname{rect}\tfrac{\omega+\omega_a}{2\omega_b} \\
&= \begin{cases} 1 & \text{for } \omega_a-\omega_b \le |\omega| \le \omega_a+\omega_b \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

The frequency response $H_{BP}(\omega)$ is plotted in Figure 5.18(c). Since $Y(\omega)=H_{BP}(\omega)X(\omega)$ and $H_{BP}(\omega)=0$ for $|\omega|<\omega_a-\omega_b$ or $|\omega|>\omega_a+\omega_b$, $Y(\omega)$ will contain only those frequency components in $X(\omega)$ that lie in the frequency range $\omega_a-\omega_b \le |\omega| \le \omega_a+\omega_b$. In other words, only the middle frequency components of $X(\omega)$ are kept. Thus, the system represents a bandpass filter. $\square$

Example 5.33 (Lowpass filtering). Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, where

$$h(t) = 300\operatorname{sinc}300\pi t.$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t)=x_1(t)$, where

$$x_1(t) = \tfrac{1}{2}+\tfrac{3}{4}\cos 200\pi t+\tfrac{1}{2}\cos 400\pi t-\tfrac{1}{4}\cos 600\pi t.$$

Solution. To begin, we must find the frequency spectrum $X_1(\omega)$ of the signal $x_1(t)$. Computing $X_1(\omega)$, we have

$$
\begin{aligned}
X_1(\omega) &= \mathscr{F}\left\{\tfrac{1}{2}+\tfrac{3}{4}\cos 200\pi t+\tfrac{1}{2}\cos 400\pi t-\tfrac{1}{4}\cos 600\pi t\right\} \\
&= \tfrac{1}{2}\mathscr{F}\{1\}+\tfrac{3}{4}\mathscr{F}\{\cos 200\pi t\}+\tfrac{1}{2}\mathscr{F}\{\cos 400\pi t\}-\tfrac{1}{4}\mathscr{F}\{\cos 600\pi t\} \\
&= \tfrac{1}{2}[2\pi\delta(\omega)]+\tfrac{3\pi}{4}[\delta(\omega+200\pi)+\delta(\omega-200\pi)]+\tfrac{\pi}{2}[\delta(\omega+400\pi)+\delta(\omega-400\pi)] \\
&\quad -\tfrac{\pi}{4}[\delta(\omega+600\pi)+\delta(\omega-600\pi)] \\
&= -\tfrac{\pi}{4}\delta(\omega+600\pi)+\tfrac{\pi}{2}\delta(\omega+400\pi)+\tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi) \\
&\quad +\tfrac{\pi}{2}\delta(\omega-400\pi)-\tfrac{\pi}{4}\delta(\omega-600\pi).
\end{aligned}
$$

A plot of the frequency spectrum $X_1(\omega)$ is shown in Figure 5.19(a). Using the results of Example 5.32, we can determine the frequency response $H(\omega)$ of the system to be

$$
\begin{aligned}
H(\omega) &= \mathscr{F}\{300\operatorname{sinc}300\pi t\} \\
&= \operatorname{rect}\tfrac{\omega}{2(300\pi)} \\
&= \begin{cases} 1 & \text{for } |\omega| \le 300\pi \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

The frequency response $H(\omega)$ is shown in Figure 5.19(b). The frequency spectrum $Y(\omega)$ of the output can be computed as

$$
\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= \tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi).
\end{aligned}
$$
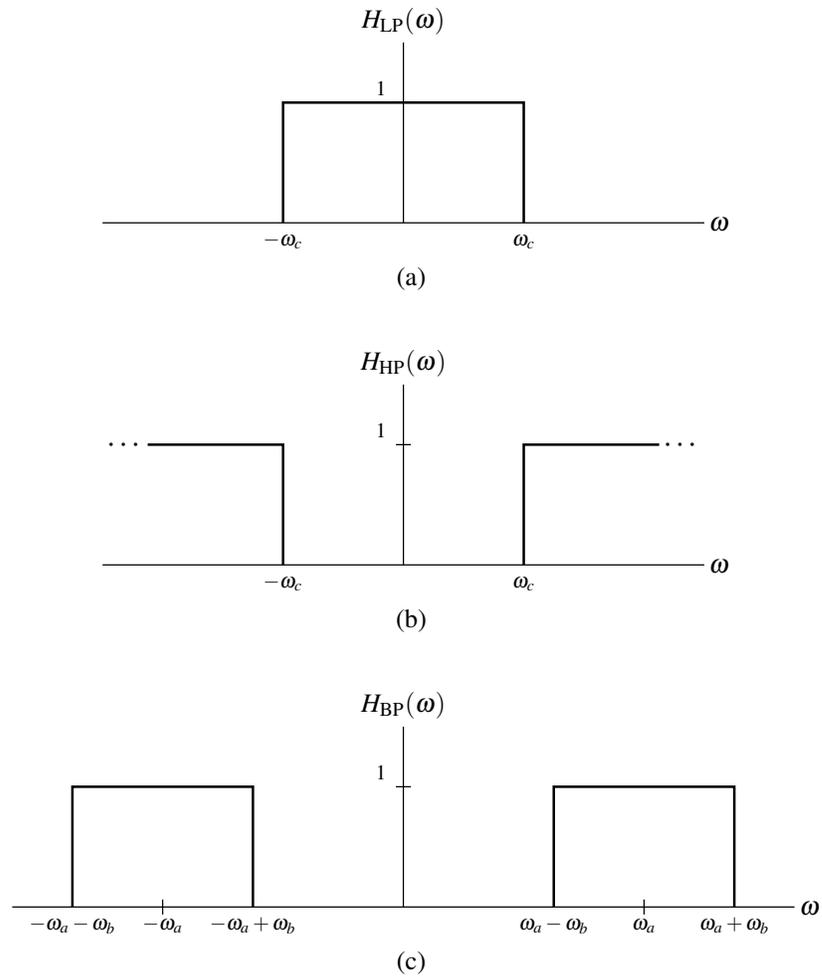
Figure 5.18: Frequency responses of systems from example.
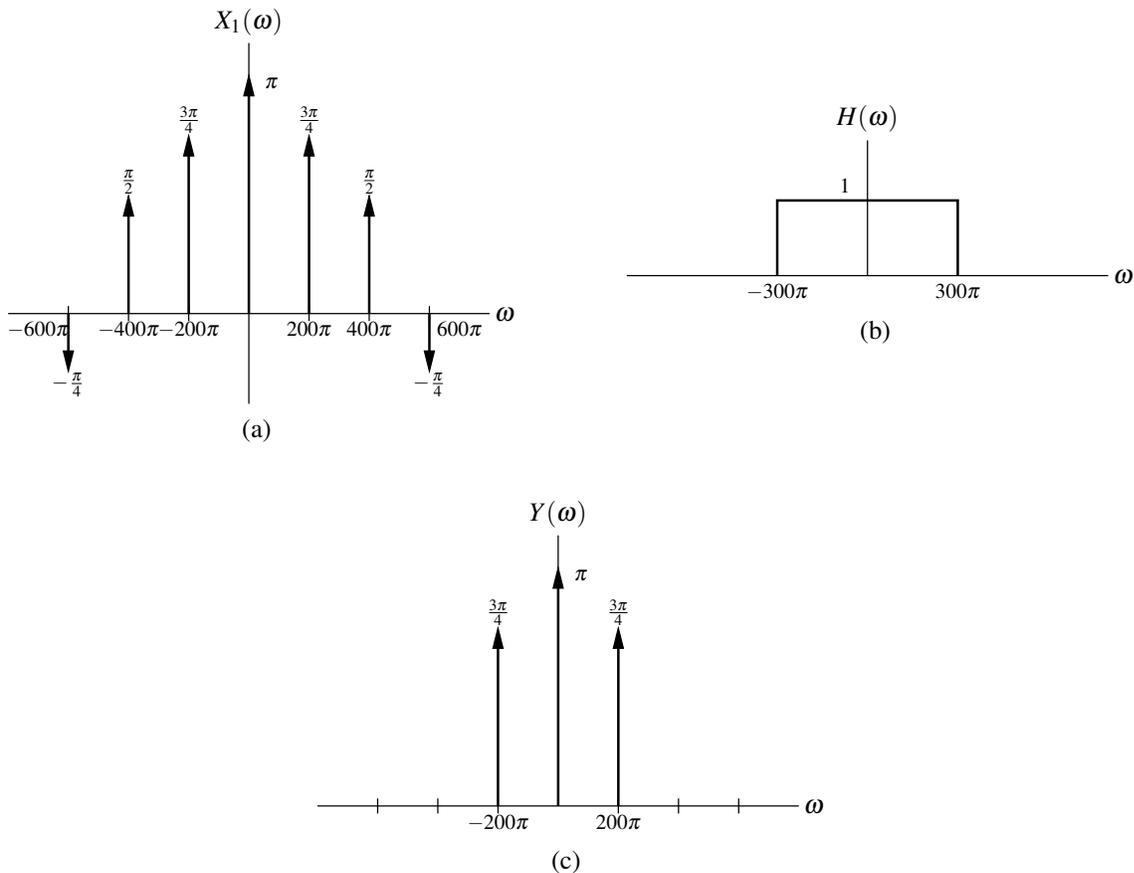
Figure 5.19: Frequency spectra for lowpass filtering example.

The frequency spectrum $Y(\omega)$ is shown in Figure 5.19(c). Taking the inverse Fourier transform of $Y(\omega)$ yields

$$
\begin{aligned}
y(t) &= \mathscr{F}^{-1}\left\{\tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi)\right\} \\
&= \pi\mathscr{F}^{-1}\{\delta(\omega)\}+\tfrac{3}{4}\mathscr{F}^{-1}\{\pi[\delta(\omega+200\pi)+\delta(\omega-200\pi)]\} \\
&= \pi(\tfrac{1}{2\pi})+\tfrac{3}{4}\cos 200\pi t \\
&= \tfrac{1}{2}+\tfrac{3}{4}\cos 200\pi t.
\end{aligned}
$$

$\square$

Example 5.34 (Bandpass filtering). Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$, where

$$
h(t) = (200\operatorname{sinc} 100\pi t)\cos 400\pi t.
$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t) = x_1(t)$ where $x_1(t)$ is as defined in Example 5.33.

Solution. From Example 5.33, we already know the frequency spectrum $X_1(\omega)$. In particular, we previously found that

$$
\begin{aligned}
X_1(\omega) = &-\tfrac{\pi}{4}\delta(\omega+600\pi)+\tfrac{\pi}{2}\delta(\omega+400\pi)+\tfrac{3\pi}{4}\delta(\omega+200\pi)+\pi\delta(\omega)+\tfrac{3\pi}{4}\delta(\omega-200\pi) \\
&+\tfrac{\pi}{2}\delta(\omega-400\pi)-\tfrac{\pi}{4}\delta(\omega-600\pi).
\end{aligned}
$$

The frequency spectrum $X_1(\omega)$ is shown in Figure 5.20(a). Now, we compute the frequency response $H(\omega)$ of the system. Using the results of Example 5.32, we can determine $H(\omega)$ to be

$$
\begin{aligned}
H(\omega) &= \mathscr{F}\{(200\,\text{sinc}\,100\pi t)\cos 400\pi t\} \\
&= \text{rect}\,\tfrac{\omega - 400\pi}{2(100\pi)} + \text{rect}\,\tfrac{\omega + 400\pi}{2(100\pi)} \\
&= \begin{cases} 1 & \text{for } 300\pi \le |\omega| \le 500\pi \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

The frequency response $H(\omega)$ is shown in Figure 5.20(b). The frequency spectrum $Y(\omega)$ of the output is given by

$$
\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= \tfrac{\pi}{2}\delta(\omega + 400\pi) + \tfrac{\pi}{2}\delta(\omega - 400\pi).
\end{aligned}
$$

Taking the inverse Fourier transform, we obtain

$$
\begin{aligned}
y(t) &= \mathscr{F}^{-1}\left\{\tfrac{\pi}{2}\delta(\omega + 400\pi) + \tfrac{\pi}{2}\delta(\omega - 400\pi)\right\} \\
&= \tfrac{1}{2}\mathscr{F}^{-1}\left\{\pi[\delta(\omega + 400\pi) + \delta(\omega - 400\pi)]\right\} \\
&= \tfrac{1}{2}\cos 400\pi t.
\end{aligned}
$$

$\square$

5.15 Sampling and Interpolation

Often, we encounter situations in which we would like to process a continuous-time signal in the discrete-time domain or vice versa. For example, we might have a continuous-time audio signal that we would like to process using a digital computer (which is a discrete-time system), or we might have a discrete-time audio signal that we wish to play on a loudspeaker (which is a continuous-time system). Clearly, some means is needed to link the continuous- and discrete-time domains. This connection is established through processes known as sampling and interpolation. In what follows, we will formally introduce these processes and study them in some detail.

Sampling allows us to create a discrete-time signal from a continuous-time signal. Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**. With this scheme, a sequence $y[n]$ of samples is obtained from a continuous-time signal $x(t)$ according to the relation

$$ y[n] = x(nT) \quad \text{for all integer } n, \tag{5.45} $$

where T is a positive real constant. As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = 2\pi/T$ as the (angular) **sampling frequency**. A system such as that described by (5.45) is known as an **ideal continuous-to-discrete-time (C/D) converter**, and is shown diagrammatically in Figure 5.21. An example of periodic sampling is shown in Figure 5.22. In Figure 5.22(a), we have the original continuous-time signal $x(t)$. This signal is then sampled with sampling period $T = 10$, yielding the sequence $y[n]$ in Figure 5.22(b).

Interpolation allows us to construct a continuous-time signal from a discrete-time signal. In effect, this process is one of assigning values to a signal between its sample points. Although there are many different ways in which to perform interpolation, we will focus our attention in subsequent sections on one particular scheme known as bandlimited interpolation. Interpolation produces a continuous-time signal $\hat{x}(t)$ from a sequence $y[n]$ according to the relation

$$ \hat{x}(t) = f(y[n]), $$

where f is some function of the sample values $y[n]$. The precise form of the function f depends on the particular interpolation scheme employed. The interpolation process is performed by a system known as an **ideal discrete-to-continuous-time (D/C) converter**, as shown in Figure 5.23.

(a)

(b)

(c)
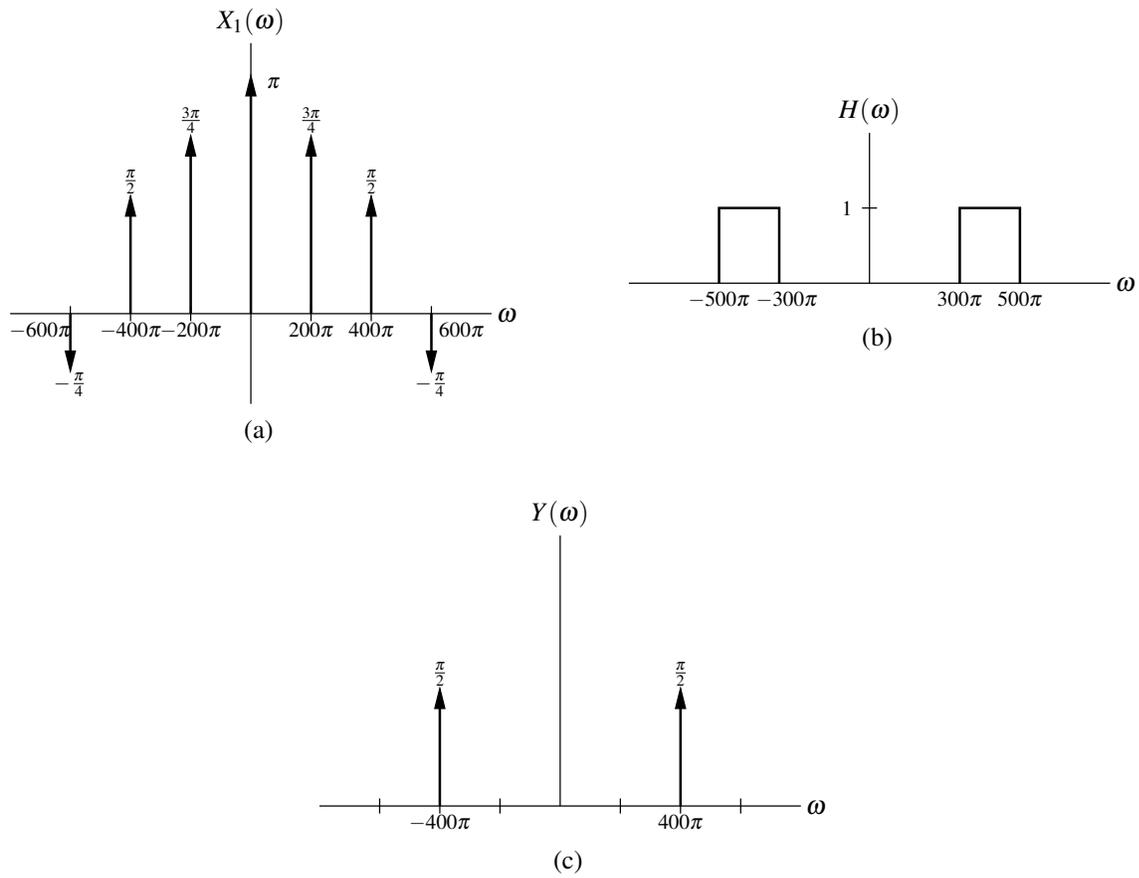
Figure 5.20: Frequency spectra for bandpass filtering example.
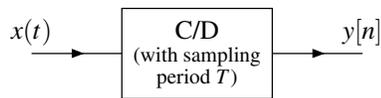
Figure 5.21: Ideal C/D converter.

(a)

(b)
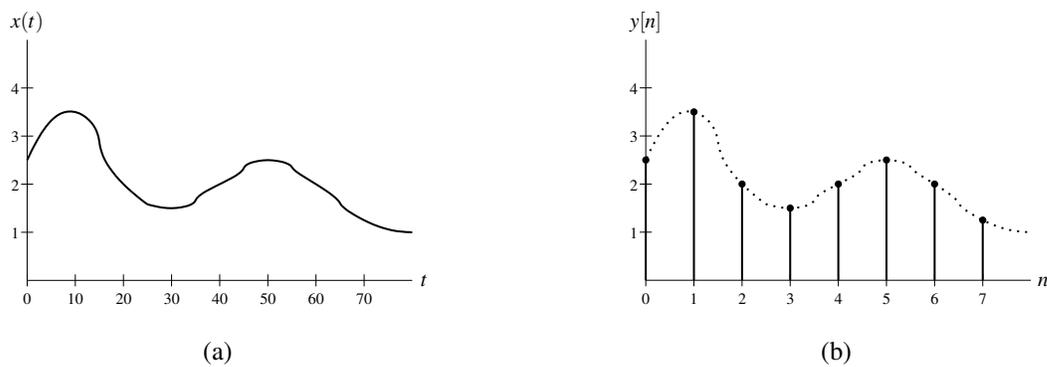
Figure 5.22: Periodic sampling of signal.
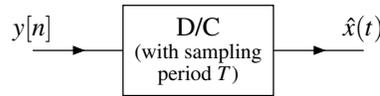
Figure 5.23: Ideal D/C converter.

In the absence of any constraints, a continuous-time signal cannot usually be uniquely determined from a sequence of its equally-spaced samples. In other words, the sampling process is not generally invertible. Consider, for example, the continuous-time signals $x_1(t)$ and $x_2(t)$ given by

$$x_1(t) = 0 \quad \text{and}$$
$$x_2(t) = \sin(2\pi t).$$

If we sample each of these signals with the sampling period $T = 1$, we obtain the respective sequences

$$y_1[n] = x_1(nT) = x_1(n) = 0 \quad \text{and}$$
$$y_2[n] = x_2(nT) = \sin(2\pi n) = 0.$$

Thus, $y_1[n] = y_2[n]$ for all n, although $x_1(t) \neq x_2(t)$ for all noninteger t. This example trivially shows that if no constraints are placed upon a continuous-time signal, then the signal cannot be uniquely determined from its samples.

Fortunately, under certain circumstances, a continuous-time signal can be recovered exactly from its samples. In particular, in the case that the signal being sampled is bandlimited, we can show that a sequence of its equally-spaced samples uniquely determines the signal if the sampling period is sufficiently small. This result, known as the sampling theorem, is of paramount importance in the study of signals and systems.

5.15.1 Sampling

In order to gain some insight into sampling, we need a way in which to mathematically model this process. To this end, we employ the simple model for the ideal C/D converter shown in Figure 5.24. In short, we may view the process of sampling as impulse train modulation followed by conversion of an impulse train to a sequence of sample values. More specifically, to sample a signal $x(t)$ with sampling period T, we first multiply the signal $x(t)$ by the periodic impulse train $p(t)$ to obtain

$$s(t) = x(t)p(t) \tag{5.46}$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

Then, we take the weights of successive impulses in $s(t)$ to form a sequence $y[n]$ of samples. The sampling frequency is given by $\omega_s = 2\pi/T$. As a matter of terminology, $p(t)$ is referred to as a sampling function. From the diagram, we can see that the signals $s(t)$ and $y[n]$, although very closely related, have some key differences. The impulse train $s(t)$ is a *continuous-time* signal that is zero everywhere except at integer multiples of T (i.e., at sample points), while $y[n]$ is a *discrete-time* signal, defined only on integers with its values corresponding to the weights of successive impulses in $s(t)$. The various signals involved in sampling are illustrated in Figure 5.25.

In passing, we note that the above model of sampling is only a mathematical convenience. That is, the model provides us with a relatively simple way in which to study the mathematical behavior of sampling. The above model, however, is not directly useful as a means for actually realizing sampling in a real world system. Obviously, the impulse train employed in the above model poses some insurmountable problems as far as implementation is concerned.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original signal $x(t)$ and its impulse-train sampled version $s(t)$. Since $p(t)$ is T-periodic, it can be represented in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \tag{5.47}$$
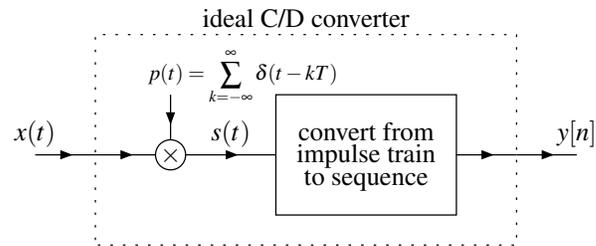
Figure 5.24: Model of ideal C/D converter.
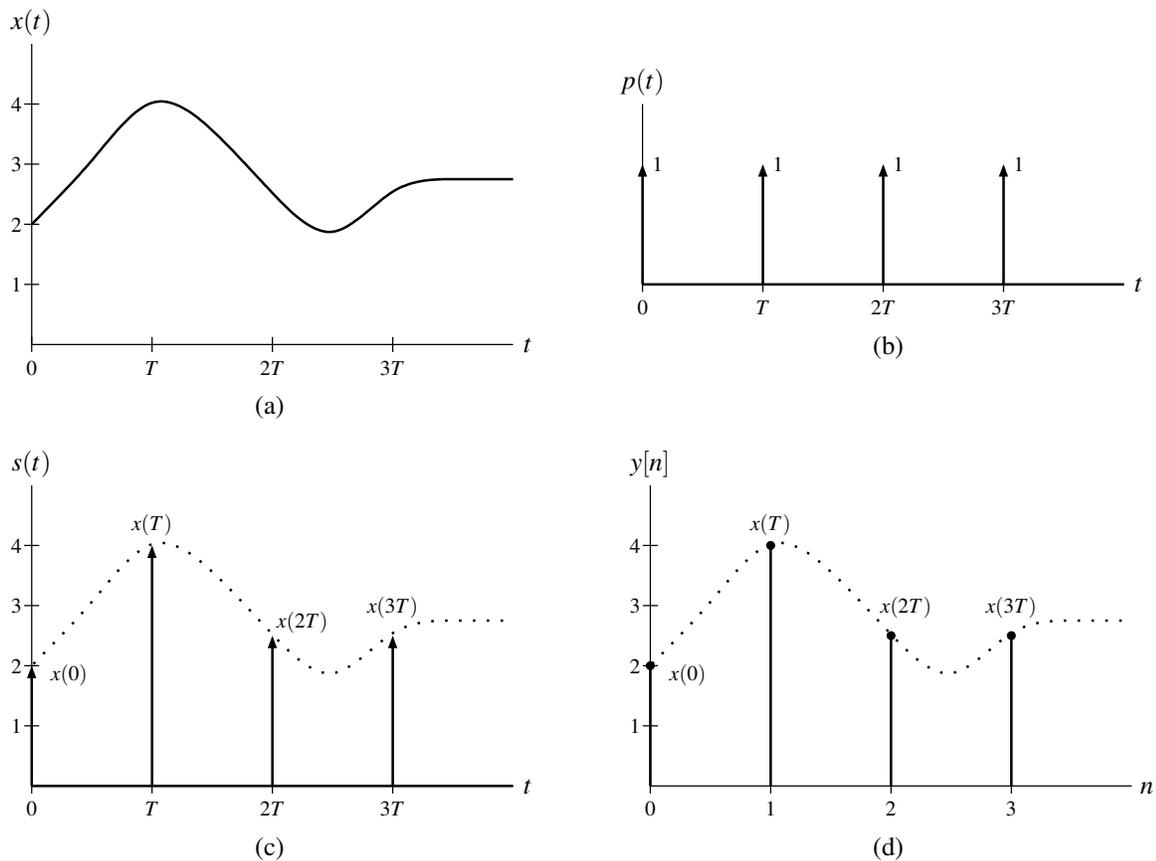
Figure 5.25: The various signals involved in the sampling process. (a) The original continuous-time signal $x(t)$. (b) The sampling function $p(t)$. (c) The impulse-modulated signal $s(t)$. (d) The discrete-time signal $y[n]$.

Using the Fourier series analysis equation, we calculate the coefficients c_k to be

$$
\begin{aligned}
c_k &= \tfrac{1}{T} \int_{-T/2}^{T/2} p(t)e^{-jk\omega_s t}\,dt \\
&= \tfrac{1}{T} \int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_s t}\,dt \\
&= \tfrac{1}{T} \\
&= \tfrac{\omega_s}{2\pi}.
\end{aligned} \tag{5.48}
$$

Substituting (5.47) and (5.48) into (5.46), we obtain

$$
\begin{aligned}
s(t) &= x(t) \sum_{k=-\infty}^{\infty} \tfrac{\omega_s}{2\pi} e^{jk\omega_s t} \\
&= \tfrac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}.
\end{aligned}
$$

Taking the Fourier transform of $s(t)$ yields

$$
S(\omega) = \tfrac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{5.49}
$$

Thus, the spectrum of the impulse-train sampled signal $s(t)$ is a scaled sum of an infinite number of shifted copies of the spectrum of the original signal $x(t)$.

Now, we consider a simple example to further illustrate the behavior of the sampling process in the frequency domain. Suppose that we have a signal $x(t)$ with the Fourier transform $X(\omega)$ where $|X(\omega)| = 0$ for $|\omega| > \omega_m$ (i.e., $x(t)$ is bandlimited). To simplify the visualization process, we will assume $X(\omega)$ has the particular form shown in Figure 5.26(a). In what follows, however, we only actually rely on the bandlimited nature of $x(t)$ and not the particular shape of $X(\omega)$. So, the results that we derive in what follows generally apply to any bandlimited signal. Let $S(\omega)$ denote the Fourier transform of $s(t)$. From (5.49), we know that $S(\omega)$ is formed by the superposition of an infinite number of shifted copies of $X(\omega)$. Upon more careful consideration, we can see that two distinct situations can arise. That is, the shifted copies of $X(\omega)$ used to form $S(\omega)$ can either: 1) overlap or, 2) not overlap. These two cases are illustrated in Figures 5.26(b) and 5.26(c), respectively. From these graphs, we can see that the shifted copies of $X(\omega)$ will not overlap if

$$
\omega_m < \omega_s - \omega_m \quad \text{and} \quad -\omega_m > -\omega_s + \omega_m
$$

or equivalently

$$
\omega_s > 2\omega_m.
$$

Consider the case in which the copies of the original spectrum $X(\omega)$ in $S(\omega)$ do not overlap, as depicted in Figure 5.26(b). In this situation, the spectrum $X(\omega)$ of the original signal is clearly discernable in the spectrum $S(\omega)$. In fact, one can see that the original spectrum $X(\omega)$ can be obtained directly from $S(\omega)$ through a lowpass filtering operation. Thus, the original signal $x(t)$ can be exactly recovered from $s(t)$.

Now, consider the case in which copies of the original spectrum $X(\omega)$ in $S(\omega)$ do overlap. In this situation, multiple frequencies in the spectrum $X(\omega)$ of the original signal are mapped to the same frequency in $S(\omega)$. This phenomenon is referred to as **aliasing**. Clearly, aliasing leads to individual periods of $S(\omega)$ having a different shape than the original spectrum $X(\omega)$. When aliasing occurs, the shape of the original spectrum $X(\omega)$ is no longer discernable from $S(\omega)$. Consequently, we are unable to recover the original signal $x(t)$ from $s(t)$ in this case.
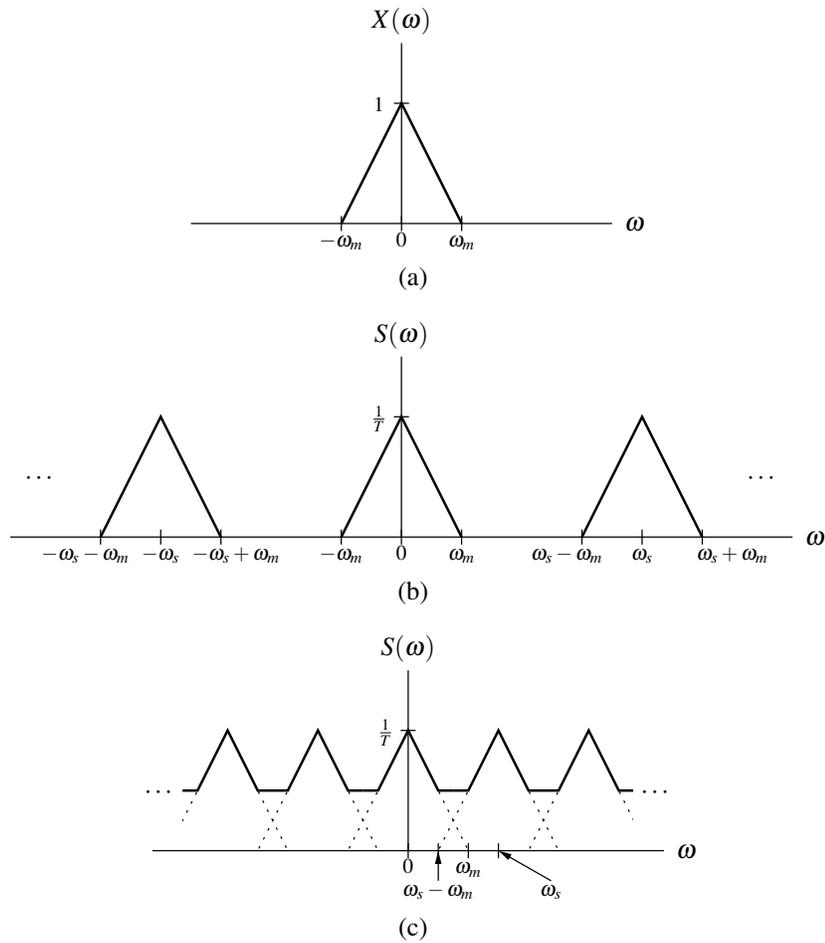
Figure 5.26: Effect of impulse-train sampling on frequency spectrum. (a) Spectrum of original signal $x(t)$. (b) Spectrum of $s(t)$ in the absence of aliasing. (c) Spectrum of $s(t)$ in the presence of aliasing.

ideal D/C converter

$y[n]$ → convert from sequence to impulse train → $s(t)$ → $H(\omega)$ → $\hat{x}(t)$

$H(\omega) = T \operatorname{rect}\left(\frac{T\omega}{2\pi}\right)$
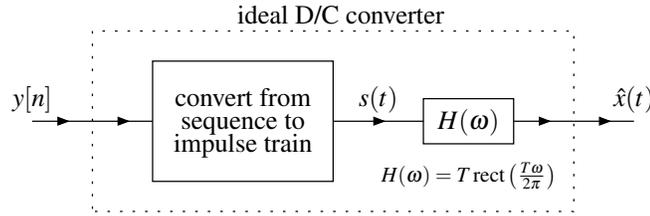
Figure 5.27: Model of ideal D/C converter.

5.15.2 Interpolation and Reconstruction of a Signal From Its Samples

Interpolation allows us to construct a continuous-time signal from a discrete-time one. This process is essentially responsible for determining the value of a continuous-time signal between sample points. Except in very special circumstances, it is not generally possible to exactly reproduce a continuous-time signal from its samples. Although many interpolation schemes exist, we shall focus our attention shortly on one particular scheme. The interpolation process can be modeled with the simple ideal D/C converter system, shown in Figure 5.27.

Recall the ideal C/D converter of Figure 5.24. Since the process of converting an impulse train to a sequence is invertible, we can reconstruct the original signal $x(t)$ from its sampled version $y[n]$ if we can somehow recover $x(t)$ from $s(t)$. Let us suppose now that $x(t)$ is bandlimited. As we saw in the previous section, we can recover $x(t)$ from $s(t)$ provided that $x(t)$ is bandlimited and sampled at a sufficiently high rate so as to avoid aliasing. In the case that aliasing does not occur, we can reconstruct the original continuous-time signal $x(t)$ from $y[n]$ using the ideal D/C converter shown in Figure 5.27. In what follows, we will derive a formula for computing the original continuous-time signal $\hat{x}(t)$ from its samples $y[n]$. Consider the model of the D/C converter. We have a lowpass filter with frequency response

$$H(\omega) = T \operatorname{rect}\left(\frac{T\omega}{2\pi}\right) = \frac{2\pi}{\omega_s} \operatorname{rect}\left(\frac{\omega}{\omega_s}\right) = \begin{cases} T & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

and impulse response

$$h(t) = \operatorname{sinc}\left(\frac{\pi t}{T}\right) = \operatorname{sinc}\left(\frac{\omega_s t}{2}\right).$$

First, we convert the sequence $y[n]$ to the impulse train $s(t)$ to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y[n]\delta(t-nT).$$

Then, we filter the resulting signal $s(t)$ with the lowpass filter having impulse response $h(t)$, yielding

$$\begin{aligned} \hat{x}(t) &= s(t) * h(t) \\ &= \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(t-\tau) \sum_{n=-\infty}^{\infty} y[n]\delta(\tau-nT)d\tau \\ &= \sum_{n=-\infty}^{\infty} y[n] \int_{-\infty}^{\infty} h(t-\tau)\delta(\tau-nT)d\tau \\ &= \sum_{n=-\infty}^{\infty} y[n]h(t-nT) \\ &= \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\pi}{T}(t-nT)). \end{aligned}$$

If $x(t)$ is bandlimited and aliasing is avoided, $\hat{x}(t) = x(t)$ and we have a formula for exactly reproducing $x(t)$ from its samples $y[n]$.

5.15.3   Sampling Theorem

In the preceding sections, we have established the important result given by the theorem below.

Theorem 5.3 (Sampling Theorem). *Let $x(t)$ be a signal with Fourier transform $X(\omega)$, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., $x(t)$ is bandlimited to the interval $[-\omega_M, \omega_M]$). Then, $x(t)$ is uniquely determined by its samples $y[n] = x(nT)$ for $n = 0, \pm 1, \pm 2, \ldots$, if*

$$\omega_s > 2\omega_M, \tag{5.50}$$

where $\omega_s = 2\pi/T$. In particular, if (5.50) is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\pi}{T}(t - nT)),$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y[n] \operatorname{sinc}(\tfrac{\omega_s}{2}t - \pi n).$$

As a matter of terminology, we refer to (5.50) as the **Nyquist condition** (or Nyquist criterion). Also, we call $\omega_s/2$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**. It is important to note that the Nyquist condition is a strict inequality. Therefore, to ensure aliasing does not occur in the most general case, one must choose the sampling rate larger than the Nyquist rate. One can show, however, that if the frequency spectrum does not have impulses at the Nyquist frequency, it is sufficient to sample at exactly the Nyquist rate.

Example 5.35. Let $x(t)$ denote a continuous-time audio signal with Fourier transform $X(\omega)$. Suppose that $|X(\omega)| = 0$ for all $|\omega| \geq 44100\pi$. Determine the largest period T with which $x(t)$ can be sampled that will allow $x(t)$ to be exactly recovered from its samples.

Solution. The signal $x(t)$ is bandlimited to frequencies in the range $(-\omega_m, \omega_m)$, where $\omega_m = 44100\pi$. From the sampling theorem, we know that the minimum sampling rate required is given by

$$\begin{aligned} \omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi. \end{aligned}$$

Thus, the largest permissible sampling period is given by

$$\begin{aligned} T &= \tfrac{2\pi}{\omega_s} \\ &= \tfrac{2\pi}{88200\pi} \\ &= \tfrac{1}{44100}. \end{aligned}$$

$\square$

Although the sampling theorem provides an upper bound on the sampling rate that holds in the case of arbitrary bandlimited signals, in some special cases it may be possible to employ an even smaller sampling rate. This point is further illustrated by way of the example below.

Example 5.36. Suppose that we have a signal $x(t)$ with the Fourier transform $X(\omega)$ shown in Figure 5.28 (where $\omega_c \gg \omega_a$). (a) Using the sampling theorem directly, determine the largest permissible sampling period T that will allow $x(t)$ to be exactly reconstructed from its samples. (b) Explain how one can exploit the fact that $X(\omega) = 0$ for a large portion of the interval $[-\omega_c - \omega_a, \omega_c + \omega_a]$ in order to reduce the rate at which $x(t)$ must be sampled.

Solution. (a) The signal $x(t)$ is bandlimited to $(-\omega_m, \omega_m)$, where $\omega_m = \omega_c + \omega_a$. Thus, the minimum sampling rate required is given by

$$\omega_s = 2\omega_m$$
$$= 2(\omega_c + \omega_a)$$
$$= 2\omega_c + 2\omega_a.$$

and the maximum sampling period is calculated as

$$T = \frac{2\pi}{\omega_s}$$
$$= \frac{2\pi}{2\omega_c + 2\omega_a}$$
$$= \frac{\pi}{\omega_c + \omega_a}.$$

(b) We can modulate and lowpass filter $x(t)$ in order to compress all of its spectral information into the frequency range $[-2\omega_a, 2\omega_a]$, yielding the signal $x_1(t)$. That is, we have

$$x_1(t) = [x(t)\cos([\omega_c - \omega_a]t)] * h(t)$$

where

$$h(t) = \frac{4\omega_a}{\pi}\operatorname{sinc}(2\omega_a t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad H(\omega) = 2\operatorname{rect}\left(\frac{\omega}{4\omega_a}\right).$$

This process can be inverted (by modulation and filtering) to obtain $x(t)$ from $x_1(t)$. In particular, we have that

$$x(t) = (x_1(t)\cos([\omega_c - \omega_a]t)) * h_2(t)$$

where

$$h_2(t) = \delta(t) - \frac{2(\omega_c - \omega_a)}{\pi}\operatorname{sinc}([\omega_c - \omega_a]t) \quad \overset{\mathscr{F}}{\longleftrightarrow} \quad H_2(\omega) = 2 - 2\operatorname{rect}\left(\frac{\omega}{4(\omega_c - \omega_a)}\right).$$

Let $X_1(\omega)$ denote the Fourier transform of $x_1(t)$. The Fourier transform $X_1(\omega)$ is as shown in Figure 5.29 Applying the sampling theorem to $x_1(t)$ we find that the minimum sampling rate is given by

$$\omega_s = 2(2\omega_a)$$
$$= 4\omega_a$$

and the largest sampling period is given by

$$T = \frac{2\pi}{\omega_s}$$
$$= \frac{2\pi}{4\omega_a}$$
$$= \frac{\pi}{2\omega_a}.$$

Since $\omega_c \gg \omega_a$ (by assumption), this new sampling period is larger than the one computed in the first part of this problem. $\qquad\square$

5.16 Amplitude Modulation

In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal. For example, voice/audio signals typically have information in the range of 0 to 20 kHz. Often, it is not practical to transmit such a signal using its original frequency range. Two potential problems with such an approach
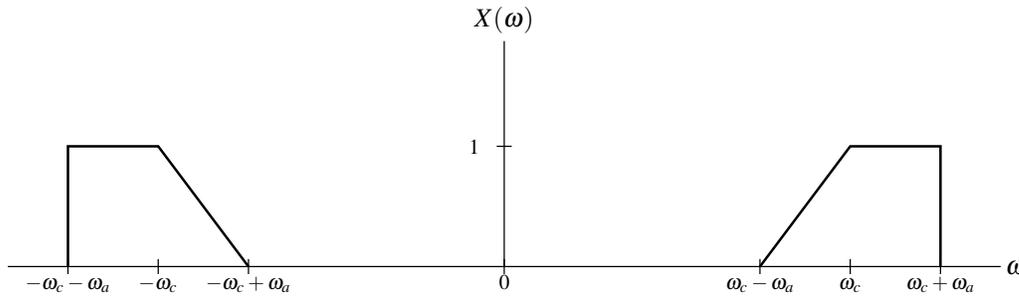
Figure 5.28: Frequency spectrum of signal $x(t)$.
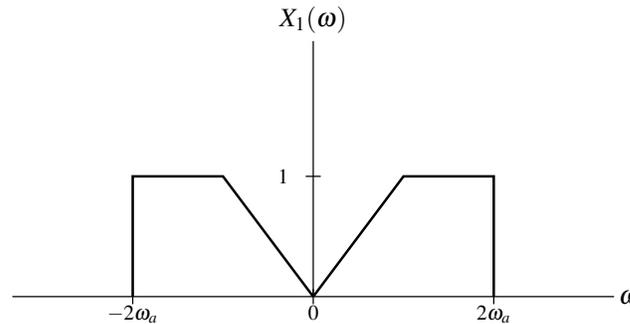
Figure 5.29: Frequency spectrum of signal $x_1(t)$.

are: 1) interference and 2) constraints on antenna length. Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference. Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals. For the preceding reasons, we often need to change the frequency range associated with a signal before transmission. In what follows, we consider one possible scheme for accomplishing this. This scheme is known as amplitude modulation.

Amplitude modulation (AM) is used in many communication systems. Numerous variations on amplitude modulation are possible. Here, we consider two of the simplest variations: double-side-band/suppressed-carrier (DSB/SC) and single-side-band/suppressed-carrier (SSB/SC).

5.16.1 Modulation With a Complex Sinusoid

Suppose that we have the communication system shown in Figure 5.30. First, let us consider the transmitter in Figure 5.30(a). The transmitter is a system with input $x(t)$ and output $y(t)$. Mathematically, the behavior of this system is given by

$$y(t) = x(t)c_1(t) \tag{5.51}$$

where

$$c_1(t) = e^{j\omega_c t}.$$

Let $X(\omega)$, $Y(\omega)$, and $C_1(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, and $c_1(t)$, respectively. Taking the Fourier transform of both sides of (5.51), we obtain

$$
\begin{aligned}
Y(\omega) &= \mathscr{F}\{c_1(t)x(t)\} \\
&= \mathscr{F}\{e^{j\omega_c t}x(t)\} \\
&= X(\omega - \omega_c). \tag{5.52}
\end{aligned}
$$
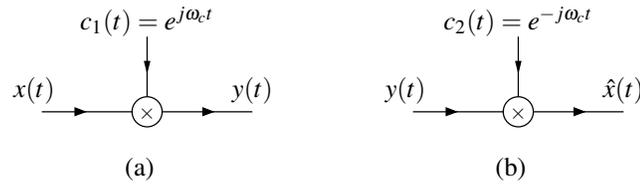
Figure 5.30: Simple communication system. (a) Transmitter and (b) receiver.

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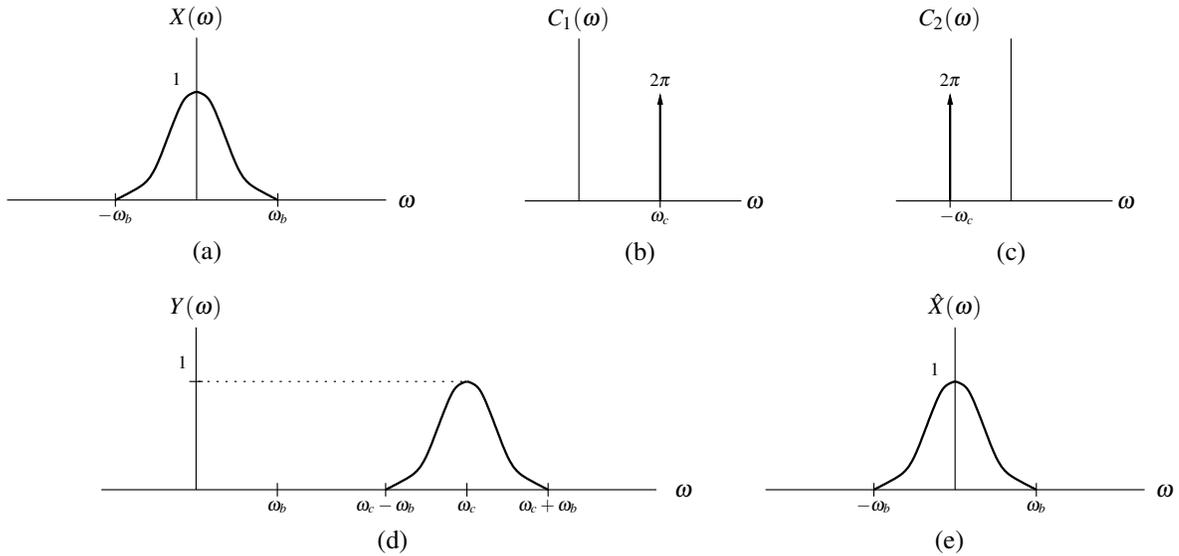
Figure 5.31: Frequency spectra for modulation with a complex sinusoid.

.

Thus, the frequency spectrum of the output is simply the frequency spectrum of the input shifted by ω_c. The relationship between the frequency spectra of the input and output is illustrated in Figure 5.31. Clearly, the output signal has been shifted to a different frequency range as desired. Now, we need to determine whether the receiver can recover the original signal $x(t)$ from the transmitted signal $y(t)$.

Now, let us consider the receiver shown in Figure 5.30(b). The receiver is a system with input $y(t)$ and output $\hat{x}(t)$. Mathematically, this system is given by

$$\hat{x}(t) = y(t)c_2(t) \tag{5.53}$$

where

$$c_2(t) = e^{-j\omega_c t}$$

(i.e., $c_2(t) = c_1^*(t)$). In order for the communication system to be useful, we need for the received signal $\hat{x}(t)$ to be equal to the original signal $x(t)$ from the transmitter. Let $Y(\omega)$, $\hat{X}(\omega)$, and $C_2(\omega)$ denote the Fourier transform of $y(t)$, $\hat{x}(t)$, and $c_2(t)$, respectively. Taking the Fourier transform of both sides of (5.53), we obtain

$$\hat{X}(\omega) = \mathscr{F}\{c_2(t)y(t)\}$$
$$= \mathscr{F}\{e^{-j\omega_c t}y(t)\}$$
$$= Y(\omega + \omega_c).$$

Substituting the expression for $Y(\omega)$ in (5.52) into this equation, we obtain

$$\hat{X}(\omega) = X([\omega + \omega_c] - \omega_c)$$
$$= X(\omega).$$

Since $\hat{X}(\omega) = X(\omega)$, we have that the received signal $\hat{x}(t)$ is equal to the original signal $x(t)$ from the transmitter. Thus, the communication system has the desired behavior. The relationship between the frequency spectra of the various signals in the AM system is illustrated in Figure 5.31.

Although the above result is quite interesting mathematically, it does not have direct practical application. The difficulty here is that $c_1(t)$, $c_2(t)$, and $y(t)$ are complex signals, and we cannot realize complex signals in the physical world. This communication system is not completely without value, however, as it leads to the development of the practically useful system that we consider next.

5.16.2   DSB/SC Amplitude Modulation

Now, let us consider the communication system shown in Figure 5.32. This system is known as a double-side-band/suppressed-carrier (DSB/SC) amplitude modulation (AM) system. This system is very similar to the one in Figure 5.30. In the new system, however, we have replaced the complex sinusoid $c_1(t)$ with a real sinusoid $c(t)$. The new system also requires that the input signal $x(t)$ be bandlimited to frequencies in the interval $[-\omega_b, \omega_b]$ and that $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$. The reasons for this restriction will become clear after having studied this system in more detail.

Consider the transmitter shown in Figure 5.32(a). The transmitter is a system with input $x(t)$ and output $y(t)$. Mathematically, we can describe the behavior of the system as

$$y(t) = x(t)c(t) \tag{5.54}$$

where $c(t) = \cos \omega_c t$. (Note that we can rewrite $c(t)$ as $c(t) = \frac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]$.) Taking the Fourier transform of both sides of (5.54), we obtain

$$\begin{aligned} Y(\omega) &= \mathscr{F}\{x(t)c(t)\} \\ &= \mathscr{F}\left\{ \tfrac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]x(t) \right\} \\ &= \tfrac{1}{2}\left[\mathscr{F}\{e^{j\omega_c t}x(t)\} + \mathscr{F}\{e^{-j\omega_c t}x(t)\} \right] \\ &= \tfrac{1}{2}[X(\omega - \omega_c) + X(\omega + \omega_c)]. \end{aligned} \tag{5.55}$$

Thus, the frequency spectrum of the output is the average of two shifted versions of the frequency spectrum of the input. The relationship between the frequency spectra of the input and output is illustrated in Figure 5.33(d). Observe that we have managed to shift the frequency spectrum of the input signal into a different range of frequencies for transmission as desired. Now, we must determine whether the receiver can recover the original signal $x(t)$.

Consider the receiver shown in Figure 5.32(b). The receiver is a system with input $y(t)$ and output $\hat{x}(t)$. Let $Y(\omega)$, $V(\omega)$ and $\hat{X}(\omega)$ denote the Fourier transforms of $y(t)$, $v(t)$ and $\hat{x}(t)$, respectively. Then, the input-output behavior of the system is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and} \tag{5.56}$$
$$\hat{X}(\omega) = H(\omega)V(\omega) \tag{5.57}$$

where

$$H(\omega) = \begin{cases} 2 & \text{for } |\omega| \le \omega_{c0} \\ 0 & \text{otherwise} \end{cases}$$

and $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$. Taking the Fourier transform of both sides of (5.56) yields

$$
\begin{aligned}
V(\omega) &= \mathscr{F}\{c(t)y(t)\} \\
&= \mathscr{F}\left\{\tfrac{1}{2}\left[e^{j\omega_c t} + e^{-j\omega_c t}\right]y(t)\right\} \\
&= \tfrac{1}{2}\left[\mathscr{F}\left\{e^{j\omega_c t}y(t)\right\} + \mathscr{F}\left\{e^{-j\omega_c t}y(t)\right\}\right] \\
&= \tfrac{1}{2}[Y(\omega - \omega_c) + Y(\omega + \omega_c)].
\end{aligned}
$$

Substituting the expression for $Y(\omega)$ in (5.55) into this equation, we obtain

$$
\begin{aligned}
V(\omega) &= \tfrac{1}{2}\left[\tfrac{1}{2}\left[X([\omega - \omega_c] - \omega_c) + X([\omega - \omega_c] + \omega_c)\right] + \tfrac{1}{2}\left[X([\omega + \omega_c] - \omega_c) + X([\omega + \omega_c] + \omega_c)\right]\right] \\
&= \tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega - 2\omega_c) + \tfrac{1}{4}X(\omega + 2\omega_c).
\end{aligned}
\tag{5.58}
$$

The relationship between $V(\omega)$ and $X(\omega)$ is depicted graphically in Figure 5.33(e). Substituting the above expression for $V(\omega)$ into (5.57) and simplifying, we obtain

$$
\begin{aligned}
\hat{X}(\omega) &= H(\omega)V(\omega) \\
&= H(\omega)\left[\tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega - 2\omega_c) + \tfrac{1}{4}X(\omega + 2\omega_c)\right] \\
&= \tfrac{1}{2}H(\omega)X(\omega) + \tfrac{1}{4}H(\omega)X(\omega - 2\omega_c) + \tfrac{1}{4}H(\omega)X(\omega + 2\omega_c) \\
&= \tfrac{1}{2}\left[2X(\omega)\right] + \tfrac{1}{4}(0) + \tfrac{1}{4}(0) \\
&= X(\omega).
\end{aligned}
$$

(In the above simplification, since $H(\omega) = 2\operatorname{rect}(\frac{\omega}{2\omega_{c0}})$ and $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$, we were able to deduce that $H(\omega)X(\omega) = 2X(\omega)$, $H(\omega)X(\omega - 2\omega_c) = 0$, and $H(\omega)X(\omega + 2\omega_c) = 0$.) The relationship between $\hat{X}(\omega)$ and $X(\omega)$ is depicted in Figure 5.33(f). Thus, we have that $\hat{X}(\omega) = X(\omega)$ which implies $\hat{x}(t) = x(t)$. So, we have recovered the original signal $x(t)$ at the receiver. This system has managed to shift $x(t)$ into a different frequency range before transmission and then recover $x(t)$ at the receiver. This is exactly what we wanted to accomplish.

5.16.3 SSB/SC Amplitude Modulation

By making a minor modification to the DSB/SC amplitude modulation system, we can reduce the bandwidth requirements of the system by half. The resulting system is referred to as a SSB/SC amplitude modulation system. This modified system is illustrated in Figure 5.34. In this new system, $G(\omega)$ and $H(\omega)$ are given by

$$
G(\omega) = \begin{cases} 1 & \text{for } |\omega| \ge \omega_c \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}
$$

$$
H(\omega) = \begin{cases} 4 & \text{for } |\omega| \le \omega_{c0} \\ 0 & \text{otherwise.} \end{cases}
$$

Let $X(\omega)$, $Y(\omega)$, $Q(\omega)$, $V(\omega)$, $\hat{X}(\omega)$, and $C(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, $q(t)$, $v(t)$, $\hat{x}(t)$, and $c(t)$, respectively. Figure 5.35 depicts the transformations the signal undergoes as it passes through the system. Again, the output from the receiver is equal to the input to the transmitter.

5.17 Equalization

Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand. As a result, we would like to change the frequency response of the system to be something more desirable. This process of modifying the frequency response in this way is referred to as
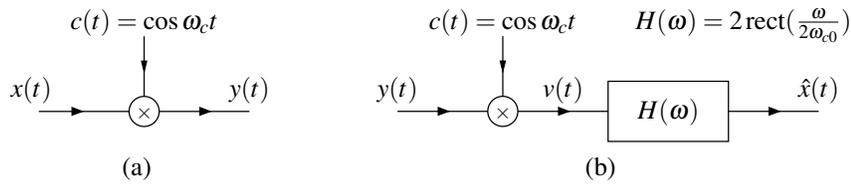
Figure 5.32: DSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.
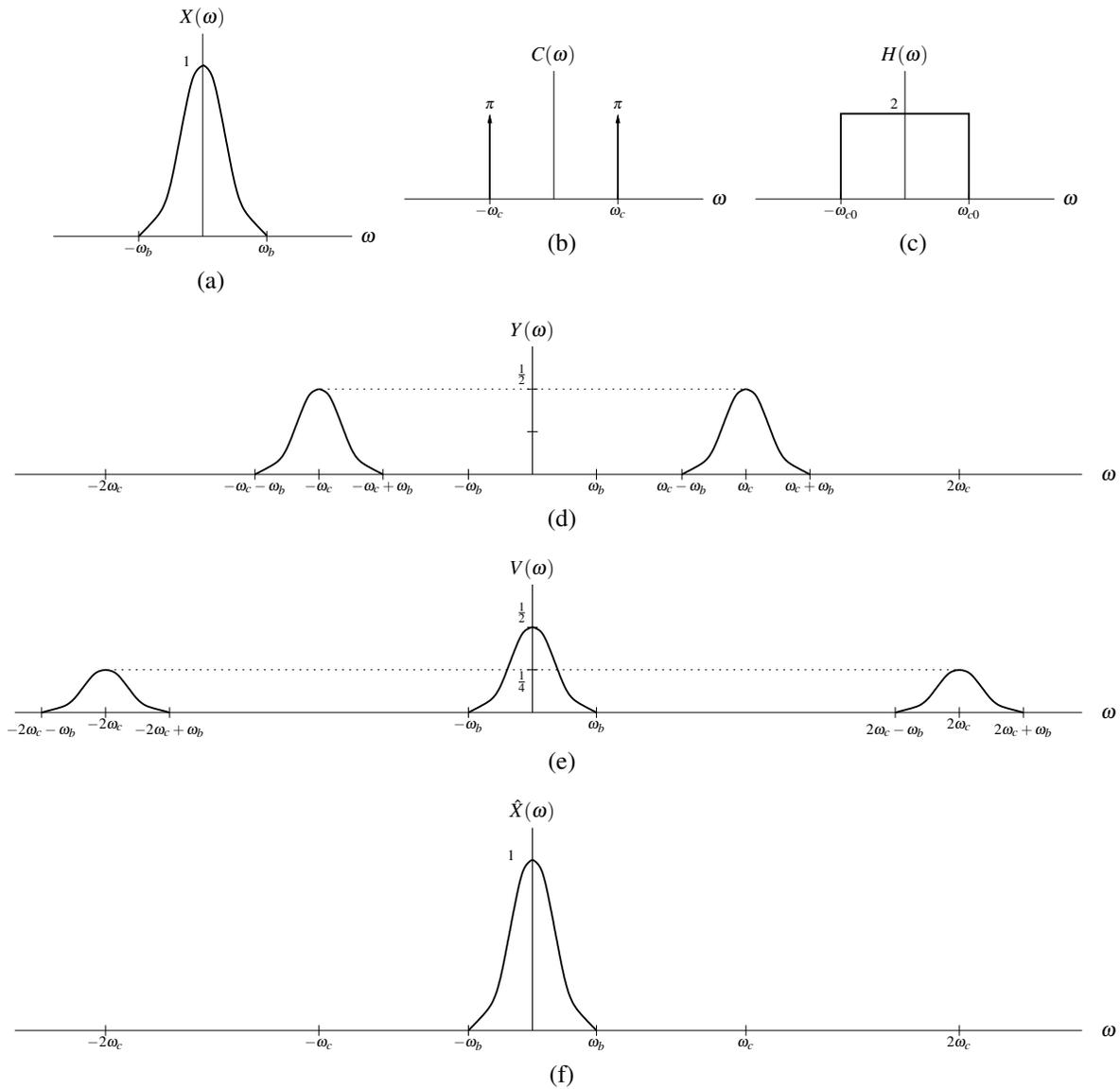
Figure 5.33: Signal spectra for DSB/SC amplitude modulation.

$c(t) = \cos \omega_c t \quad G(\omega) = 1 - \text{rect}(\frac{\omega}{2\omega_c})$

$x(t) \quad \times \quad q(t) \quad \boxed{G(\omega)} \quad y(t)$

(a)

$c(t) = \cos \omega_c t \quad H(\omega) = 4\,\text{rect}(\frac{\omega}{2\omega_{c0}})$

$y(t) \quad \times \quad v(t) \quad \boxed{H(\omega)} \quad \hat{x}(t)$

(b)
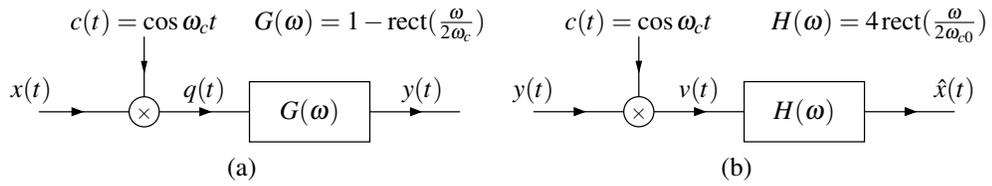
Figure 5.34: SSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

$X(\omega)$

(a)

$C(\omega)$

(b)

$G(\omega)$

(c)

$H(\omega)$

(d)

$Q(\omega)$

(e)

$Y(\omega)$

(f)

$V(\omega)$

(g)

$\hat{X}(\omega)$

(h)
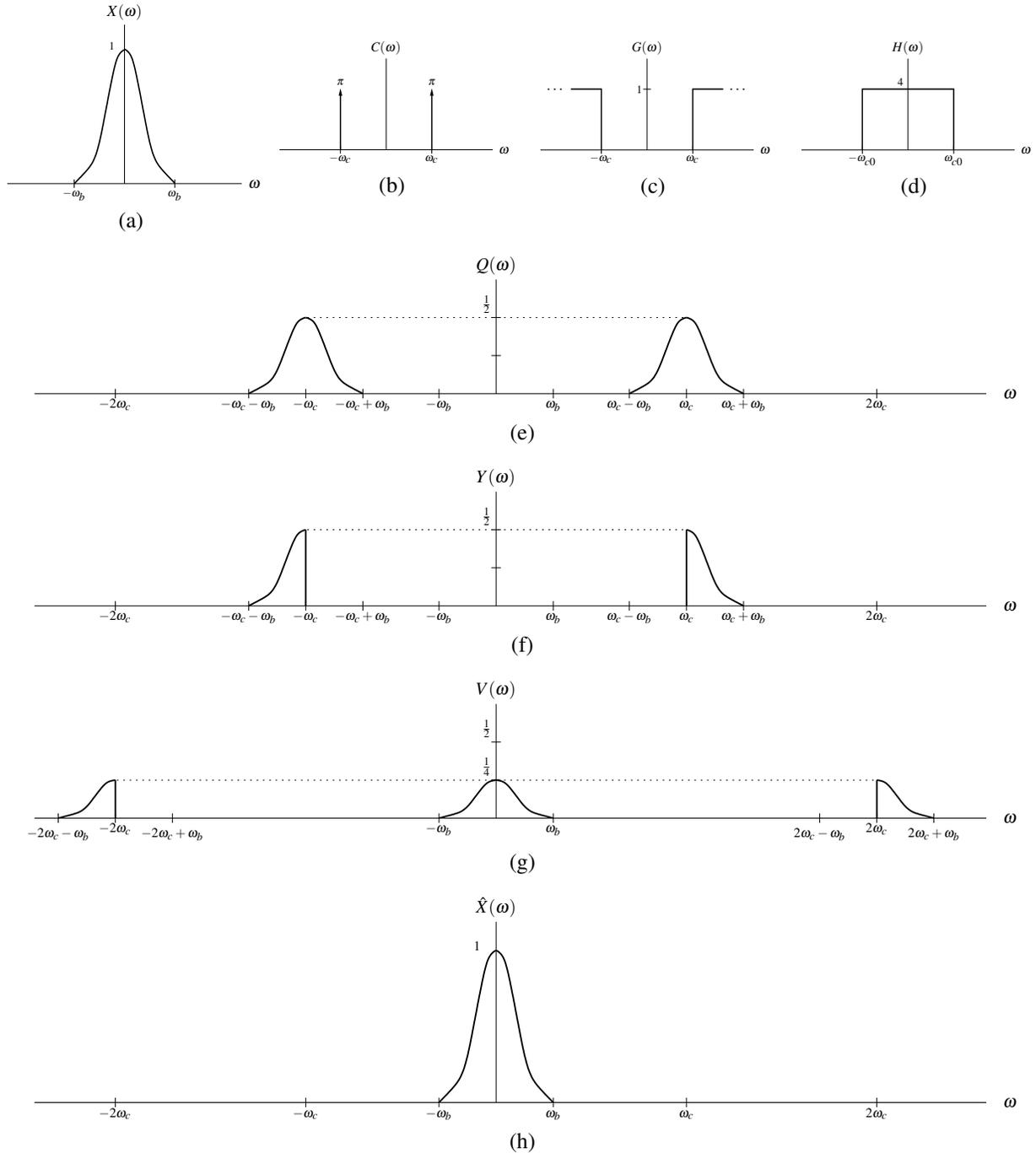
Figure 5.35: Signal spectra for SSB/SC amplitude modulation.
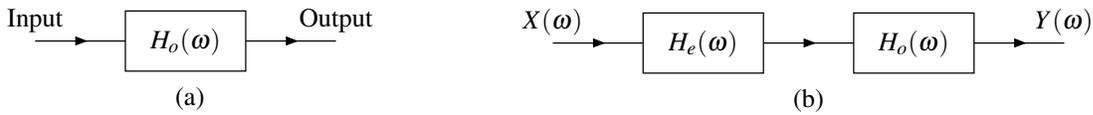
Figure 5.36: Equalization example. (a) Original system. (b) New system with equalization.
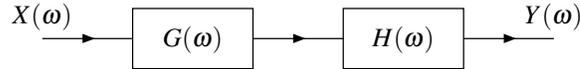
Figure 5.37: Equalization system.

equalization. Essentially, equalization is just a filtering operation, where the filtering is applied with the specific goal of obtaining a more desirable frequency response.

Let us now examine the mathematics behind equalization. Consider the LTI system with frequency response $H_o(\omega)$ as shown in Figure 5.36(a). Suppose that the frequency response $H_o(\omega)$ is undesirable for some reason (i.e., the system does not behave in a way that is good for the application at hand). Consequently, we would instead like to have a system with frequency response $H_d(\omega)$. In effect, we would like to somehow change the frequency response $H_o(\omega)$ of the original system to $H_d(\omega)$. This can be accomplished by using another system called an **equalizer**. More specifically, consider the new system shown in Figure 5.36(b) which consists of a LTI equalizer with frequency response $H_e(\omega)$ connected in series with the original system having frequency response $H_o(\omega)$. From the block diagram, we have

$$Y(\omega) = H(\omega)X(\omega),$$

where $H(\omega) = H_o(\omega)H_e(\omega)$. In effect, we want to force $H(\omega)$ to be equal to $H_d(\omega)$ so that the overall (i.e., series-interconnected) system has the frequency response desired. So, we choose the equalizer to be such that $H_e(\omega) = \frac{H_d(\omega)}{H_o(\omega)}$. Then, we have

$$H(\omega) = H_o(\omega)H_e(\omega)$$
$$= H_o(\omega)\left[\frac{H_d(\omega)}{H_o(\omega)}\right]$$
$$= H_d(\omega).$$

Thus, the system in Figure 5.36(b) has the frequency response $H_d(\omega)$ as desired.

Equalization is used in many applications. In real-world communication systems, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel. In audio applications, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, often we like to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Example 5.37 (Communication channel equalization)**.** Suppose that we have a LTI communication channel with frequency response $H(\omega) = \frac{1}{3+j\omega}$. Unfortunately, this channel has the undesirable effect of attenuating higher frequencies. Find the frequency response $G(\omega)$ of an equalizer that when connected in series with the communication channel yields an ideal (i.e., distortionless) channel. The new system with equalization is shown in Figure 5.37.

Solution. An ideal communication channel has a frequency response equal to one for all frequencies. Consequently, we want $H(\omega)G(\omega) = 1$ or equivalently $G(\omega) = 1/H(\omega)$. Thus, we conclude that

$$G(\omega) = \frac{1}{H(\omega)} = \frac{1}{\frac{1}{3+j\omega}} = 3 + j\omega.$$

$\square$

5.18 Problems

5.1 Using the Fourier transform analysis equation, find the Fourier transform of each of the following signals:
(a) $x(t) = A\delta(t - t_0)$ where t_0 and A are real and complex constants, respectively;
(b) $x(t) = \text{rect}(t - t_0)$ where t_0 is a constant;
(c) $x(t) = e^{-4t}u(t - 1)$;
(d) $x(t) = 3[u(t) - u(t - 2)]$; and
(e) $x(t) = e^{-|t|}$.

5.2 Use a Fourier transform table and properties of the Fourier transform to find the Fourier transform of each of the signals below.
(a) $x(t) = \cos(t - 5)$;
(b) $x(t) = e^{-j5t}u(t + 2)$;
(c) $x(t) = [\cos t]u(t)$;
(d) $x(t) = 6[u(t) - u(t - 3)]$;
(e) $x(t) = 1/t$;
(f) $x(t) = t\,\text{rect}(2t)$;
(g) $x(t) = e^{-j3t}\sin(5t - 2)$;
(h) $x(t) = \cos(5t - 2)$;
(i) $x(t) = e^{-j2t}\frac{1}{3t+1}$;
(j) $x(t) = \int_{-\infty}^{5t} e^{-\tau-1}u(\tau - 1)d\tau$;
(k) $x(t) = (t + 1)\sin(5t - 3)$;
(l) $x(t) = (\sin 2\pi t)\delta(t - \frac{\pi}{2})$;
(m) $x(t) = e^{-jt}\frac{1}{3t-2}$;
(n) $x(t) = e^{j5t}(\cos 2t)u(t)$; and
(o) $x(t) = e^{-j2t}\text{sgn}(-t - 1)$.

5.3 Compute the Fourier transform $X(\omega)$ of the signal $x(t)$ given by

$$x(t) = \sum_{k=0}^{\infty} a^k \delta(t - kT),$$

where a is a constant satisfying $|a| < 1$. (Hint: Recall the formula for the sum of an infinite geometric series. That is, $b + br + br^2 + \ldots = \frac{b}{1-r}$ if $|r| < 1$.)

5.4 The ideal Hilbert transformer is a LTI system with the frequency response

$$H(\omega) = \begin{cases} -j & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ j & \text{for } \omega < 0. \end{cases}$$

This type of system is useful in a variety of signal processing applications (e.g., SSB/SC amplitude modulation). By using the duality property of the Fourier transform, find the impulse response $h(t)$ of this system.

5.5 Given that $x(t) \xleftrightarrow{\mathscr{F}} X(\omega)$ and $y(t) \xleftrightarrow{\mathscr{F}} Y(\omega)$, express $Y(\omega)$ in terms of $X(\omega)$ for each of the following:
(a) $y(t) = x(at - b)$ where a and b are constants and $a \neq 0$;
(b) $y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$;
(c) $y(t) = \int_{-\infty}^{t} x^2(\tau)d\tau$;
(d) $y(t) = \frac{d}{dt}[x(t) * x(t)]$;
(e) $y(t) = tx(2t - 1)$;
(f) $y(t) = e^{j2t}x(t - 1)$;
(g) $y(t) = \left(te^{-j5t}x(t)\right)^*$;

(h) $y(t) = \left[\frac{d}{dt}x(t)\right] * \left[e^{-jt}x(t)\right]$;

(i) $y(t) = \int_{-\infty}^{3t} x^*(\tau - 1)d\tau$;

(j) $y(t) = [\cos(3t - 1)]x(t)$;

(k) $y(t) = \left[\frac{d}{dt}x(t)\right]\sin(t - 2)$;

(l) $y(t) = tx(t)\sin 3t$; and

(m) $y(t) = e^{j7t}\left[x(\lambda) * x(\lambda)\right]\big|_{\lambda = t-1}$.

5.6 Find the Fourier transform of each of the periodic signals shown below.

(a)

(b)
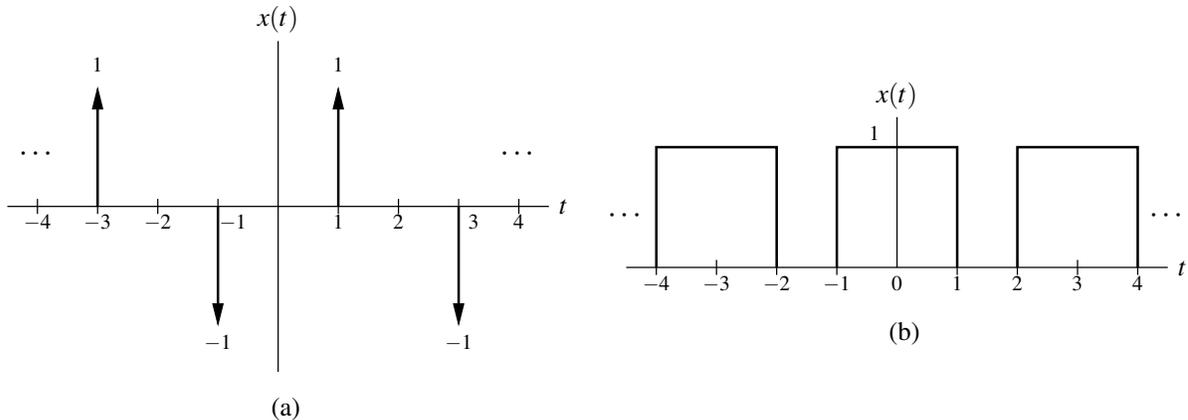
5.7 Using the time-domain convolution property of the Fourier transform, compute the convolution $h(t) = h_1(t) * h_2(t)$ where

$$h_1(t) = 2000\,\text{sinc}(2000\pi t) \quad \text{and} \quad h_2(t) = \delta(t) - 1000\,\text{sinc}(1000\pi t).$$

5.8 Compute the energy contained in the signal $x(t) = 200\,\text{sinc}(200\pi t)$.

5.9 Compute the frequency spectrum of each of the signals specified below. In each case, also find and plot the corresponding magnitude and phase spectra.

(a) $x(t) = e^{-at}u(t)$, where a is a positive real constant; and

(b) $x(t) = \text{sinc}\,\frac{t-1}{200}$.

5.10 Suppose that we have the LTI systems defined by the differential/integral equations given below, where $x(t)$ and $y(t)$ denote the system input and output, respectively. Find the frequency response of each of these systems.

(a) $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + y(t) + 3\frac{d}{dt}x(t) - x(t) = 0$; and

(b) $\frac{d}{dt}y(t) + 2y(t) + \int_{-\infty}^{t} 3y(\tau)d\tau + 5\frac{d}{dt}x(t) - x(t) = 0$.

5.11 Suppose that we have the LTI systems with the frequency responses given below. Find the differential equation that characterizes each of these systems.

(a) $H(\omega) = \dfrac{j\omega}{1 + j\omega}$; and

(b) $H(\omega) = \dfrac{j\omega + \frac{1}{2}}{-j\omega^3 - 6\omega^2 + 11j\omega + 6}$.

5.12 Suppose that we have a LTI system with input $x(t)$ and output $y(t)$, and impulse response $h(t)$, where

$$h(t) = \delta(t) - 300\,\text{sinc}\,300\pi t.$$

Using frequency-domain methods, find the response $y(t)$ of the system to the input $x(t) = x_1(t)$, where

$$x_1(t) = \tfrac{1}{2} + \tfrac{3}{4}\cos 200\pi t + \tfrac{1}{2}\cos 400\pi t - \tfrac{1}{4}\cos 600\pi t.$$

5.13 Consider the LTI system with input $v_0(t)$ and output $v_1(t)$ as shown in the figure below, where $R = 1$ and $L = 1$.
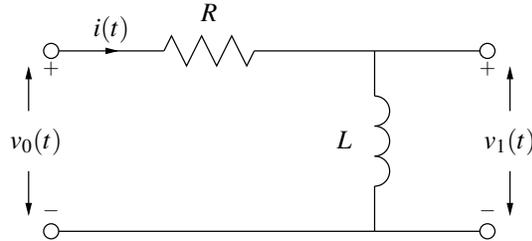
(a) Find the frequency response $H(\omega)$ of the system.
(b) Determine the magnitude and phase responses of the system.
(c) Find the impulse response $h(t)$ of the system.

5.14 Consider the LTI system with input $v_0(t)$ and output $v_1(t)$ as shown in the figure below, where $R = 1$, $C = \frac{1}{1000}$, and $L = \frac{1}{1000}$.
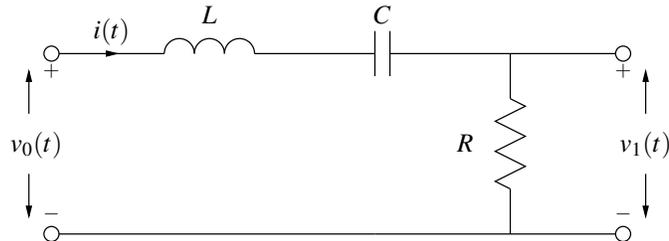
(a) Find the frequency response $H(\omega)$ of the system.
(b) Use a computer to plot the magnitude and phase responses of the system.
(c) From the plots in part (b), identify the type of ideal filter that this system approximates.

5.15 Let $x(t)$ be a real signal with Fourier transform $X(\omega)$ satisfying $X(\omega) = 0$ for $|\omega| > \omega_b$. We use amplitude modulation to produce the signal $y(t) = x(t)\sin\omega_c t$. Note that $\omega_c \gg \omega_b$. In order to recover the original signal $x(t)$, it is proposed that the system shown in the figure below be used. The frequency response $H(\omega)$ is given by

$$H(\omega) = \begin{cases} 2 & \text{for } |\omega| < \omega_b \\ 0 & \text{otherwise.} \end{cases}$$
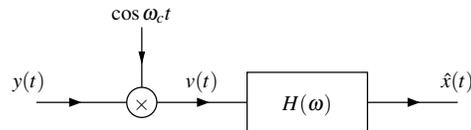
Let $Y(\omega)$, $V(\omega)$, and $\hat{X}(\omega)$ denote the Fourier transforms of $y(t)$, $v(t)$, and $\hat{x}(t)$, respectively.
(a) Find an expression for $Y(\omega)$ in terms of $X(\omega)$. Find an expression for $\hat{X}(\omega)$ in terms of $V(\omega)$. Find a simplified expression for $\hat{X}(\omega)$.
(b) Compare $\hat{x}(t)$ and $x(t)$. Comment on the utility of the proposed system.

5.16 When discussing DSB/SC amplitude modulation, we saw that a system of the form shown below in Figure A is often useful. In practice, however, the multiplier unit needed by this system is not always easy to implement. For this reason, we sometimes employ a system like that shown below in Figure B. In this second system, we sum the sinusoidal carrier and modulating signal $x(t)$ and then pass the result through a nonlinear squaring device (i.e., $v_2(t) = [v_1(t)]^2$).

(a) (b)
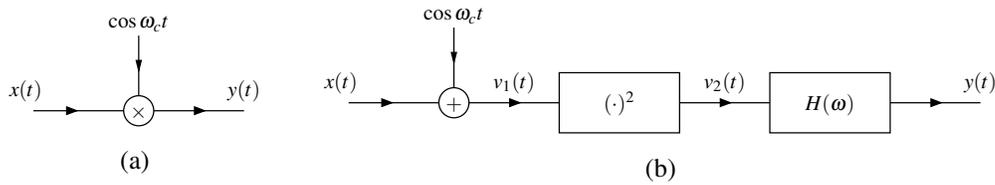
Let $X(\omega)$, $V_1(\omega)$, and $V_2(\omega)$ denote the Fourier transforms of $x(t)$, $v_1(t)$, and $v_2(t)$, respectively. Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$ (i.e., $x(t)$ is bandlimited).

(a) Find an expression for $v_1(t)$, $v_2(t)$ and $V_2(\omega)$. (Hint: If $X(\omega) = 0$ for $|\omega| > \omega_b$, then using the time-domain convolution property of the Fourier transform, we can deduce that the Fourier transform of $x^2(t)$ is zero for $|\omega| > 2\omega_b$.)

(b) Determine the frequency response $H(\omega)$ required for the system shown in Figure B to be equivalent to the system in Figure A. State any assumptions made with regard to the relationship between ω_c and ω_b. (Hint: It might be helpful to sketch $X(\omega)$ and $V_2(\omega)$ for the case of some simple $X(\omega)$. Then, compare $V_2(\omega)$ to $X(\omega)$ in order to deduce your answer.)

5.17 Consider the system with input $x(t)$ and output $y(t)$ as shown in Figure A below. The frequency response $H(\omega)$ is that of an ideal Hilbert transformer, which is given by

$$H(\omega) = -j \operatorname{sgn} \omega.$$

Let $X(\omega)$, $Y(\omega)$, $V_1(\omega)$, $V_2(\omega)$, and $V_3(\omega)$ denote the Fourier transforms of $x(t)$, $y(t)$, $v_1(t)$, $v_2(t)$, and $v_3(t)$, respectively.

(a) (b)
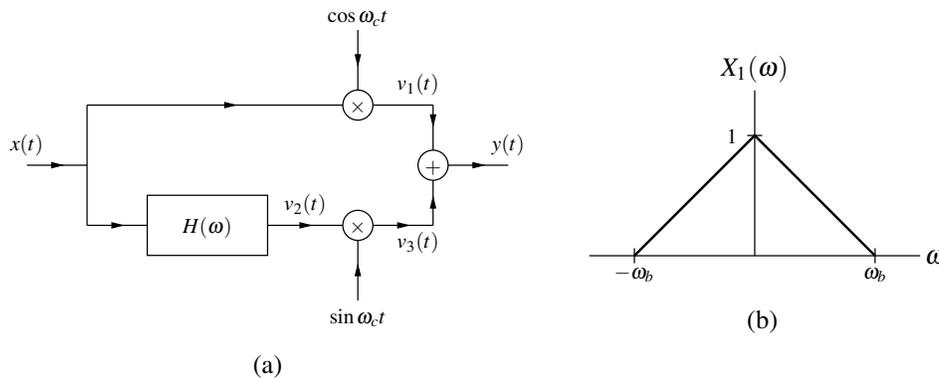
(a) Suppose that $X(\omega) = 0$ for $|\omega| > \omega_b$, where $\omega_b \ll \omega_c$. Find expressions for $V_1(\omega)$, $V_2(\omega)$, $V_3(\omega)$, and $Y(\omega)$ in terms of $X(\omega)$.

(b) Suppose that $X(\omega) = X_1(\omega)$ where $X_1(\omega)$ is as shown in Figure B. Sketch $V_1(\omega)$, $V_2(\omega)$, $V_3(\omega)$, and $Y(\omega)$ in this case.

(c) Draw the block diagram of a system that could be used to recover $x(t)$ from $y(t)$.

5.18 Consider the system shown below in Figure A with input $x(t)$ and output $\hat{x}(t)$, where

$$G(\omega) = \begin{cases} 2 & \text{for } |\omega| \le 100\pi \\ 0 & \text{otherwise.} \end{cases}$$

Let $X(\omega)$, $\hat{X}(\omega)$, $Y(\omega)$, and $Q(\omega)$ denote the Fourier transforms of $x(t)$, $\hat{x}(t)$, $y(t)$, and $q(t)$, respectively.

(a)

(b)
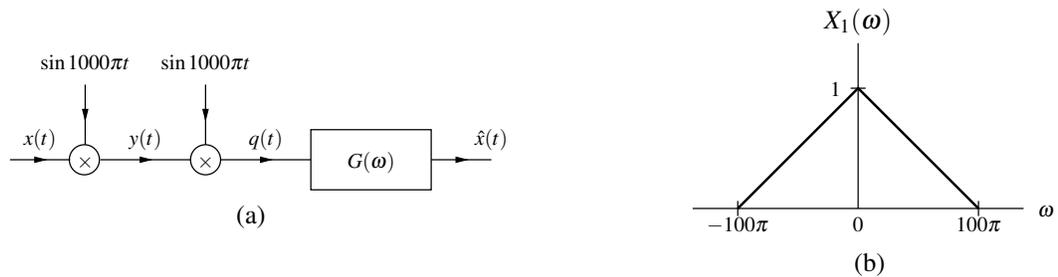
(a) Suppose that $X(\omega) = 0$ for $|\omega| > 100\pi$. Find expressions for $Y(\omega)$, $Q(\omega)$, and $\hat{X}(\omega)$ in terms of $X(\omega)$.

(b) If $X(\omega) = X_1(\omega)$ where $X_1(\omega)$ is as shown in Figure B, sketch $Y(\omega)$, $Q(\omega)$, and $\hat{X}(\omega)$.

5.19 Consider the system shown below in Figure A with input $x(t)$ and output $y(t)$. Let $X(\omega)$, $P(\omega)$, $S(\omega)$, $H(\omega)$, and $Y(\omega)$ denote the Fourier transforms of $x(t)$, $p(t)$, $s(t)$, $h(t)$, and $y(t)$, respectively. Suppose that

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - \tfrac{n}{1000}) \quad \text{and} \quad H(\omega) = \tfrac{1}{1000} \operatorname{rect}(\tfrac{\omega}{2000\pi}).$$

(a) Derive an expression for $S(\omega)$ in terms of $X(\omega)$. Derive an expression for $Y(\omega)$ in terms of $S(\omega)$ and $H(\omega)$.

(b) Suppose that $X(\omega) = X_1(\omega)$, where $X_1(\omega)$ is as shown in Figure B. Using the results of part (a), plot $S(\omega)$ and $Y(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.

(c) Suppose that $X(\omega) = X_2(\omega)$, where $X_2(\omega)$ is as shown in Figure C. Using the results of part (a), plot $S(\omega)$ and $Y(\omega)$. Indicate the relationship (if any) between the input $x(t)$ and output $y(t)$ of the system.

(a)

(b)

(c)
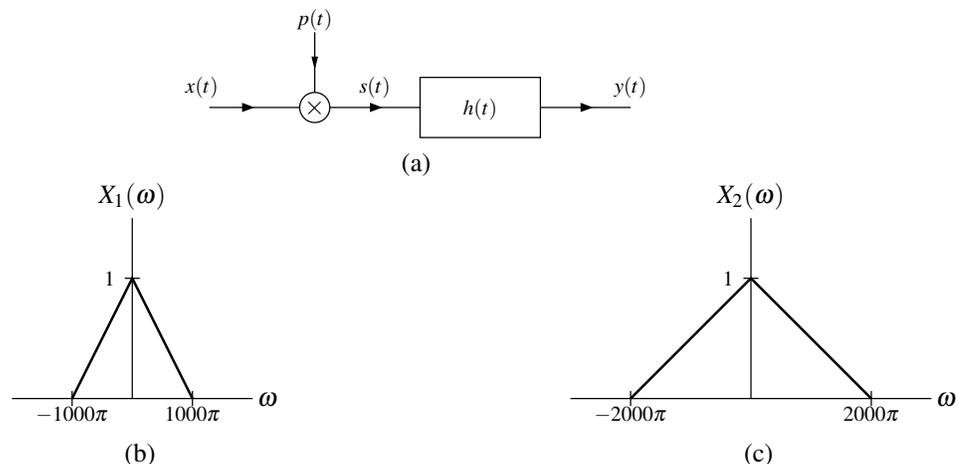
5.19 MATLAB Problems

5.101 (a) Consider a frequency response $H(\omega)$ of the form

$$H(\omega) = \frac{\sum_{k=0}^{M-1} a_k \omega^k}{\sum_{k=0}^{N-1} b_k \omega^k},$$

where a_k and b_k are complex constants. Write a MATLAB function called `freqw` that evaluates a function of the above form at an arbitrary number of specified points. The function should take three input arguments: 1) a vector containing the a_k coefficients, 2) a vector containing the b_k coefficients, 3) a vector containing the values of ω at which to evaluate $H(\omega)$. The function should generate two return values: 1) a vector of function values, and 2) a vector of points at which the function was evaluated. If the function is called with no output arguments

(i.e., the `nargout` variable is zero), then the function should plot the magnitude and phase responses before returning. [Hint: The `polyval` function may be helpful.]

(b) Use the function developed in part (a) to plot the magnitude and phase responses of the system with the frequency response

$$H(\omega) = \frac{16.0000}{1.0000\omega^4 - j5.2263\omega^3 - 13.6569\omega^2 + j20.9050\omega + 16.0000}.$$

For each of the plots, use the frequency range $[-5, 5]$.

(c) What type of ideal frequency-selective filter does this system approximate?

5.102 Consider the filter associated with each of the frequency responses given below. In each case, plot the magnitude and phase responses of the filter, and indicate what type of ideal frequency-selective filter it best approximates.

(a) $H(\omega) = \dfrac{\omega_b^3}{(j\omega)^3 + 2\omega_b(j\omega)^2 + 2\omega_b(j\omega) + \omega_b^3}$ where $\omega_b = 1$;

(b) $H(\omega) = \dfrac{(j\omega)^5}{(j\omega)^5 + 17.527635(j\omega)^4 + 146.32995(j\omega)^3 + 845.73205(j\omega)^2 + 2661.6442(j\omega) + 7631.0209}$; and

(c) $H(\omega) = \dfrac{13.104406(j\omega)^3}{(j\omega)^6 + 3.8776228(j\omega)^5 + 34.517979(j\omega)^4 + 75.146371(j\omega)^3 + 276.14383(j\omega)^2 + 248.16786(j\omega) + 512}$.

Hint: Use the `freqs` function with $s = j\omega$ to compute the frequency response. The `abs`, `angle`, `linspace`, `plot`, `xlabel`, `ylabel`, and `print` functions may also prove useful for this problem.

5.103 (a) Use the `butter` and `besself` functions to design a tenth-order Butterworth lowpass filter and tenth-order Bessel lowpass filter, each with a cutoff frequency of 10 rad/s.

(b) For each of the filters designed in part (a), plot the magnitude and phase responses using a linear scale for the frequency axis. In the case of the phase response, plot the unwrapped phase (as this will be helpful later in part (d) of this problem). (Hint: The `freqs` and `unwrap` functions may be helpful.)

(c) Consider the magnitude responses for each of the filters. Recall that an ideal lowpass filter has a magnitude response that is constant in the passband. Which of the two filters more closely approximates this ideal behavior?

(d) Consider the phase responses for each of the filters. An ideal lowpass filter has a phase response that is a linear function. Which of the two filters has a phase response that best approximates a linear (i.e., straight line) function in the passband?

Chapter 6

Laplace Transform

6.1 Introduction

In this chapter, we introduce another important mathematical tool in the study of signals and systems known as the Laplace transform. The Laplace transform can be viewed as a generalization of the Fourier transform. Due to its more general nature, the Laplace transform has a number of advantages over the Fourier transform. First, the Laplace transform representation exists for some signals that do not have Fourier transform representations. So, we can handle a larger class of signals with the Laplace transform. Second, since the Laplace transform is a more general tool, it can provide additional insights beyond those facilitated by the Fourier transform.

6.2 Motivation Behind the Laplace Transform

In Section 3.10, we showed that complex exponentials are eigenfunctions of LTI systems. Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. This eigenfunction property leads to the result that the response $y(t)$ to the input $x(t) = e^{st}$ is

$$y(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt. \tag{6.1}$$

Previously, we referred to $H(s)$ as the system function. In this chapter, we will learn that $H(s)$ is, in fact, the Laplace transform of $h(t)$. That is, the integral in (6.1) is simply the definition of the Laplace transform. In the case that $s = j\omega$ where ω is real (i.e., s is purely imaginary), (6.1) becomes the Fourier transform integral (studied in Chapter 5). From our earlier reading, we know that $H(j\omega)$ is the Fourier transform of $h(t)$, which we refer to as the frequency response of the LTI system. In (6.1), we consider a more general class of transform where s is an arbitrary complex value (i.e., s is not just purely imaginary as in the case of the Fourier transform). This generalization is known as the Laplace transform.

6.3 Definition of the Laplace Transform

The (bilateral) **Laplace transform** of the function $x(t)$ is denoted as $\mathscr{L}\{x(t)\}$ or $X(s)$ and is defined as

$$X(s) = \mathscr{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt. \tag{6.2}$$

The **inverse Laplace transform** is then given by

$$x(t) = \mathscr{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds, \tag{6.3}$$

where $\sigma = \mathrm{Re}\{s\}$. We refer to $x(t)$ and $X(s)$ as a **Laplace transform pair** and denote this relationship as

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s).$$

As we can see from (6.3), the calculation of the inverse Laplace transform requires a contour integration (since s is a complex variable). In particular, we must integrate along the vertical line $s = \sigma$ in the complex plane. Such a contour integration is often not so easy to compute. Therefore, in practice, we do not usually compute the inverse Laplace transform using (6.3) directly. Instead, we resort to other means (to be discussed later).

Two different versions of the Laplace transform are commonly used. The first is the bilateral version, as introduced above. The second is the unilateral version. The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions. As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the lower limit of integration. In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0. In the remainder of this chapter, we will focus our attention primarily on the bilateral Laplace transform. We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations. Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean bilateral Laplace transform.

6.4 Relationship Between Laplace Transform and Continuous-Time Fourier Transform

In the previous section, we introduced the Laplace transform, and in the previous chapter, we studied the Fourier transform. As it turns out, the Laplace transform and Fourier transform are very closely related. Recall the definition of the Laplace transform in (6.2). Consider now the special case of (6.2) where $s = j\omega$ and ω is real (i.e., $\mathrm{Re}(s) = 0$). In this case, (6.2) becomes

$$\begin{aligned}
X(s)|_{s=j\omega} &= \left[\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right]\bigg|_{s=j\omega} \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\
&= \mathscr{F}\{x(t)\}.
\end{aligned}$$

Thus, the Fourier transform is simply the Laplace transform evaluated at $s = j\omega$. In other words,

$$X(s)|_{s=j\omega} = \mathscr{F}\{x(t)\}. \tag{6.4}$$

Incidentally, it is due to the preceding relationship that the Fourier transform of $x(t)$ is sometimes written as $X(j\omega)$. When this notation is used, the function X actually corresponds to the Laplace transform of $x(t)$ rather than its Fourier transform (i.e., the expression $X(j\omega)$ corresponds to the Laplace transform evaluated at points on the imaginary axis).

Now, consider the general case of an arbitrary complex value for s in (6.2). Let us express s in Cartesian form as $s = \sigma + j\omega$ where σ and ω are real. Substituting $s = \sigma + j\omega$ into (6.2), we obtain

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt \\
&= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt \\
&= \mathscr{F}\{e^{-\sigma t}x(t)\}.
\end{aligned}$$

Thus, the Laplace transform of $x(t)$ can be viewed as the Fourier transform of $x(t)e^{-\sigma t}$ (i.e., $x(t)$ weighted by a real exponential signal). As a consequence of multiplying by the real exponential $e^{-\sigma t}$, the Laplace transform of a signal may exist when the Fourier transform of the same signal does not.

By using the above relationship, we can derive the formula for the inverse Laplace transform given in (6.3). Suppose we have $x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$. Let $s = \sigma + j\omega$ where σ and ω are real. From the relationship between the Fourier and Laplace transforms, we have

$$\mathscr{L}\{x(t)\} = X(s) = X(\sigma + j\omega) = \mathscr{F}\{x(t)e^{-\sigma t}\}.$$

Taking the inverse Fourier transform of both sides of the preceding equation yields

$$\mathscr{F}^{-1}\{X(\sigma + j\omega)\} = x(t)e^{-\sigma t}.$$

Multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = e^{\sigma t}\mathscr{F}^{-1}\{X(\sigma + j\omega)\}.$$

From the definition of the inverse Fourier transform, we have

$$x(t) = e^{\sigma t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega\right]$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega.$$

Since $s = \sigma + j\omega$, we have that $ds = jd\omega$, and consequently,

$$x(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}(1/j)ds$$
$$= \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds.$$

Thus, we have just derived the inverse Laplace transform formula in (6.3).

6.5 Laplace Transform Examples

In this section, we calculate the Laplace transform of several relatively simple signals. In the process, we gain some important insights into the Laplace transform.

Example 6.1. Find the Laplace transform $X(s)$ of the signal $x(t) = e^{-at}u(t)$.

Solution. Let $s = \sigma + j\omega$ where σ and ω are real. From the definition of the Laplace transform, we have

$$X(s) = \mathscr{L}\{e^{-at}u(t)\}$$
$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt$$
$$= \int_{0}^{\infty} e^{-(s+a)t}dt$$
$$= \left[\left(-\frac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{0}^{\infty}.$$

At this point, we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges to a finite value. This yields

$$X(s) = \left[\left(-\frac{1}{\sigma + a + j\omega}\right)e^{-(\sigma + a + j\omega)t}\right]\Big|_{0}^{\infty}$$
$$= \left(\frac{-1}{\sigma + a + j\omega}\right)\left[e^{-(\sigma + a)t}e^{-j\omega t}\right]\Big|_{0}^{\infty}$$
$$= \left(\frac{-1}{\sigma + a + j\omega}\right)\left[e^{-(\sigma + a)\infty}e^{-j\omega\infty} - 1\right].$$
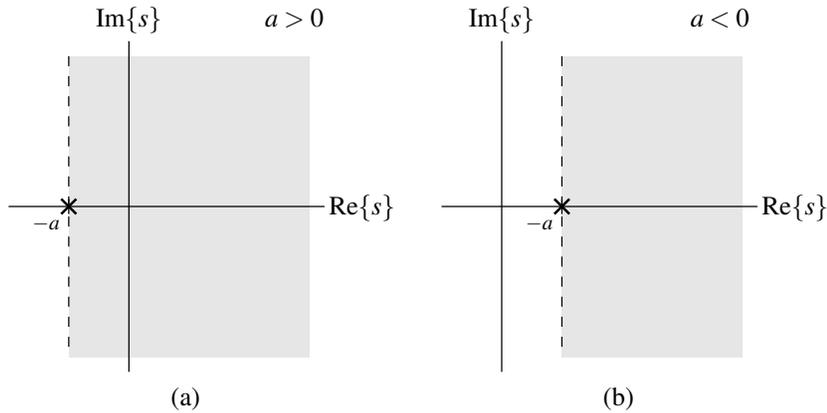
Figure 6.1: Regions of convergence.

Thus, we can see that the above expression only converges for $\sigma + a > 0$ (i.e., $\text{Re}\{s\} > -a$). In this case, we have that

$$X(s) = \left(\frac{-1}{\sigma+a+j\omega}\right)[0-1]$$
$$= \left(\frac{-1}{s+a}\right)(-1)$$
$$= \frac{1}{s+a}.$$

Thus, we have that

$$e^{-at}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \text{Re}\{s\} > -a.$$

The region of convergence for $X(s)$ is illustrated in Figures 6.1(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.

$\square$

Example 6.2. Find the Laplace transform $X(s)$ of the signal $x(t) = -e^{-at}u(-t)$.

Solution. Let $s = \sigma + j\omega$ where σ and ω are real. From the definition of the Laplace transform, we can write

$$X(s) = \mathscr{L}\{-e^{-at}u(-t)\}$$
$$= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$
$$= \int_{-\infty}^{0} -e^{-at}e^{-st}dt$$
$$= \int_{-\infty}^{0} -e^{-(s+a)t}dt$$
$$= \left[\left(\frac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}.$$

In order to more easily determine when the above expression converges to a finite value, we substitute $s = \sigma + j\omega$. This yields

$$X(s) = \left[\left(\frac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0}$$
$$= \left(\frac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{-\infty}^{0}$$
$$= \left(\frac{1}{\sigma+a+j\omega}\right)\left[1 - e^{(\sigma+a)\infty}e^{j\omega\infty}\right].$$
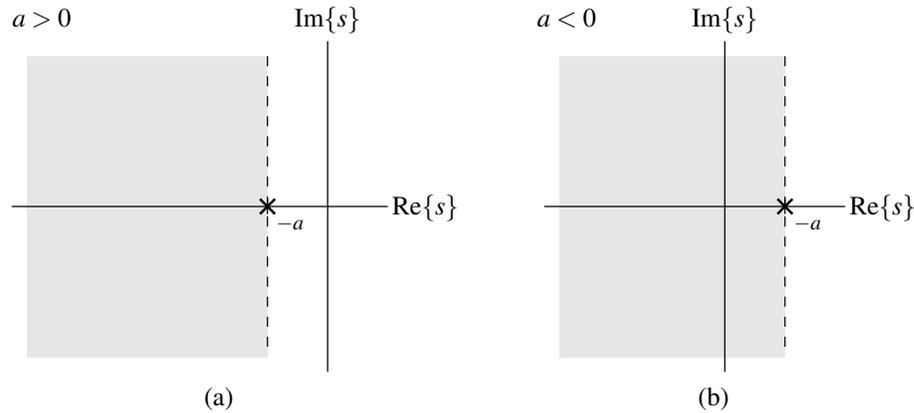
Figure 6.2: Regions of convergence.

Thus, we can see that the above expression only converges for $\sigma + a < 0$ (i.e., $\text{Re}\{s\} < -a$). In this case, we have

$$X(s) = \left(\frac{1}{\sigma + a + j\omega}\right)[1 - 0]$$

$$= \frac{1}{s + a}.$$

Thus, we have that

$$-e^{-at}u(-t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s + a} \quad \text{for } \text{Re}\{s\} < -a.$$

The region of convergence for $X(s)$ is illustrated in Figures 6.2(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.
□

At this point, we compare the results of Examples 6.1 and 6.2, and make an important observation. Notice that the same algebraic expression for $X(s)$ was obtained in both of these examples (i.e., $X(s) = \frac{1}{s+a}$). The only difference is in the convergence properties of $X(s)$. In one case, $X(s)$ converges for $\text{Re}\{s\} > -a$ while in the other it converges for $\text{Re}\{s\} < -a$. As it turns out, one must specify both the algebraic expression for $X(s)$ and its region of convergence in order to uniquely determine $x(t) = \mathscr{L}^{-1}\{X(s)\}$.

Example 6.3 (Laplace transform of the unit-step function). Find the Laplace transform $X(s)$ of the signal $x(t) = u(t)$.

Solution. Let $s = \sigma + j\omega$ where σ and ω are real. From the definition of the Laplace transform, we have

$$X(s) = \mathscr{L}\{u(t)\}$$

$$= \int_{-\infty}^{\infty} u(t)e^{-st}\,dt$$

$$= \int_{0}^{\infty} e^{-st}\,dt$$

$$= \left[\left(-\tfrac{1}{s}\right)e^{-st}\right]\Big|_{0}^{\infty}.$$

At this point, we substitute $s = \sigma + j\omega$ in order to more clearly see the region of convergence for this expression. This yields

$$X(s) = \left[\left(-\tfrac{1}{\sigma + j\omega}\right)e^{-(\sigma + j\omega)t}\right]\Big|_{0}^{\infty}$$

$$= \left[\left(-\tfrac{1}{\sigma + j\omega}\right)e^{-\sigma t}e^{-j\omega t}\right]\Big|_{0}^{\infty}.$$

Thus, we can see that the above expression converges only for $\sigma > 0$ (i.e., $\text{Re}\{s\} > 0$). In this case, we have

$$\begin{aligned}
X(s) &= \left(-\tfrac{1}{\sigma + j\omega}\right)[0 - 1] \\
&= \left(-\tfrac{1}{s}\right)(-1) \\
&= \frac{1}{s}.
\end{aligned}$$

Thus, we have that

$$u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s} \quad \text{for } \text{Re}\{s\} > 0.$$

$\square$

Example 6.4 (Laplace transform of the unit-impulse function). Find the Laplace transform of the signal $x(t) = A\delta(t - t_0)$ where t_0 is an arbitrary real constant.

Solution. From the definition of the Laplace transform, we can write

$$\begin{aligned}
X(s) &= \mathscr{L}\{x(t)\} \\
&= \mathscr{L}\{A\delta(t - t_0)\} \\
&= \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-st}\,dt \\
&= A\int_{-\infty}^{\infty} \delta(t - t_0)e^{-st}\,dt.
\end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify this result to obtain

$$X(s) = Ae^{-st_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \overset{\mathscr{L}}{\longleftrightarrow} Ae^{-st_0} \quad \text{for all } s.$$

$\square$

6.6   Region of Convergence for the Laplace Transform

Before discussing the region of convergence (ROC) of the Laplace transform in detail, we need to introduce some terminology involving sets in the complex plane. Let R denote a set in the complex plane. A set R comprised of all complex numbers s such that

$$\text{Re}\{s\} < a,$$

for some real constant a (or $a = \infty$), is said to be a **left-half plane** (LHP). A set R comprised of all complex numbers s such that

$$\text{Re}\{s\} > a,$$

for some real constant a (or $a = -\infty$), is said to be a **right-half plane** (RHP). Examples of LHPs and RHPs are given in Figure 6.3. Note that the set consisting of the entire complex plane is trivially both a LHP and RHP (i.e., the entire complex plane satisfies $\text{Re}\{s\} > -\infty$ and $\text{Re}\{s\} < \infty$).

Since the ROC is a set (of points in the complex plane), we often need to employ some basic set operations when dealing with ROCs. For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B. An illustrative example of set intersection is shown in Figure 6.4.
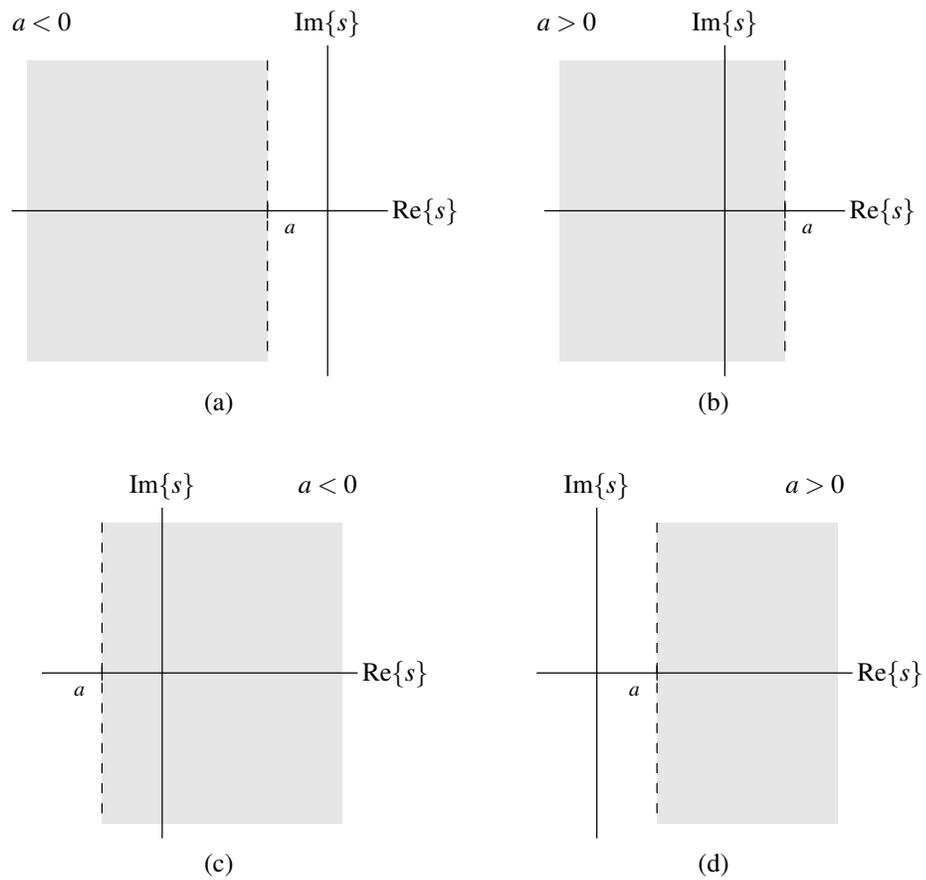
Figure 6.3: Examples of LHPs and RHPs. (a) and (b) LHPs, and (c) and (d) RHPs.
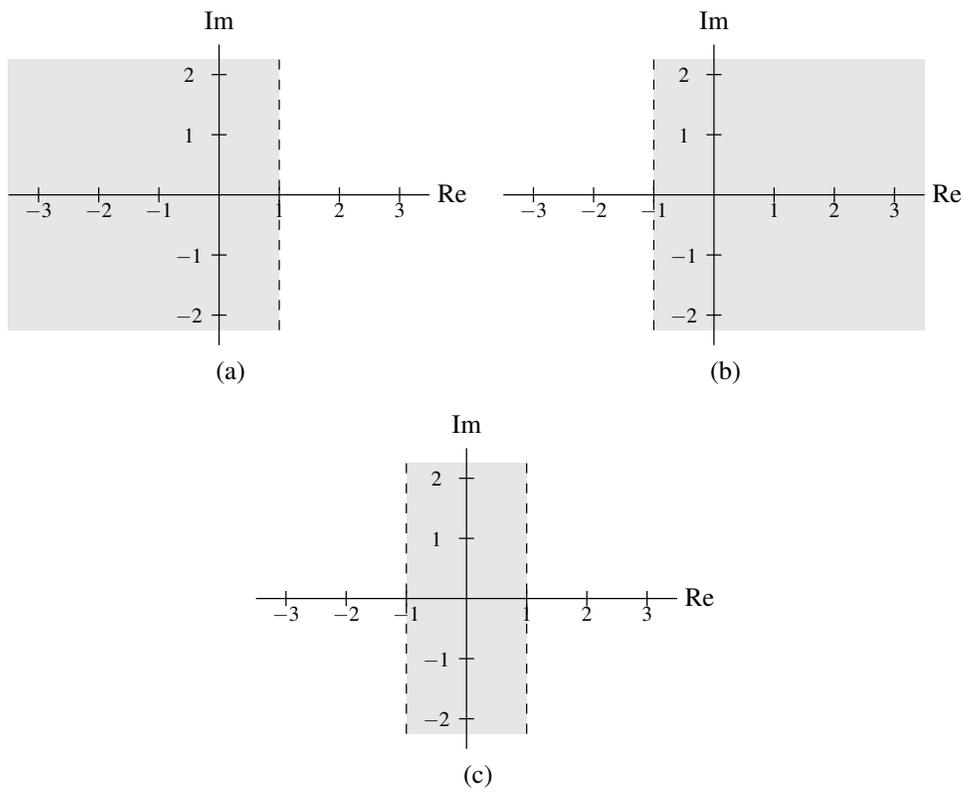
Figure 6.4: Intersection of sets. (a) First set R_1, (b) second set R_2, and (c) their intersection $R_1 \cap R_2$.

As we saw earlier, for a signal $x(t)$, the complete specification of its Laplace transform $X(s)$ requires not only an algebraic expression for $X(s)$, but also the ROC associated with $X(s)$. Two very different signals can have the same algebraic expressions for $X(s)$. In this section, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of signals.

One can show that the ROC for the Laplace transform has the following properties:

1. The ROC of the Laplace transform $X(s)$ consists of strips parallel to the imaginary axis in the complex plane.

   Justification: The Laplace transform $X(s)$ of the signal $x(t)$ is simply the Fourier transform of $x(t)e^{-\text{Re}\{s\}t}$. Thus, $X(s)$ converges whenever this Fourier transform converges. Since the convergence of the Fourier transform only depends on $\text{Re}\{s\}$, the convergence of the Laplace transform only depends on $\text{Re}\{s\}$.

2. If the Laplace transform $X(s)$ is a rational function, the ROC does not contain any poles, and the ROC is bounded by poles or extends to infinity.

   Partial justification: Since $X(s)$ is rational, its value becomes infinite at a pole. So obviously, $X(s)$ does not converge at a pole. Therefore, it follows that the ROC cannot contain a pole.

3. If the signal $x(t)$ is finite duration and its Laplace transform $X(s)$ converges for some value of s, then $X(s)$ converges for all values of s (i.e., the ROC is the entire complex plane).

4. If the signal $x(t)$ is right sided and the (vertical) line $\text{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X(s) = \mathscr{L}\{x(t)\}$, then all values of s for which $\text{Re}\{s\} > \sigma_0$ must also be in the ROC (i.e., the ROC is a right-half plane including $\text{Re}\{s\} = \sigma_0$).

5. If the signal $x(t)$ is left sided and the (vertical) line $\text{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X(s) = \mathscr{L}\{x(t)\}$, then all values of s for which $\text{Re}\{s\} < \sigma_0$ must also be in the ROC (i.e., the ROC is a left-half plane including $\text{Re}\{s\} = \sigma_0$).

6. If the signal $x(t)$ is two sided and the (vertical) line $\text{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X(s) = \mathscr{L}\{x(t)\}$, then the ROC will consist of a strip in the complex plane that includes the line $\text{Re}\{s\} = \sigma_0$.

7. If the Laplace transform $X(s)$ of the signal $x(t)$ is rational, then:

   (a) If $x(t)$ is right sided, the ROC of $X(s)$ is to the right of the rightmost pole of $X(s)$ (i.e., the right-half plane to the right of the rightmost pole).

   (b) If $x(t)$ is left sided, the ROC of $X(s)$ is to the left of the leftmost pole of $X(s)$ (i.e., the left-half plane to the left of the leftmost pole).

Note that some of the above properties are redundant. For example, properties 1, 2, and 4 imply property 7(a). Also, properties 1, 2, and 5 imply property 7(b). Moreover, since every function can be classified as one of left sided (but not right sided), right sided (but not left sided), two sided, or finite duration, we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a left-half plane, a right-half plane, a (single) vertical strip, the entire complex plane, or the empty set. Thus, the ROC must be a connected region. That is, it cannot consist of multiple (unconnected) vertical strips.

Example 6.5. The Laplace transform $X(s)$ of the signal $x(t)$ has the algebraic expression

$$X(s) = \frac{s + \frac{1}{2}}{(s^2 + 2s + 2)(s^2 + s - 2)}.$$

Identify all of the possible ROCs of $X(s)$. For each ROC, indicate whether the corresponding signal $x(t)$ is left sided, right sided, two sided, or finite duration.
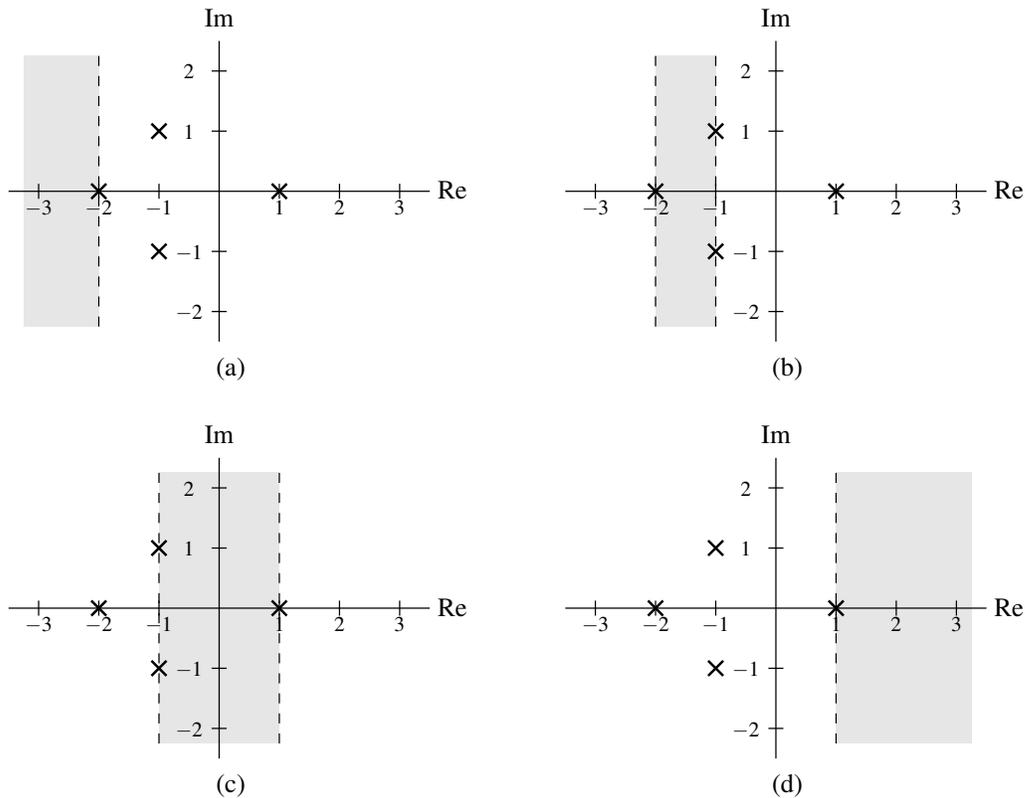
Figure 6.5: ROCs for example.

Solution. The possible ROCs associated with $X(s)$ are determined by the poles of this function. So, we must find the poles of $X(s)$. Factoring the denominator of $X(s)$, we obtain

$$X(s) = \frac{s + \frac{1}{2}}{(s+1-j)(s+1+j)(s+2)(s-1)}.$$

Thus, $X(s)$ has poles at $-1-j$, $-1+j$, -2, and 1. Since these poles only have three distinct real parts, there are four possible ROCs: i) $\operatorname{Re}\{s\} < -2$, ii) $-2 < \operatorname{Re}\{s\} < -1$, iii) $-1 < \operatorname{Re}\{s\} < 1$, and iv) $\operatorname{Re}\{s\} > 1$. These ROCs are plotted in Figures 6.5(a), (b), (c), and (d), respectively. The first ROC is a left-half plane, so the corresponding $x(t)$ must be a left-sided signal. The second ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding $x(t)$ must be a two-sided signal. The third ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding $x(t)$ must be a two-sided signal. The fourth ROC is a right-half plane, so the corresponding $x(t)$ must be a right-sided signal.   □

6.7 Properties of the Laplace Transform

The Laplace transform has a number of important properties. In the sections that follow, we introduce several of these properties. For the convenience of the reader, the properties described in the subsequent sections are also summarized in Table 6.1 (on page 190).

6.7.1 Linearity

If $x_1(t) \overset{\mathscr{L}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

$$a_1 x_1(t) + a_2 x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} a_1 X_1(s) + a_2 X_2(s) \quad \text{with ROC } R \text{ containing } R_1 \cap R_2,$$

where a_1 and a_2 are arbitrary complex constants. This is known as the linearity property of the Laplace transform.

The above property is easily proven as follows. Using the definition of the Laplace transform and straightforward algebraic manipulation, we have

$$\mathscr{L}\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-st} dt$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t) e^{-st} dt + \int_{-\infty}^{\infty} a_2 x_2(t) e^{-st} dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

$$= a_1 X_1(s) + a_2 X_2(s).$$

The ROC R can be deduced as follows. If $X_1(s)$ and $X_2(s)$ both converge for some s, say $s = \lambda$, then any linear combination of these functions must also converge for $s = \lambda$. Therefore, the ROC R must contain the intersection of R_1 and R_2. Thus, we have shown that the linearity property holds.

Example 6.6 (Linearity property of the Laplace transform). Find the Laplace transform of the signal $x(t) = x_1(t) + x_2(t)$ where

$$x_1(t) = e^{-t} u(t) \quad \text{and}$$
$$x_2(t) = e^{-t} u(t) - e^{-2t} u(t).$$

Solution. Using Table 6.2, we have

$$X_1(s) = \mathscr{L}\{e^{-t} u(t)\}$$

$$= \frac{1}{s+1} \quad \text{for } \text{Re}\{s\} > -1 \quad \text{and}$$

$$X_2(s) = \mathscr{L}\{e^{-t} u(t) - e^{-2t} u(t)\}$$

$$= \mathscr{L}\{e^{-t} u(t)\} - \mathscr{L}\{e^{-2t} u(t)\}$$

$$= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \text{Re}\{s\} > -1$$

$$= \frac{1}{(s+1)(s+2)} \quad \text{for } \text{Re}\{s\} > -1.$$

So, from the definition of $X(s)$, we can write

$$X(s) = \mathscr{L}\{x_1(t) + x_2(t)\}$$

$$= X_1(s) + X_2(s)$$

$$= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)}$$

$$= \frac{s+2+1}{(s+1)(s+2)}$$

$$= \frac{s+3}{(s+1)(s+2)}.$$

Now, we must determine the ROC of $X(s)$. We know that the ROC of $X(s)$ must contain the intersection of the ROCs of $X_1(s)$ and $X_2(s)$. So, the ROC must contain $\text{Re}\{s\} > -1$. Furthermore, the ROC cannot be larger than this

(a)

(b)

(c)

(d)
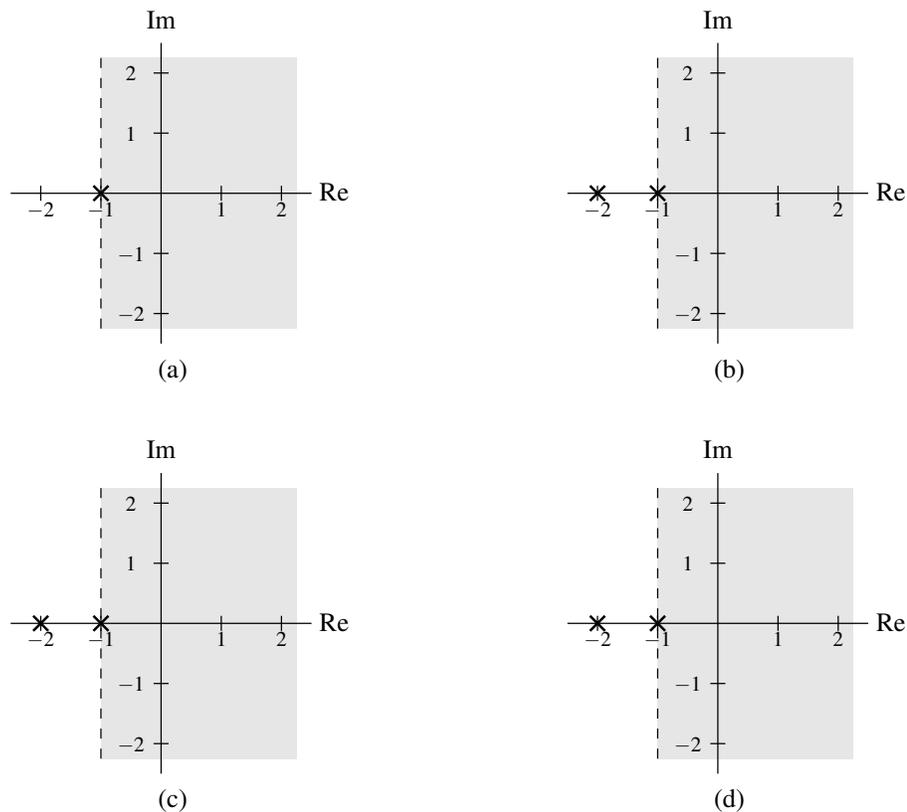
Figure 6.6: ROCs for linearity example. (a) ROC of $X_1(s)$, (b) ROC of $X_2(s)$, (c) ROC associated with the intersection of the ROCs of $X_1(s)$ and $X_2(s)$, and (d) ROC of $X(s)$.

intersection, since $X(s)$ has a pole at -1. (The various ROCs are illustrated in Figure 6.6.) Therefore, the ROC of $X(s)$ is $\text{Re}\{s\} > -1$. Thus, we have

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \text{Re}\{s\} > -1.$$

$\square$

Example 6.7 (Linearity property of the Laplace transform and pole-zero cancellation). Find the Laplace transform $X(s)$ of the signal $x(t) = x_1(t) - x_2(t)$ where $x_1(t)$ and $x_2(t)$ are as defined in the previous example.

Solution. From the previous example, we know that

$$X_1(s) = \frac{1}{s+1} \quad \text{for } \text{Re}\{s\} > -1 \quad \text{and}$$

$$X_2(s) = \frac{1}{(s+1)(s+2)} \quad \text{for } \text{Re}\{s\} > -1.$$
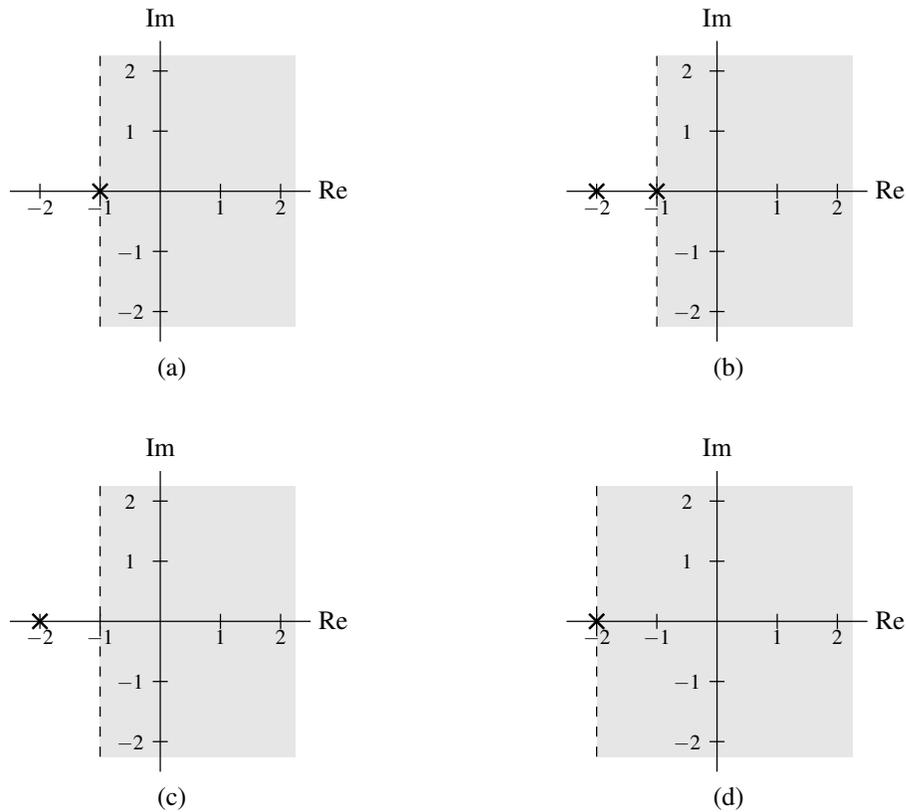
Figure 6.7: ROCs for linearity example. (a) ROC of $X_1(s)$, (b) ROC of $X_2(s)$, (c) ROC associated with the intersection of the ROCs of $X_1(s)$ and $X_2(s)$, and (d) ROC of $X(s)$.

From the definition of $X(s)$, we have

$$
\begin{aligned}
X(s) &= \mathcal{L}\{x_1(t) - x_2(t)\} \\
&= X_1(s) - X_2(s) \\
&= \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} \\
&= \frac{s+2-1}{(s+1)(s+2)} \\
&= \frac{s+1}{(s+1)(s+2)} \\
&= \frac{1}{s+2}.
\end{aligned}
$$

Now, we must determine the ROC of $X(s)$. We know that the ROC of $X(s)$ must at least contain the intersection of the ROCs of $X_1(s)$ and $X_2(s)$. Therefore, the ROC must contain $\text{Re}\{s\} > -1$. Since $X(s)$ is rational, we also know that the ROC must be bounded by poles or extend to infinity. Since $X(s)$ has only one pole and this pole is at -2, the ROC must also include $-2 < \text{Re}\{s\} < -1$. Therefore, the ROC of $X(s)$ is $\text{Re}\{s\} > -2$. In effect, the pole at -1 has been cancelled by a zero at the same location. As a result, the ROC of $X(s)$ is larger than the intersection of the ROCs of $X_1(s)$ and $X_2(s)$. The various ROCs are illustrated in Figure 6.7. □

6.7.2 Time-Domain Shifting

If $x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x(t - t_0) \overset{\mathscr{L}}{\longleftrightarrow} e^{-st_0} X(s) \quad \text{with ROC } R,$$

where t_0 is an arbitrary real constant. This is known as the time-domain shifting property of the Laplace transform.

To prove the above property, we proceed as follows. From the definition of the Laplace transform, we have

$$\mathscr{L}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0) e^{-st} dt.$$

Now, we perform a change of variable. Let $\tau = t - t_0$ so that $t = \tau + t_0$ and $d\tau = dt$. Applying this change of variable, we obtain

$$\begin{aligned} Y(s) = \mathscr{L}\{x(t - t_0)\} &= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau + t_0)} d\tau \\ &= e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau \\ &= e^{-st_0} X(s). \end{aligned}$$

The ROC of $Y(s)$ is the same as the ROC of $X(s)$, since $Y(s)$ and $X(s)$ differ only by a finite constant factor (i.e., e^{-st_0}). Thus, we have proven that the time-domain shifting property holds.

Example 6.8 (Time-domain shifting property). Find the Laplace transform of

$$x(t) = u(t - 1).$$

Solution. From a Laplace transform table, we can write

$$u(t) \overset{\mathscr{L}}{\longleftrightarrow} 1/s \quad \text{for } \text{Re}\{s\} > 0.$$

Using the time-domain shifting property, we can deduce

$$u(t - 1) \overset{\mathscr{L}}{\longleftrightarrow} e^{-s} \left(\tfrac{1}{s}\right) \quad \text{for } \text{Re}\{s\} > 0.$$

Therefore, we have

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{e^{-s}}{s} \quad \text{for } \text{Re}\{s\} > 0.$$

$\square$

6.7.3 Laplace-Domain Shifting

If $x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$e^{s_0 t} x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s - s_0) \quad \text{with ROC } R + \text{Re}\{s_0\},$$

where s_0 is an arbitrary complex constant. This is known as the Laplace-domain shifting property of the Laplace transform. The ROCs are illustrated in Figure 6.8.

To prove the above property, we proceed as follows. Using the definition of the Laplace transform and straightforward manipulation, we obtain

$$\begin{aligned} Y(s) = \mathscr{L}\{e^{s_0 t} x(t)\} &= \int_{-\infty}^{\infty} e^{s_0 t} x(t) e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-(s - s_0)t} dt \\ &= X(s - s_0). \end{aligned}$$
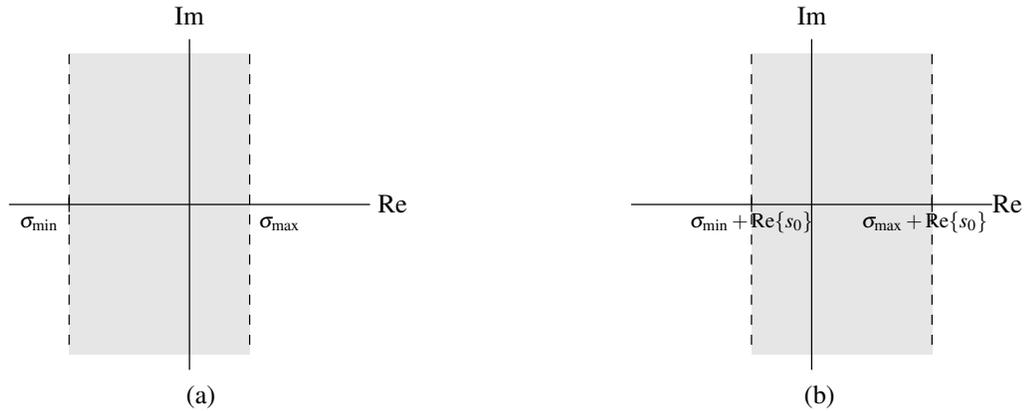
Figure 6.8: Regions of convergence for Laplace-domain shifting. (a) Before shift. (b) After shift.

Since $Y(s+s_0) = X(s)$, $Y(s)$ converges at $s = \lambda + s_0$ if and only if $X(s)$ converges at $s = \lambda$. Since the convergence properties of a Laplace transform only depend on the real part of the s parameter, $Y(s)$ converges at $s = \lambda + \operatorname{Re}\{s_0\}$ if and only if $X(s)$ converges at $s = \lambda$. Consequently, the ROC of $Y(s)$ is simply the ROC of $X(s)$ shifted by $\operatorname{Re}\{s_0\}$. Thus, we have shown that the Laplace-domain shifting property holds.

Example 6.9 (Laplace-domain shifting property). Using only the properties of the Laplace transform and the transform pair

$$e^{-|t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}\{s\} < 1,$$

find the Laplace transform of

$$x(t) = e^{5t} e^{-|t|}.$$

Solution. We are given

$$e^{-|t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}\{s\} < 1.$$

Using the Laplace-domain shifting property, we can deduce

$$e^{5t} e^{-|t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{2}{1-(s-5)^2} \quad \text{for } -1+5 < \operatorname{Re}\{s\} < 1+5.$$

Simplifying, we have

$$\frac{2}{1-(s-5)^2} = \frac{2}{1-(s^2-10s+25)} = \frac{2}{-s^2+10s-24} = \frac{-2}{s^2-10s+24} = \frac{-2}{(s-6)(s-4)}.$$

Therefore, we have

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{-2}{(s-4)(s-6)} \quad \text{for } 4 < \operatorname{Re}\{s\} < 6.$$

$\square$

6.7.4 Time-Domain/Laplace-Domain Scaling

If $x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x(at) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC } R_1 = aR,$$

where a is a nonzero real constant. This is known as the (time-domain/Laplace-domain) scaling property of the Laplace transform. The ROCs associated with scaling are illustrated in Figure 6.9.

To prove the above property, we proceed as below. From the definition of the Laplace transform, we have

$$Y(s) = \mathscr{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt.$$

Now, we perform a change of variable. Let $\tau = at$ so that $t = \tau/a$ and $d\tau = adt$. Performing the change of variable (and being mindful of the change in the limits of integration), we obtain

$$Y(s) = \begin{cases} \int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}\left(\frac{1}{a}\right)d\tau & \text{for } a > 0 \\ \int_{\infty}^{-\infty} x(\tau)e^{-s\tau/a}\left(\frac{1}{a}\right)d\tau & \text{for } a < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}d\tau & \text{for } a > 0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}d\tau & \text{for } a < 0. \end{cases}$$

Combining the two cases for a (i.e., $a > 0$ and $a < 0$), we obtain

$$Y(s) = \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-s\tau/a}d\tau$$

$$= \frac{1}{|a|}X(s/a).$$

Since $|a|Y(as) = X(s)$, $Y(s)$ converges at $s = a\lambda$ if and only if $X(s)$ converges at $s = \lambda$. Thus, the ROC of $Y(s)$ is aR. Thus, we have proven that the scaling property holds.

Example 6.10 (Time-domain scaling property). Using only properties of the Laplace transform and the transform pair

$$e^{-|t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}\{s\} < 1,$$

find the Laplace transform of

$$x(t) = e^{-|3t|}.$$

Solution. We are given

$$e^{-|t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}\{s\} < 1.$$

Using the time-domain scaling property, we can deduce

$$e^{-|3t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{|3|}\frac{2}{1-(\frac{s}{3})^2} \quad \text{for } 3(-1) < \text{Re}\{s\} < 3(1).$$

Simplifying, we have

$$\frac{1}{3}\frac{2}{1-(\frac{s}{3})^2} = \frac{2}{3(1-\frac{s^2}{9})} = \frac{2}{3(\frac{9-s^2}{9})} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{-6}{(s+3)(s-3)} \quad \text{for } -3 < \text{Re}\{s\} < 3.$$

$\square$
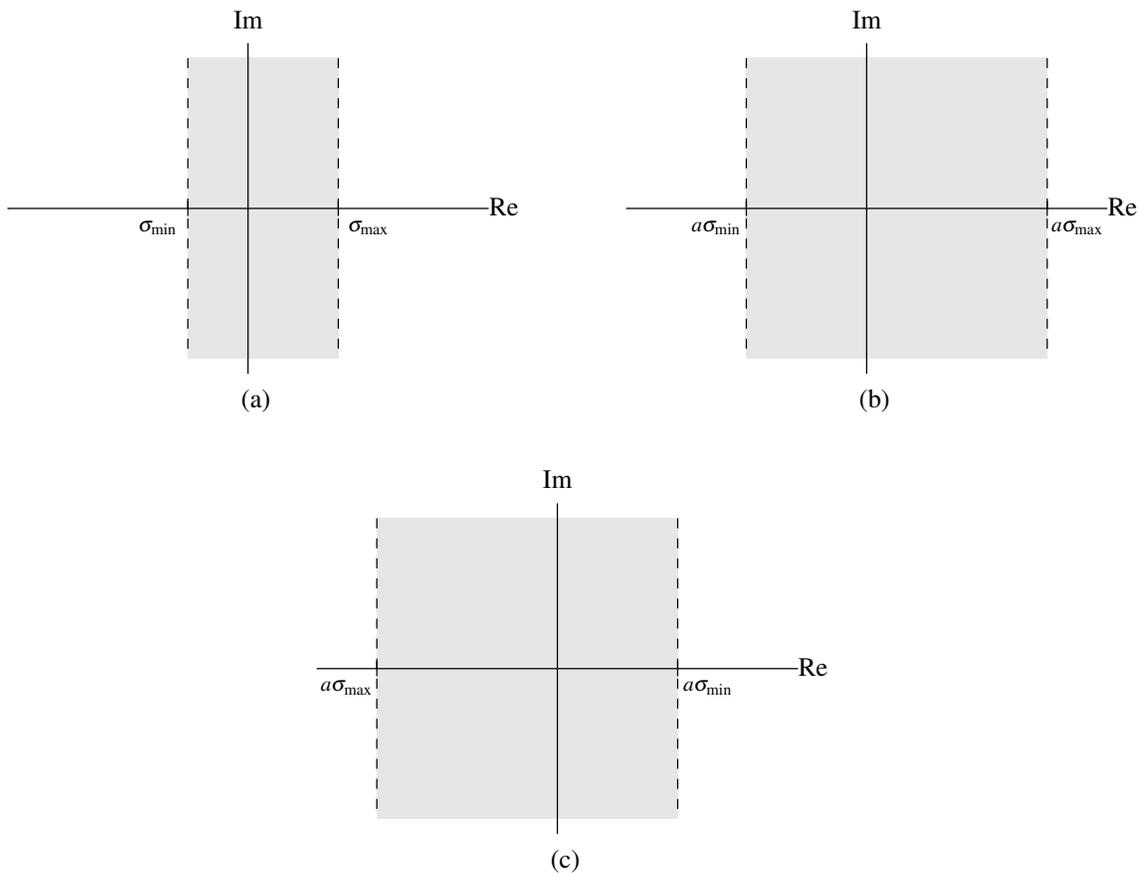
Figure 6.9: Regions of convergence for time-domain/Laplace-domain scaling. (a) Before scaling. After scaling for (b) $a > 0$ and (c) $a < 0$.

6.7.5 Conjugation

If $x(t) \xleftrightarrow{\mathscr{L}} X(s)$ with ROC R, then

$$x^*(t) \xleftrightarrow{\mathscr{L}} X^*(s^*) \quad \text{with ROC } R.$$

This is known as the conjugation property of the Laplace transform.

To prove the above property, we proceed as follows. Let $s = \sigma + j\omega$. From the definition of the Laplace transform and the properties of conjugation, we can write

$$Y(s) = \mathscr{L}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)e^{-st}\,dt$$

$$= \left[\left[\left[\int_{-\infty}^{\infty} x^*(t)e^{-st}\,dt\right]^*\right]\right]^*$$

$$= \left[\int_{-\infty}^{\infty} [x^*(t)]^* (e^{-st})^*\,dt\right]^*$$

$$= \left[\int_{-\infty}^{\infty} x(t)(e^{-st})^*\,dt\right]^*.$$

Now, we observe that $(e^{-st})^* = e^{-s^*t}$. Thus, we can write

$$\mathscr{L}\{x^*(t)\} = \left[\int_{-\infty}^{\infty} x(t)e^{-s^*t}\,dt\right]^*$$

$$= X^*(s^*).$$

We determine the ROC of $Y(s)$ as follows. First, we observe that $X(s) = Y^*(s^*)$. Since $Y^*(s^*) = X(s)$, $Y(s)$ converges at $s = \lambda$ if and only if $X(s)$ converges at $s = \lambda^*$. We know, however, that convergence only depends on the real part of λ. So, $Y(s)$ converges at $s = \lambda$ if and only if $X(s)$ converges at $s = \lambda$. From these results, we have that the ROC of $Y(s)$ must be identical to the ROC of $X(s)$. Thus, we have shown that the conjugation property holds.

Example 6.11 (Conjugation property). Using only properties of the Laplace transform and the transform pair

$$e^{(-1-j)t}u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s+1+j} \quad \text{for } \text{Re}\{s\} > -1,$$

find the Laplace transform of

$$x(t) = e^{(-1+j)t}u(t).$$

Solution. We have

$$x(t) = \left(\left(e^{(-1+j)t}u(t)\right)^*\right)^*$$

$$= \left(\left(e^{(-1+j)t}\right)^* u^*(t)\right)^*$$

$$= \left[e^{(-1-j)t}u(t)\right]^*.$$

We are given

$$e^{(-1-j)t}u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s+1+j} \quad \text{for } \text{Re}\{s\} > -1.$$

Using the conjugation property, we can deduce

$$x(t) \xleftrightarrow{\mathscr{L}} \left[\frac{1}{s^*+1+j}\right]^* \quad \text{for } \text{Re}\{s\} > -1.$$

Simplifying, we have

$$\left[\frac{1}{s^*+1+j}\right]^* = \frac{1^*}{[s^*+1+j]^*} = \frac{1}{s+1-j}.$$

Therefore, we can conclude

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+1-j} \quad \text{for } \operatorname{Re}\{s\} > -1.$$

$\square$

6.7.6 Time-Domain Convolution

If $x_1(t) \overset{\mathscr{L}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

$$x_1(t) * x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} X_1(s)X_2(s) \quad \text{with ROC containing } R_1 \cap R_2.$$

This is known as the time-domain convolution property of the Laplace transform.

To prove the above property, we proceed as below. From the definition of the Laplace transform and convolution, we have

$$Y(s) = \mathscr{L}\{x_1(t) * x_2(t)\} = \mathscr{L}\left\{\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right\}$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-st}dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st}d\tau dt.$$

Changing the order of integration, we have

$$Y(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)e^{-st}dt d\tau$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st}dt\right]x_1(\tau)d\tau.$$

Now, we perform a change of variable. Let $v = t - \tau$ so that $t = v + \tau$ and $dv = dt$. Applying the change of variable and simplifying, we obtain

$$Y(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(v)e^{-s(v+\tau)}dv\right]x_1(\tau)d\tau$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2(v)e^{-sv}dv\right]e^{-s\tau}x_1(\tau)d\tau$$

$$= \left[\int_{-\infty}^{\infty} x_2(v)e^{-sv}dv\right]\left[\int_{-\infty}^{\infty} e^{-s\tau}x_1(\tau)d\tau\right]$$

$$= X_1(s)X_2(s).$$

Now, we consider the ROC of $Y(s)$. If $X_1(s)$ and $X_2(s)$ both converge for some $s = \lambda$, then $Y(s) = X_1(s)X_2(s)$ must also converge for $s = \lambda$. Therefore, the ROC of $Y(s)$ must contain the intersection of the ROCs of $X_1(s)$ and $X_2(s)$. Thus, we have shown that the time-domain convolution property holds.

Example 6.12 (Time-domain convolution property). Find the Laplace transform of

$$x(t) = ([\sin 3t]u(t)) * (tu(t)).$$

Solution. From a Laplace transform table, we have

$$[\sin 3t]u(t) \xleftrightarrow{\mathscr{L}} \frac{3}{s^2+9} \quad \text{for Re}\{s\} > 0 \quad \text{and}$$

$$tu(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s^2} \quad \text{for Re}\{s\} > 0.$$

Using the time-domain convolution property, we have

$$([\sin 3t]u(t)) * (tu(t)) \xleftrightarrow{\mathscr{L}} \left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right) \quad \text{for Re}\{s\} > 0 \cap \text{Re}\{s\} > 0.$$

Simplifying, we have

$$\left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right) = \frac{3}{s^2(s^2+9)}.$$

Therefore, we can conclude

$$x(t) \xleftrightarrow{\mathscr{L}} \frac{3}{s^2(s^2+9)} \quad \text{for Re}\{s\} > 0.$$

$\square$

6.7.7   Time-Domain Differentiation

If $x(t) \xleftrightarrow{\mathscr{L}} X(s)$ with ROC R, then

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathscr{L}} sX(s) \quad \text{with ROC containing } R.$$

This is known as the time-domain differentiation property of the Laplace transform.

To prove the above property, we proceed as follows. From the definition of the inverse Laplace transform, we have

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds.$$

Differentiating both sides of this equation with respect to t, we have

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st} ds.$$

Observing that the right-hand side of the above equation is simply the inverse Laplace transform of $sX(s)$, we can write

$$\frac{d}{dt}x(t) = \mathscr{L}^{-1}\{sX(s)\}.$$

Taking the Laplace transform of both sides yields

$$\mathscr{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s).$$

Now, we consider the ROC of $Y(s) = sX(s)$. Clearly, $Y(s)$ must converge for $s = \lambda$ if $X(s)$ converges for $s = \lambda$. Since multiplication by s has the potential to cancel a pole in $X(s)$, it is possible that the ROC of $Y(s)$ may be larger than the ROC of $X(s)$. Consequently, the ROC of $Y(s)$ must at least contain R. Thus, we have shown that the time-domain differentiation property holds.

Example 6.13 (Time-domain differentiation property)**.** Find the Laplace transform of

$$x(t) = \frac{d}{dt}\delta(t).$$

Solution. From a Laplace transform table, we can write

$$\delta(t) \overset{\mathscr{L}}{\longleftrightarrow} 1 \quad \text{for all } s.$$

Using the time-domain differentiation property, we can deduce

$$\frac{d}{dt}\delta(t) \overset{\mathscr{L}}{\longleftrightarrow} s(1) \quad \text{for all } s.$$

Therefore, we have

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} s \quad \text{for all } s.$$

$\square$

6.7.8 Laplace-Domain Differentiation

If $x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s)$ with ROC R, then

$$-tx(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{dX(s)}{ds} \quad \text{with ROC } R.$$

This is known as the Laplace-domain differentiation property of the Laplace transform.

To prove the above property, we proceed as follows. From the definition of the Laplace transform, we have

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

Differentiating both sides of the above equation with respect to s and simplifying, we obtain

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} -tx(t)e^{-st}dt$$
$$= \mathscr{L}\{-tx(t)\}.$$

Thus, we have shown that the Laplace-domain differentiation property holds.

Example 6.14 (Laplace-domain differentiation property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-2t}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for Re}\{s\} > -2,$$

find the Laplace transform of

$$x(t) = te^{-2t}u(t).$$

Solution. We are given

$$e^{-2t}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for Re}\{s\} > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

$$te^{-2t}u(t) \overset{\mathscr{L}}{\longleftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+2}\right] \quad \text{for Re}\{s\} > -2.$$

Simplifying, we have

$$-\frac{d}{ds}\left[\frac{1}{s+2}\right] = -\frac{d}{ds}(s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we can conclude

$$x(t) \xleftrightarrow{\mathscr{L}} \frac{1}{(s+2)^2} \quad \text{for } \text{Re}\{s\} > -2.$$

$\square$

6.7.9 Time-Domain Integration

If $x(t) \xleftrightarrow{\mathscr{L}} X(s)$ with ROC R, then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathscr{L}} \frac{1}{s}X(s) \quad \text{with ROC containing } R \cap \{\text{Re}\{s\} > 0\}.$$

This is known as the time-domain integration property of the Laplace transform.

To prove the above property, we proceed as below. First, we observe that

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t).$$

Taking the Laplace transform of both sides of this equation, we have

$$Y(s) = \mathscr{L}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \mathscr{L}\{x(t) * u(t)\}.$$

From the time-domain convolution property of the Laplace transform, we have

$$Y(s) = \mathscr{L}\{x(t)\}\mathscr{L}\{u(t)\}$$
$$= X(s)\mathscr{L}\{u(t)\}.$$

From Example 6.3, we know that $\mathscr{L}\{u(t)\} = U(s) = \frac{1}{s}$ for $\text{Re}\{s\} > 0$. So,

$$Y(s) = \frac{1}{s}X(s).$$

Now, we need to consider the ROC of $Y(s)$. Clearly, $Y(s)$ must converge at $s = \lambda$ if $X(s)$ and $U(s)$ both converge at $s = \lambda$. Consequently, the ROC of $Y(s)$ must contain the intersection of the ROCs of $X(s)$ and $U(s)$. Since $U(s)$ converges for $\text{Re}\{s\} > 0$, the ROC of $Y(s)$ must contain $R \cap \{\text{Re}\{s\} > 0\}$. Thus, we have shown that the time-domain integration property holds.

Example 6.15 (Time-domain integration property). Find the Laplace transform of

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}[\sin\tau]u(\tau)d\tau.$$

Solution. From a Laplace transform table, we can write

$$e^{-2t}[\sin t]u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{(s+2)^2+1} \quad \text{for } \text{Re}\{s\} > -2.$$

Using the time-domain integration property, we can deduce

$$\int_{-\infty}^{t} e^{-2\tau}[\sin\tau]u(\tau)d\tau \xleftrightarrow{\mathscr{L}} \frac{1}{s}\left(\frac{1}{(s+2)^2+1}\right) \quad \text{for } \text{Re}\{s\} > -2 \cap \text{Re}\{s\} > 0.$$

Simplifying, we have

$$\frac{1}{s}\left(\frac{1}{(s+2)^2+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+4+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+5}\right).$$

Therefore, we have

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s(s^2+4s+5)} \quad \text{for } \mathrm{Re}\{s\} > 0.$$

[Note: $s^2+4s+5 = (s+2-j)(s+2+j)$.] □

6.7.10 Initial Value Theorem

If $x(t) = 0$ for all $t < 0$ and $x(t)$ contains no impulses or higher order singularities at the origin, then

$$x(0^+) = \lim_{s\to\infty} sX(s),$$

where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t. This result is known as the **initial value theorem**.

To prove the above property, we proceed as below. First, we expand $x(t)$ as a Taylor series at $t = 0^+$.

$$x(t) = \left[x(0^+) + x^{(1)}(0^+)t + \ldots + x^{(n)}(0^+)\frac{t^n}{n!} + \ldots\right]u(t).$$

where $x^{(n)}(0^+)$ denotes the nth derivative of $x(t)$ evaluated at $t = 0^+$. Taking the Laplace transform of the above equation, and using the fact that $\mathscr{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}$, we can write

$$\mathscr{L}\{x(t)\} = X(s) = x(0^+)\frac{1}{s} + x^{(1)}(0^+)\frac{1}{s^2} + \ldots + x^{(n)}(0^+)\frac{1}{s^{n+1}} + \ldots$$

Taking the limit, we have

$$\lim_{s\to\infty} sX(s) = x(0^+).$$

Thus, the initial value theorem holds.

6.7.11 Final Value Theorem

If $x(t) = 0$ for all $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

This result is known as the **final value theorem**.

The above property can be proven by using a partial fraction expansion of $X(s)$. This proof, however, is left as an exercise for the reader.

Example 6.16 (Initial and final value theorems). Consider the signal $x(t)$ given by

$$x(t) = [1 + e^{-t}\cos t]u(t).$$

This signal can be shown to have the Laplace transform $X(s)$ given by

$$X(s) = \frac{2s^2+3s+2}{s^3+2s^2+2s}.$$

Use the initial and final value theorems in order to determine the quantities $x(0^+)$ and $\lim_{t\to\infty} x(t)$.

Table 6.1: Laplace Transform Properties

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + \text{Re}\{s_0\}$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{|a|}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\text{Re}\{s\} > 0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim\limits_{s\to\infty} sX(s)$
Final Value Theorem	$\lim\limits_{t\to\infty} x(t) = \lim\limits_{s\to 0} sX(s)$

Solution. Since $x(t)$ is causal (i.e., $x(t) = 0$ for all $t < 0$) and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$
\begin{aligned}
x(0^+) &= \lim_{s\to\infty} sX(s) \\
&= \lim_{s\to\infty} s\left[\frac{2s^2 + 3s + 2}{s^3 + 2s^2 + 2s}\right] \\
&= \lim_{s\to\infty} \frac{2s^2 + 3s + 2}{s^2 + 2s + 2} \\
&= 2.
\end{aligned}
$$

We can easily verify that this result is correct from the expression given for $x(t)$ in the original problem statement.

Since $x(t)$ is causal and has a finite limit at infinity, we can apply the final value theorem. From this theorem, we have

$$
\begin{aligned}
\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \\
&= \lim_{s\to 0} s\left[\frac{2s^2 + 3s + 2}{s^3 + 2s^2 + 2s}\right] \\
&= \left.\frac{2s^2 + 3s + 2}{s^2 + 2s + 2}\right|_{s=0} \\
&= 1.
\end{aligned}
$$

Again, we can easily verify that this result is correct from the expression given for $x(t)$ in the original problem statement.

Sometimes the initial and final value theorems are useful for checking for errors in Laplace transform calculations. For example, if we had made a mistake in computing $X(s)$, the values obtained from the initial and final value theorems would most likely disagree with the values obtained directly from the original expression for $x(t)$. □

Table 6.2: Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} < 0$
6	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > -a$
7	$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < -a$
8	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > -a$
9	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} < -a$
10	$[\cos \omega_0 t] u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
11	$[\sin \omega_0 t] u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
12	$[e^{-at} \cos \omega_0 t] u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$
13	$[e^{-at} \sin \omega_0 t] u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$

6.8 More Laplace Transform Examples

Earlier in this chapter, we derived a number of Laplace transform pairs. Some of these and other important transform pairs are listed in Table 6.2. Using the various Laplace transform properties listed in Table 6.1 and the Laplace transform pairs listed in Table 6.2, we can more easily determine the Laplace transform of more complicated signals.

Example 6.17. Using properties of the Laplace transform and the Laplace transform pair $e^{-a|t|} \overset{\mathscr{L}}{\longleftrightarrow} \frac{-2a}{(s+a)(s-a)}$ for $-a < \mathrm{Re}\{s\} < a$, find the Laplace transform $X(s)$ of the signal $x(t) = e^{-5|3t-7|}$.

Solution. We begin by re-expressing $x(t)$ in terms of the following equations:

$$v_1(t) = e^{-5|t|},$$
$$v_2(t) = v_1(t-7), \quad \text{and}$$
$$x(t) = v_2(3t).$$

In what follows, let R_{V_1}, R_{V_2}, and R_X denote the ROCs of $V_1(s)$, $V_2(s)$, and $X(s)$, respectively. Taking the Laplace transform of the above three equations, we obtain:

$$V_1(s) = \frac{-10}{(s+5)(s-5)}, \quad R_{V_1} = (-5 < \mathrm{Re}\{s\} < 5)$$
$$V_2(s) = e^{-7s}V_1(s), \quad R_{V_2} = R_{V_1}$$
$$X(s) = \tfrac{1}{3}V_2(s/3), \quad R_X = 3R_{V_2}.$$

Combining the above equations, we have

$$\begin{aligned}
X(s) &= \tfrac{1}{3}V_2(s/3) \\
&= \tfrac{1}{3}e^{-7(s/3)}V_1(s/3) \\
&= \tfrac{1}{3}e^{-7s/3}V_1(s/3) \\
&= \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)}, \quad \text{and}
\end{aligned}$$

$$R_X = 3R_{V_2}$$
$$= 3R_{V_1}$$
$$= 3(-5 < \mathrm{Re}\{s\} < 5)$$
$$= -15 < \mathrm{Re}\{s\} < 15.$$

Thus, we have shown that

$$e^{-5|3t-7|} \overset{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \quad \text{for } -15 < \mathrm{Re}\{s\} < 15.$$

$\square$

Example 6.18. Find the Laplace transform $X(s)$ of the signal $x(t) = [e^{-t} + e^{-2t}]u(t)$.

Solution. We can calculate $X(s)$ as

$$
\begin{aligned}
X(s) = \mathscr{L}\{x(t)\} &= \mathscr{L}\{[e^{-t} + e^{-2t}]u(t)\} \\
&= \mathscr{L}\{e^{-t}u(t)\} + \mathscr{L}\{e^{-2t}u(t)\} \\
&= \frac{1}{s+1} + \frac{1}{s+2} \quad \text{for } \mathrm{Re}\{s\} > -1 \cap \mathrm{Re}\{s\} > -2 \\
&= \frac{s+2+s+1}{(s+1)(s+2)} \\
&= \frac{2s+3}{(s+1)(s+2)} \quad \text{for } \mathrm{Re}\{s\} > -1.
\end{aligned}
$$

Thus, we have that

$$[e^{-t} + e^{-2t}]u(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{2s+3}{(s+1)(s+2)} \quad \text{for } \mathrm{Re}\{s\} > -1.$$

$\square$

Example 6.19. Find the Laplace transform $X(s)$ of the signal $x(t) = [e^{-2t} + e^{-3t}]u(t-1)$.

Solution. To begin, we rewrite $x(t)$ as

$$x(t) = [e^{-2t} + e^{-3t}]v_1(t)$$

where

$$v_1(t) = u(t-1).$$

Taking the Laplace transform of the preceding equations, we have

$$
\begin{aligned}
V_1(s) = \mathscr{L}\{v_1(t)\} &= \mathscr{L}\{u(t-1)\} \\
&= e^{-s}\left(\tfrac{1}{s}\right) \quad \text{for } \mathrm{Re}\{s\} > 0.
\end{aligned}
$$

and

$$
\begin{aligned}
X(s) = \mathscr{L}\{x(t)\} &= \mathscr{L}\{[e^{-2t} + e^{-3t}]v_1(t)\} \\
&= \mathscr{L}\{e^{-2t}v_1(t)\} + \mathscr{L}\{e^{-3t}v_1(t)\}.
\end{aligned}
$$

Now, we focus on simplifying the preceding expression for $X(s)$. Let R_{V_1} denote the ROC of $V_1(s)$. Then, we have

$$\mathscr{L}\{e^{-2t}v_1(t)\} = V_1(s+2) \quad \text{for } s \in R_{V_1} - 2 \quad \text{and}$$
$$\mathscr{L}\{e^{-3t}v_1(t)\} = V_1(s+3) \quad \text{for } s \in R_{V_1} - 3.$$

Thus, we have

$$X(s) = \mathcal{L}\{e^{-2t}v_1(t)\} + \mathcal{L}\{e^{-3t}v_1(t)\}$$
$$= V_1(s+2) + V_1(s+3) \quad \text{for } (R_{V_1} - 2) \cap (R_{V_1} - 3).$$

Substituting the earlier expression for $V_1(s)$ into the above equation yields

$$X(s) = e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } (\text{Re}\{s\} > -2) \cap (\text{Re}\{s\} > -3)$$
$$= e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } \text{Re}\{s\} > -2.$$

Thus, we have that

$$[e^{-2t} + e^{-3t}]u(t-1) \overset{\mathcal{L}}{\longleftrightarrow} e^{-(s+2)}\frac{1}{s+2} + e^{-(s+3)}\frac{1}{s+3} \quad \text{for } \text{Re}\{s\} > -2.$$

$\square$

Example 6.20. Find the Laplace transform $X(s)$ of the signal $x(t) = \delta(t) + u(t)$.

Solution. Taking the Laplace transform of $x(t)$, we have

$$X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t) + u(t)\}$$
$$= \mathcal{L}\{\delta(t)\} + \mathcal{L}\{u(t)\}.$$

From Table 6.2, we have

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1 \quad \text{for all } s \quad \text{and}$$
$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \tfrac{1}{s} \quad \text{for } \text{Re}\{s\} > 0.$$

Substituting the above results, we obtain

$$X(s) = 1 + \tfrac{1}{s} \quad \text{for } \text{Re}\{s\} > 0$$
$$= \frac{s+1}{s}.$$

Thus, we have that

$$\delta(t) + u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s+1}{s} \quad \text{for } \text{Re}\{s\} > 0.$$

$\square$

Example 6.21. Find the Laplace transform $X(s)$ of the signal $x(t) = te^{-3|t|}$.

Solution. To begin, we rewrite $x(t)$ as

$$x(t) = te^{3t}u(-t) + te^{-3t}u(t).$$

Taking the Laplace transform, we have

$$X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{te^{3t}u(-t) + te^{-3t}u(t)\}$$
$$= \mathcal{L}\{te^{3t}u(-t)\} + \mathcal{L}\{te^{-3t}u(t)\}$$
$$= \frac{-(1!)}{(s-3)^2} + \frac{1!}{(s+3)^2} \quad \text{for } \text{Re}\{s\} > -3 \cap \text{Re}\{s\} < 3$$
$$= \frac{-(s+3)^2 + (s-3)^2}{(s+3)^2(s-3)^2}$$
$$= \frac{-(s^2 + 6s + 9) + s^2 - 6s + 9}{(s+3)^2(s-3)^2}$$
$$= \frac{-12s}{(s+3)^2(s-3)^2}.$$
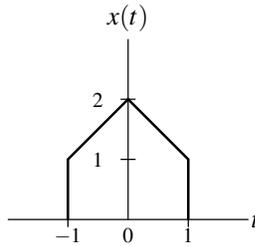
Figure 6.10: Signal for Laplace transform example.

Thus, we have that

$$te^{-3|t|} \xleftarrow{\ \mathscr{L}\ } \frac{-12s}{(s+3)^2(s-3)^2} \quad \text{for } -3 < \mathrm{Re}\{s\} < 3.$$

$\square$

Example 6.22. Suppose that we have the signal $y(t) = e^{-t}[x(t) * x(t)]$. Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively. Find an expression for $Y(s)$ in terms of $X(s)$.

Solution. To begin, we rewrite $y(t)$ as

$$y(t) = e^{-t} v_1(t)$$

where

$$v_1(t) = x(t) * x(t).$$

Taking the Laplace transform of the above equations yields

$$
\begin{aligned}
V_1(s) &= \mathscr{L}\{v_1(t)\} \\
&= \mathscr{L}\{x(t) * x(t)\} \\
&= \mathscr{L}\{x(t)\}\mathscr{L}\{x(t)\} \\
&= X^2(s) \quad \text{for } s \in R_X \quad \text{and} \\
Y(s) &= \mathscr{L}\{e^{-t} v_1(t)\} \\
&= V_1(s+1) \quad \text{for } s \in R_{V_1} - 1
\end{aligned}
$$

where R_X and R_{V_1} denote the ROCs of $X(s)$ and $V_1(s)$, respectively. Substituting the expression for $V_1(s)$ into the above equation, we have

$$
\begin{aligned}
Y(s) &= V_1(s+1) \\
&= X^2(s+1) \quad \text{for } s \in R_X - 1.
\end{aligned}
$$

$\square$

Example 6.23. Let $x(t)$ be as shown in Figure 6.10. Find the Laplace transform $X(s)$ of $x(t)$.

Solution. First, we express $x(t)$ using step functions. We have

$$
\begin{aligned}
x(t) &= (t+2)[u(t+1) - u(t)] + (-t+2)[u(t) - u(t-1)] \\
&= tu(t+1) - tu(t) + 2u(t+1) - 2u(t) - tu(t) + tu(t-1) + 2u(t) - 2u(t-1) \\
&= tu(t+1) + 2u(t+1) - 2tu(t) + tu(t-1) - 2u(t-1).
\end{aligned}
$$

Taking the Laplace transform of both sides of the preceding equation, we obtain

$$X(s) = -\mathscr{L}\{-tu(t+1)\} + 2\mathscr{L}\{u(t+1)\} + 2\mathscr{L}\{-tu(t)\} - \mathscr{L}\{-tu(t-1)\} - 2\mathscr{L}\{u(t-1)\}$$

$$= -\frac{d}{ds}\left[\frac{e^s}{s}\right] + 2\left[\frac{e^s}{s}\right] + 2\frac{d}{ds}\left[\frac{1}{s}\right] - \frac{d}{ds}\left[\frac{e^{-s}}{s}\right] - 2\left[\frac{e^{-s}}{s}\right]$$

$$= -\left(\frac{se^s - e^s}{s^2}\right) + 2\left(\frac{e^s}{s}\right) + 2\left(-\frac{1}{s^2}\right) - \left(\frac{s(-e^{-s}) - e^{-s}}{s^2}\right) - 2\left(\frac{e^{-s}}{s}\right)$$

$$= \frac{-se^s + e^s + se^{-s} + e^{-s} - 2 + 2se^s - 2e^{-s}}{s^2}$$

$$= \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2}.$$

Since $x(t)$ is a (bounded) finite-duration signal, the ROC of $X(s)$ must be the entire complex plane. Thus, we have that

$$x(t) \xleftrightarrow{\mathscr{L}} \frac{se^s + e^s - 2 - se^{-s} + e^{-s}}{s^2} \quad \text{for all } s.$$

$\square$

Alternate solution. Alternatively, we can rearrange our above expression for $x(t)$ to obtain

$$x(t) = [tu(t+1) + u(t+1) - u(t+1)] + 2u(t+1) - 2tu(t) + [tu(t-1) - u(t-1) + u(t-1)] - 2u(t-1)$$

$$= (t+1)u(t+1) + u(t+1) - 2tu(t) + (t-1)u(t-1) - u(t-1).$$

Taking the Laplace transform of both sides of the preceding equation, we have

$$X(s) = \mathscr{L}\left\{\left.\lambda u(\lambda)\right|_{\lambda=t+1}\right\} + \mathscr{L}\{u(t+1)\} - 2\mathscr{L}\{tu(t)\} + \mathscr{L}\left\{\left.\lambda u(\lambda)\right|_{\lambda=t-1}\right\} - \mathscr{L}\{u(t-1)\}$$

$$= e^s \frac{1}{s^2} + e^s \frac{1}{s} - 2\frac{1}{s^2} + e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s}$$

$$= \frac{e^s + se^s - 2 + e^{-s} - se^{-s}}{s^2}.$$

In the case of this alternate solution, the expression for $X(s)$ is considerably easier to simplify. $\square$

6.9 Determination of the Inverse Laplace Transform

As suggested earlier, in practice, we rarely use (6.3) directly in order to compute the inverse Laplace transform. This formula requires a contour integration, which is not usually very easy to compute. Instead, we employ a partial fraction expansion of the function. In so doing, we obtain a number of simpler functions for which we can usually find the inverse Laplace transform in a table (e.g., such as Table 6.2). In what follows, we assume that the reader is already familiar with partial fraction expansions. A tutorial on partial fraction expansions provided in Appendix B for those who might not be acquainted with such expansions.

Example 6.24. Find the inverse Laplace transform of

$$X(s) = \frac{2}{s^2 - s - 2} \quad \text{for } -1 < \text{Re}\{s\} < 2.$$

Solution. We begin by rewriting $X(s)$ in the factored form

$$X(s) = \frac{2}{(s+1)(s-2)}.$$

Then, we find a partial fraction expansion of $X(s)$. We know that $X(s)$ has an expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2}.$$

Calculating the coefficients of the expansion, we obtain

$$A_1 = (s+1)X(s)|_{s=-1}$$
$$= \frac{2}{s-2}\bigg|_{s=-1}$$
$$= -\frac{2}{3} \quad \text{and}$$
$$A_2 = (s-2)X(s)|_{s=2}$$
$$= \frac{2}{s+1}\bigg|_{s=2}$$
$$= \frac{2}{3}.$$

So, $X(s)$ has the expansion

$$X(s) = \frac{2}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right).$$

Taking the inverse Laplace transform of both sides of this equation, we have

$$x(t) = \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}. \tag{6.5}$$

Using Table 6.2 and the given ROC, we have

$$-e^{2t}u(-t) \xleftrightarrow{\mathscr{L}} \frac{1}{s-2} \quad \text{for Re}\{s\} < 2 \quad \text{and}$$
$$e^{-t}u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s+1} \quad \text{for Re}\{s\} > -1.$$

Substituting these results into (6.5), we obtain

$$x(t) = \frac{2}{3}[-e^{2t}u(-t)] - \frac{2}{3}[e^{-t}u(t)]$$
$$= -\frac{2}{3}e^{2t}u(-t) - \frac{2}{3}e^{-t}u(t).$$

$\square$

Example 6.25 (Rational function with a repeated pole). Find the inverse Laplace transform of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for Re}\{s\} > -1.$$

Solution. To begin, we find a partial fraction expansion of $X(s)$. We know that $X(s)$ has an expansion of the form

$$X(s) = \frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{s+2}.$$

Calculating the coefficients of the expansion, we obtain

$$
\begin{aligned}
A_{11} &= \frac{1}{(2-1)!}\left[\left(\frac{d}{ds}\right)^{2-1}[(s+1)^2X(s)]\right]\Bigg|_{s=-1} \\
&= \frac{1}{1!}\left[\frac{d}{ds}[(s+1)^2X(s)]\right]\Bigg|_{s=-1} \\
&= \left[\frac{d}{ds}\left(\frac{2s+1}{s+2}\right)\right]\Bigg|_{s=-1} \\
&= \left[\frac{(s+2)(2)-(2s+1)(1)}{(s+2)^2}\right]\Bigg|_{s=-1} \\
&= \left[\frac{2s+4-2s-1}{(s+2)^2}\right]\Bigg|_{s=-1} \\
&= \left[\frac{3}{(s+2)^2}\right]\Bigg|_{s=-1} \\
&= 3,
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= \frac{1}{(2-2)!}\left[\left(\frac{d}{ds}\right)^{2-2}[(s+1)^2X(s)]\right]\Bigg|_{s=-1} \\
&= \frac{1}{0!}\left[(s+1)^2X(s)\right]\Big|_{s=-1} \\
&= \frac{2s+1}{s+2}\Bigg|_{s=-1} \\
&= \frac{-1}{1} \\
&= -1, \quad \text{and} \\
A_{21} &= (s+2)X(s)\big|_{s=-2} \\
&= \frac{2s+1}{(s+1)^2}\Bigg|_{s=-2} \\
&= \frac{-3}{1} \\
&= -3.
\end{aligned}
$$

Thus, $X(s)$ has the expansion

$$
X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.
$$

Taking the inverse Laplace transform of both sides of this equation yields

$$
x(t) = 3\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}. \tag{6.6}
$$

From Table 6.2 and the given ROC, we have

$$
\begin{aligned}
e^{-t}u(t) &\overset{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \mathrm{Re}\{s\} > -1, \\
te^{-t}u(t) &\overset{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{(s+1)^2} \quad \text{for } \mathrm{Re}\{s\} > -1, \quad \text{and} \\
e^{-2t}u(t) &\overset{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \mathrm{Re}\{s\} > -2.
\end{aligned}
$$

Substituting these results into (6.6), we obtain

$$x(t) = 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t)$$
$$= \left[3e^{-t} - te^{-t} - 3e^{-2t}\right]u(t).$$

$\square$

Example 6.26 (Inverse Laplace transform of improper rational function). Find the inverse Laplace transform $x(t)$ of

$$X(s) = \frac{2s^2 + 4s + 5}{(s+1)(s+2)} \quad \text{for Re}\{s\} > -1.$$

Solution. We begin by noting that $X(s)$ is not a strictly proper rational function. So, we need to perform polynomial long division in order to express $X(s)$ as the sum of a polynomial and a strictly proper rational function. By long division, we have

$$
\begin{array}{r}
2 \\
s^2 + 3s + 2 \overline{\smash{)}\ 2s^2 + 4s + 5} \\
\underline{2s^2 + 6s + 4} \\
-2s + 1.
\end{array}
$$

In other words, we have

$$X(s) = 2 + \frac{-2s+1}{s^2 + 3s + 2}.$$

For convenience, we define

$$V(s) = \frac{-2s+1}{(s+1)(s+2)}$$

so that

$$X(s) = 2 + V(s).$$

Observe that $V(s)$ is a strictly proper rational function. So, we can find a partial fraction expansion of $V(s)$. Now, we find this expansion. We know that such an expansion is of the form

$$V(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients, we have

$$A_1 = (s+1)V(s)|_{s=-1}$$
$$= \left.\frac{-2s+1}{s+2}\right|_{s=-1}$$
$$= 3 \quad \text{and}$$
$$A_2 = (s+2)V(s)|_{s=-2}$$
$$= \left.\frac{-2s+1}{s+1}\right|_{s=-2}$$
$$= -5$$

So, we have

$$X(s) = 2 + V(s)$$
$$= 2 + \frac{3}{s+1} - \frac{5}{s+2}.$$

Taking the inverse Laplace transform, we obtain

$$x(t) = \mathscr{L}^{-1}\{X(s)\}$$
$$= 2\mathscr{L}^{-1}\{1\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - 5\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

Considering the ROC of $X(s)$, we can obtain the following from Table 6.2:

$$\delta(t) \overset{\mathscr{L}}{\longleftrightarrow} 1,$$
$$e^{-t}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \text{Re}\{s\} > -1, \quad \text{and}$$
$$e^{-2t}u(t) \overset{\mathscr{L}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \text{Re}\{s\} > -2.$$

Finally, we can write

$$x(t) = 2\delta(t) + 3e^{-t}u(t) - 5e^{-2t}u(t)$$
$$= 2\delta(t) + [3e^{-t} - 5e^{-2t}]u(t).$$

$\square$

Example 6.27. Find all of the possible inverse Laplace transforms of

$$X(s) = \frac{1}{s^2 + 3s + 2}. \tag{6.7}$$

Solution. We begin by rewriting $X(s)$ in factored form as

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

Then, we find the partial fraction expansion of $X(s)$. We know that such an expansion has the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the coefficients of the expansion, we have

$$A_1 = (s+1)X(s)|_{s=-1} = \left.\frac{1}{s+2}\right|_{s=-1} = 1 \quad \text{and}$$
$$A_2 = (s+2)X(s)|_{s=-2} = \left.\frac{1}{s+1}\right|_{s=-2} = -1.$$

So, we have

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}. \tag{6.8}$$

For the Laplace transform $X(s)$, three possible ROCs exist: i) $\text{Re}\{s\} < -2$, ii) $-2 < \text{Re}\{s\} < -1$, and iii) $\text{Re}\{s\} > -1$. Thus, three possible inverse Laplace transforms exist for $X(s)$, depending on the choice of ROC.

First, let us consider the case of the ROC $\text{Re}\{s\} < -2$. From Table 6.2, we have

$$-e^{-t}u(-t) \xleftrightarrow{\mathscr{L}} \tfrac{1}{s+1} \quad \text{for } \text{Re}\{s\} < -1 \quad \text{and}$$
$$-e^{-2t}u(-t) \xleftrightarrow{\mathscr{L}} \tfrac{1}{s+2} \quad \text{for } \text{Re}\{s\} < -2.$$

Substituting these results into (6.8), we have

$$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$$
$$= \left[-e^{-t} + e^{-2t}\right]u(-t).$$

Thus, we have that

$$\left[-e^{-t} + e^{-2t}\right]u(-t) \xleftrightarrow{\mathscr{L}} \frac{1}{s^2 + 3s + 2} \quad \text{for } \text{Re}\{s\} < -2.$$

Second, let us consider the case of the ROC $-2 < \text{Re}\{s\} < -1$. From Table 6.2, we have

$$-e^{-t}u(-t) \xleftrightarrow{\mathscr{L}} \tfrac{1}{s+1} \quad \text{for } \text{Re}\{s\} < -1 \quad \text{and}$$
$$e^{-2t}u(t) \xleftrightarrow{\mathscr{L}} \tfrac{1}{s+2} \quad \text{for } \text{Re}\{s\} > -2.$$

Substituting these results into (6.8), we have

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t).$$

Thus, we have that

$$-e^{-t}u(-t) - e^{-2t}u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s^2 + 3s + 2} \quad \text{for } -2 < \text{Re}\{s\} < -1.$$

Third, let us consider the case of the ROC $\text{Re}\{s\} > -1$. From Table 6.2, we have

$$e^{-t}u(t) \xleftrightarrow{\mathscr{L}} \tfrac{1}{s+1} \quad \text{for } \text{Re}\{s\} > -1 \quad \text{and}$$
$$e^{-2t}u(t) \xleftrightarrow{\mathscr{L}} \tfrac{1}{s+2} \quad \text{for } \text{Re}\{s\} > -2.$$

Substituting these results into (6.8), we have

$$x(t) = e^{-t}u(t) - e^{-2t}u(t)$$
$$= \left[e^{-t} - e^{-2t}\right]u(t).$$

Thus, we have that

$$\left[e^{-t} - e^{-2t}\right]u(t) \xleftrightarrow{\mathscr{L}} \frac{1}{s^2 + 3s + 2} \quad \text{for } \text{Re}\{s\} > -1.$$

$\square$

6.10   Characterizing LTI Systems Using the Laplace Transform

Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Such a system is characterized by the equation

$$y(t) = x(t) * h(t).$$

Let $X(s)$, $Y(s)$, and $H(s)$ denote the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively. Taking the Laplace transform of both sides of the above equation, we obtain

$$Y(s) = \mathscr{L}\{x(t) * h(t)\}.$$

From the time-domain convolution property of the Laplace transform, we have

$$Y(s) = H(s)X(s).$$

The quantity $H(s)$ is known as the **system function** or **transfer function** of the system. If the ROC of $H(s)$ includes the imaginary axis, then $H(s)|_{s=j\omega}$ is the frequency response of the LTI system.

6.11 System Function and System Properties

Many properties of a system can be readily determined from the characteristics of its system function, as we shall elaborate upon in the sections that follow.

6.11.1 Causality

If a LTI system is causal, its impulse response is zero for all $t < 0$. Such an impulse response is inherently right sided. Consequently, from the results of Section 6.6, we have the theorem below.

Theorem 6.1. *The ROC associated with the system function of a causal system is a right-half plane.*

In general, the converse of the above theorem is not necessarily true. That is, it is not always true that a right-hand plane ROC is associated with a causal system function. If $H(s)$ is rational, however, we have that the converse does hold, as indicated by the theorem below.

Theorem 6.2. *For a system with a rational system function $H(s)$, causality of the system is equivalent to the ROC of $H(s)$ being the right-half plane to the right of the rightmost pole.*

Example 6.28. For each of the following system functions, determine whether the corresponding system is causal:
(a) $H(s) = \frac{1}{s+1}$ for $\text{Re}\{s\} > -1$,
(b) $H(s) = \frac{1}{s^2-1}$ for $-1 < \text{Re}\{s\} < 1$,
(c) $H(s) = \frac{e^s}{s+1}$ for $\text{Re}\{s\} < -1$, and
(d) $H(s) = \frac{e^s}{s+1}$ for $\text{Re}\{s\} > -1$.

Solution. (a) The poles of $H(s)$ are plotted in Figure 6.11(a) and the ROC is indicated by the shaded area. The system function $H(s)$ is rational and the ROC is the right-half plane to the right of the rightmost pole. Therefore, the system is causal.

(b) The poles of $H(s)$ are plotted in Figure 6.11(b) and the ROC is indicated by the shaded area. The system function is rational but the ROC is not a right-half plane. Therefore, the system is not causal.

(c) The system function $H(s)$ has a left-half plane ROC. Therefore, $h(t)$ is a left-sided signal. Thus, the system is not causal.

(d) The system function $H(s)$ has a right-half plane ROC but is not rational. Thus, we cannot make any conclusion directly from the system function. Instead, by taking the inverse Laplace transform, we can show that

$$\mathscr{L}^{-1}\{H(s)\} = h(t) = e^{-(t+1)}u(t+1).$$

The impulse response $h(t)$ is clearly not that of a causal system. Therefore, the system is not causal. $\square$
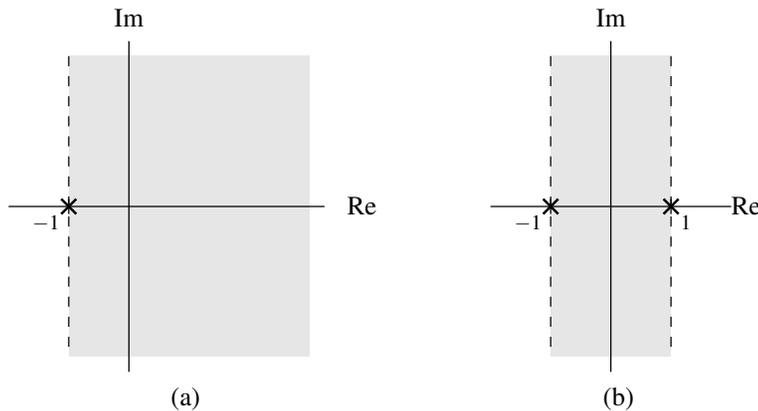
Figure 6.11: Pole diagrams for causality example.
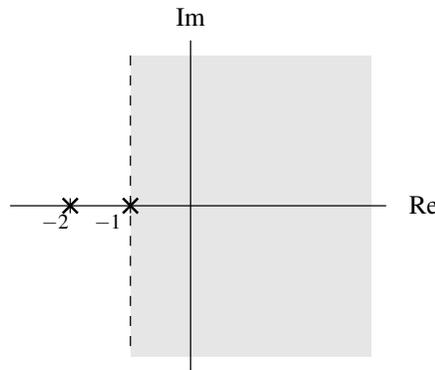
Figure 6.12: ROC for example.

6.11.2 Stability

In this section, we consider the relationship between the system function and stability. The first important result is given by the theorem below.

Theorem 6.3. *A LTI system is BIBO stable if and only if the ROC of its system function $H(s)$ includes the entire imaginary axis (i.e., $\mathrm{Re}\{s\} = 0$).*

Example 6.29. A LTI system has the system function $H(s)$ given by

$$H(s) = \frac{1}{(s+1)(s+2)}.$$

Given that the system is stable, determine the ROC of $H(s)$.

Solution. Clearly, the system function $H(s)$ is rational with poles at $s = -1$ and $s = -2$. Therefore, only three possibilities exist for the ROC: i) $\mathrm{Re}\{s\} < -2$, ii) $-2 < \mathrm{Re}\{s\} < -1$, and iii) $\mathrm{Re}\{s\} > -1$. In order for the system to be stable, however, the ROC of $H(s)$ must include the entire imaginary axis. Therefore, the ROC must be $\mathrm{Re}\{s\} > -1$. This ROC is illustrated in Figure 6.12. $\qquad\square$

In the case that the system is causal, a more specific result can be derived. This result is given by the theorem below.

Theorem 6.4. *A causal system with a rational system function $H(s)$ is BIBO stable if and only if all of the poles of $H(s)$ lie in the left half of the plane (i.e., all of the poles have negative real parts).*
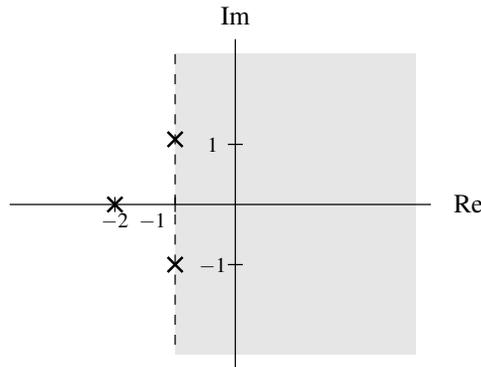
Figure 6.13: Poles of system function.

Example 6.30. A LTI system is causal and has the system function $H(s)$ given by

$$H(s) = \frac{1}{(s+2)(s^2+2s+2)}.$$

Determine whether this system is stable.

Solution. We begin by factoring $H(s)$ to obtain

$$H(s) = \frac{1}{(s+2)(s+1-j)(s+1+j)}.$$

(Using the quadratic formula, one can confirm that $s^2 + 2s + 2 = 0$ has roots at $s = -1 \pm j$.) Thus, $H(s)$ has poles at -2, $-1+j$, and $-1-j$. The poles are plotted in Figure 6.13. Since the system is causal and all of the poles of $H(s)$ are in the left half of the plane, the system is stable. $\square$

Example 6.31. For each of the following system functions, sketch its pole diagram, and indicate the ROC that corresponds to a stable system:

$$H_1(s) = \frac{s(s-1)}{(s+2)(s+1+j)(s+1-j)},$$

$$H_2(s) = \frac{s}{(s+1)(s-1)(s-1-j)(s-1+j)},$$

$$H_3(s) = \frac{(s+j)(s-j)}{(s+2-j)(s+2+j)}, \quad \text{and}$$

$$H_4(s) = \frac{s-1}{s}.$$

Solution. The function $H_1(s)$ has poles at -2, $-1+j$, and $-1-j$. The pole diagram is shown in Figure 6.14(a). Since $H_1(s)$ is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible: i) $\text{Re}\{s\} < -2$, ii) $-2 < \text{Re}\{s\} < -1$, and iii) $\text{Re}\{s\} > -1$. Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\text{Re}\{s\} > -1$. This is the shaded region on the pole diagram.

The function $H_2(s)$ has poles at -1, 1, $1+j$, and $1-j$. The pole plot is shown in Figure 6.14(b). Since $H_2(s)$ is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible: i) $\text{Re}\{s\} < -1$, ii) $-1 < \text{Re}\{s\} < 1$, and iii) $\text{Re}\{s\} > 1$. Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $-1 < \text{Re}\{s\} < 1$. This is the shaded region in the pole diagram.

The function $H_3(s)$ has poles at $-2+j$ and $-2-j$. The pole diagram is shown in Figure 6.14(c). Since $H_3(s)$ is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only two distinct ROCs are possible:
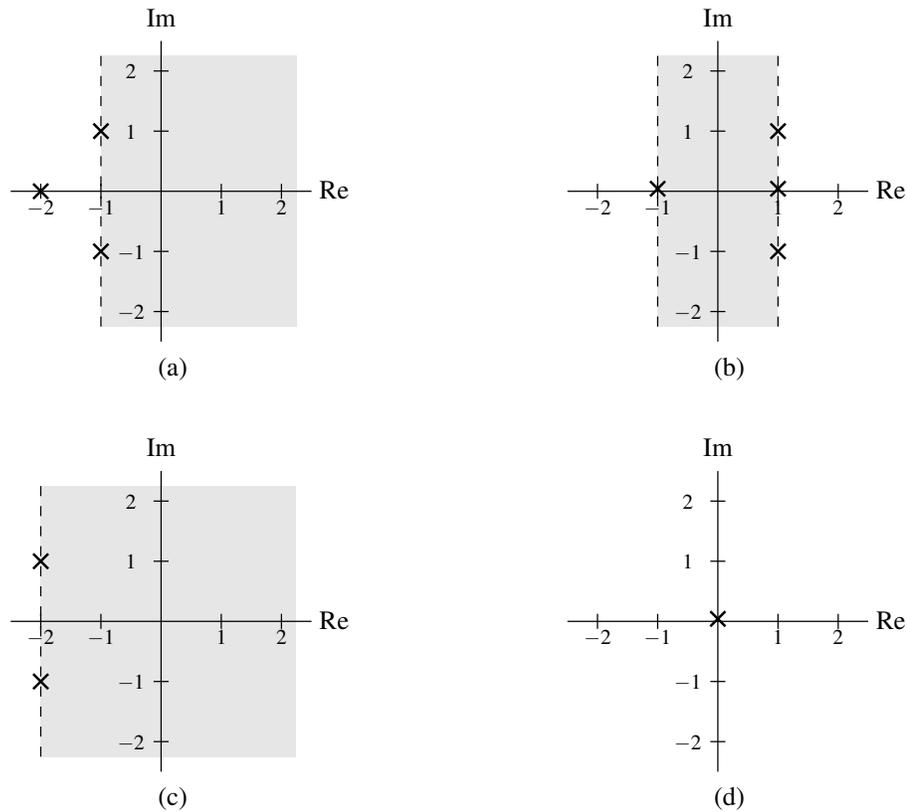
Figure 6.14: Pole diagrams for example.

i) $\text{Re}\{s\} < -2$, and ii) $\text{Re}\{s\} > -2$. Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\text{Re}\{s\} > -2$. This is the shaded region in the pole diagram.

The function $H_4(s)$ has a pole at 0. The pole plot is shown in Figure 6.14(d). Since $H_4(s)$ is rational, it cannot converge at $s = 0$ (which is a pole of $H_4(s)$). Consequently, the ROC can never include the entire imaginary axis. Therefore, the system function $H_4(s)$ can never be associated with a stable system. $\square$

6.11.3 Invertibility

Suppose that we have an LTI system $\mathcal{H}$ with impulse response $h(t)$. As discussed in Section 3.9.3, such a system is invertible if and only if there exists another system with impulse response $h^{\text{inv}}(t)$ satisfying

$$h(t) * h^{\text{inv}}(t) = \delta(t).$$

Let $H(s)$ and $H^{\text{inv}}(s)$ denote the Laplace transforms of $h(t)$ and $h^{\text{inv}}(t)$, respectively. Taking the Laplace transform of both sides of the above equation, we have

$$\mathcal{L}\{h(t) * h^{\text{inv}}(t)\} = \mathcal{L}\{\delta(t)\}.$$

From the time-domain convolution property of the Laplace transform and Table 6.2 (i.e., $\mathcal{L}\{\delta(t)\} = 1$), we have

$$H(s)H^{\text{inv}}(s) = 1. \tag{6.9}$$

Alternatively, we can express this relationship as

$$H^{\text{inv}}(s) = \frac{1}{H(s)}.$$

From the above results, we have that a LTI system $\mathscr{H}$ with system function $H(s)$ has an inverse if and only if a solution for $H^{\text{inv}}(s)$ exists in (6.9). Furthermore, if an inverse system exists, its system function is $1/H(s)$.

Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is not necessarily unique. In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in one specific choice of inverse system (due to these additional constraints of stability and/or causality).

Example 6.32. Suppose that we have the LTI system with system function $H(s)$ given by

$$H(s) = \frac{s+1}{s+2} \quad \text{for Re}\{s\} > -2.$$

Determine all possible inverses of this system. Comment on the stability of each of these inverse systems.

Solution. The system function $H^{\text{inv}}(s)$ of the inverse system is given by

$$H^{\text{inv}}(s) = \frac{1}{H(s)} = \frac{s+2}{s+1}.$$

Two ROCs are possible for $H^{\text{inv}}(s)$: i) Re$\{s\} < -1$, and ii) Re$\{s\} > -1$. Each ROC is associated with a distinct inverse system. The first ROC is associated with an unstable system since this ROC does not include the imaginary axis. The second ROC is associated with a stable system, since this ROC includes the entire imaginary axis. $\square$

6.12 LTI Systems and Differential Equations

Many LTI systems of practical interest can be described by Nth-order linear differential equations with constant coefficients. Such a system with input $x(t)$, output $y(t)$, and impulse response $h(t)$ can be characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t), \tag{6.10}$$

where $M \leq N$. Let $X(s)$, $Y(s)$, and $H(s)$ denote the Laplace transforms of $x(t)$, $y(t)$, and $h(t)$, respectively. Taking the Laplace transform of both sides of the above equation, we obtain

$$\mathscr{L}\left\{\sum_{k=0}^{N} b_k \frac{d^k}{dt^k} y(t)\right\} = \mathscr{L}\left\{\sum_{k=0}^{M} a_k \frac{d^k}{dt^k} x(t)\right\}.$$

Using the linearity property of the Laplace transform, we can rewrite this equation as

$$\sum_{k=0}^{N} b_k \mathscr{L}\left\{\frac{d^k}{dt^k} y(t)\right\} = \sum_{k=0}^{M} a_k \mathscr{L}\left\{\frac{d^k}{dt^k} x(t)\right\}.$$

Using the time differentiation property of the Laplace transform, we have

$$\sum_{k=0}^{N} b_k s^k Y(s) = \sum_{k=0}^{M} a_k s^k X(s).$$

Factoring, we have

$$Y(s)\left[\sum_{k=0}^{N} b_k s^k\right] = X(s)\left[\sum_{k=0}^{M} a_k s^k\right].$$
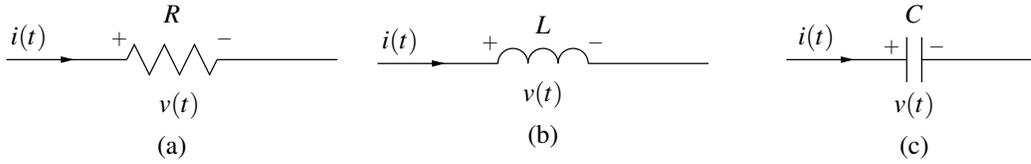
Figure 6.15: Basic electrical components. (a) Resistor, (b) inductor, and (c) capacitor.

Rearranging this equation, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

Observe that, for a system of the form considered above (i.e., a system characterized by an equation of the form of (6.10)), the system function is always rational. It is for this reason that rational functions are of particular interest to us.

Example 6.33 (Resistors, inductors, and capacitors)**.** The basic building blocks of many electrical networks are resistors, inductors, and capacitors. The resistor, shown in Figure 6.15(a), is governed by the relationship

$$v(t) = Ri(t) \quad \overset{\mathscr{L}}{\longleftrightarrow} \quad V(s) = RI(s)$$

where R, $v(t)$ and $i(t)$ denote the resistance of, voltage across, and current through the resistor, respectively. The inductor, shown in Figure 6.15(b), is governed by the relationship

$$v(t) = L\tfrac{d}{dt}i(t) \quad \overset{\mathscr{L}}{\longleftrightarrow} \quad V(s) = sLI(s)$$

or equivalently

$$i(t) = \tfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau \quad \overset{\mathscr{L}}{\longleftrightarrow} \quad I(s) = \tfrac{1}{sL}V(s)$$

where L, $v(t)$, and $i(t)$ denote the inductance of, voltage across, and current through the inductor, respectively. The capacitor, shown in Figure 6.15(c), is governed by the relationship

$$v(t) = \tfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \overset{\mathscr{L}}{\longleftrightarrow} \quad V(s) = \tfrac{1}{sC}I(s)$$

or equivalently

$$i(t) = C\tfrac{d}{dt}v(t) \quad \overset{\mathscr{L}}{\longleftrightarrow} \quad I(s) = sCV(s)$$

where C, $v(t)$, and $i(t)$ denote the capacitance of, voltage across, and current through the capacitor, respectively. Note that resistance, inductance, and capacitance are nonnegative quantities (i.e., $R \geq 0$, $L \geq 0$, and $C \geq 0$).

Example 6.34 (Simple RC network)**.** Suppose that we have the RC network shown in Figure 6.16 with input $v_1(t)$ and output $v_2(t)$. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function $H(s)$ of this system. (b) Determine whether the system is stable. (c) Determine the step response of the system.

Solution. (a) From basic circuit analysis, we have

$$v_1(t) = Ri(t) + v_2(t) \quad \text{and} \tag{6.11a}$$

$$i(t) = C\frac{d}{dt}v_2(t). \tag{6.11b}$$

Taking the Laplace transform of (6.11) yields

$$V_1(s) = RI(s) + V_2(s) \quad \text{and} \tag{6.12a}$$
$$I(s) = CsV_2(s). \tag{6.12b}$$

Using these equations, we can write:

$$V_1(s) = R[CsV_2(s)] + V_2(s)$$
$$\Rightarrow \quad V_1(s) = RCsV_2(s) + V_2(s)$$
$$\Rightarrow \quad V_1(s) = [1 + RCs]V_2(s).$$

Thus, we have that the system function $H(s)$ is given by

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + RCs} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} = \frac{\frac{1}{RC}}{s - (-\frac{1}{RC})}.$$

Since the system can be physically realized, it must be causal. Therefore, $H(s)$ must have the ROC $\text{Re}\{s\} > -\frac{1}{RC}$. So, we have

$$H(s) = \frac{1}{1 + RCs} \quad \text{for } \text{Re}\{s\} > -\frac{1}{RC}.$$

(b) Since resistance and capacitance are positive quantities, $-\frac{1}{RC}$ is a negative value. Consequently, the ROC always includes the entire imaginary axis, and the system is stable.

(c) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

$$V_2(s) = H(s)V_1(s)$$
$$= \left(\frac{1}{1 + RCs}\right)V_1(s).$$

To compute the step response, we need to consider an input equal to the unit-step function. So, $v_1(t) = u(t)$, implying that $V_1(s) = \frac{1}{s}$. Substituting this expression for $V_1(s)$ into the above expression for $V_2(s)$, we have

$$V_2(s) = \left(\frac{1}{1 + RCs}\right)\left(\frac{1}{s}\right)$$
$$= \frac{\frac{1}{RC}}{s(s + \frac{1}{RC})}.$$

Now, we need to compute the inverse Laplace transform of $V_2(s)$ in order to determine $v_2(t)$. To simplify this task, we find the partial fraction expansion for $V_2(s)$. We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s + \frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$A_1 = sV_2(s)|_{s=0}$$
$$= 1 \quad \text{and}$$
$$A_2 = (s + \tfrac{1}{RC})V_2(s)\big|_{s=-\frac{1}{RC}}$$
$$= \frac{\frac{1}{RC}}{-\frac{1}{RC}}$$
$$= -1.$$
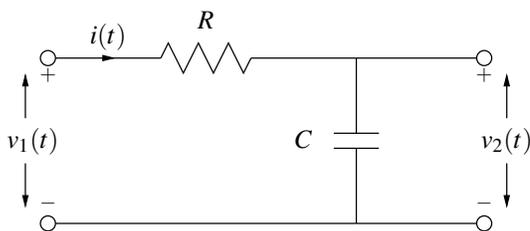
Figure 6.16: Simple RC network.

Thus, we have that $V_2(s)$ has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

$$v_2(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s + \frac{1}{RC}}\right\}.$$

Using Table 6.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

$$v_2(t) = u(t) - e^{-t/(RC)}u(t)$$
$$= \left[1 - e^{-t/(RC)}\right]u(t).$$

$\square$

Example 6.35. A LTI system with input $x(t)$ and output $y(t)$ has the system function $H(s)$ given by

$$H(s) = \frac{s}{s + R/L},$$

where L and R are positive real constants. Find the differential equation that characterizes this system.

Solution. Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively. To begin, we have

$$Y(s) = H(s)X(s)$$
$$= \left[\frac{s}{s + R/L}\right]X(s).$$

Rearranging this equation, we obtain

$$(s + \tfrac{R}{L})Y(s) = sX(s)$$
$$\Rightarrow \quad sY(s) + \tfrac{R}{L}Y(s) = sX(s).$$

Taking the inverse Laplace transform of both sides of this equation (by using the linearity and time-differentiation properties of the Laplace transform), we have

$$\mathscr{L}^{-1}\{sY(s)\} + \tfrac{R}{L}\mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\{sX(s)\}$$
$$\Rightarrow \quad \frac{dy(t)}{dt} + \tfrac{R}{L}y(t) = \frac{dx(t)}{dt}.$$

Therefore, the system is characterized by the differential equation

$$\frac{dy(t)}{dt} + \tfrac{R}{L}y(t) = \frac{dx(t)}{dt}.$$

$\square$
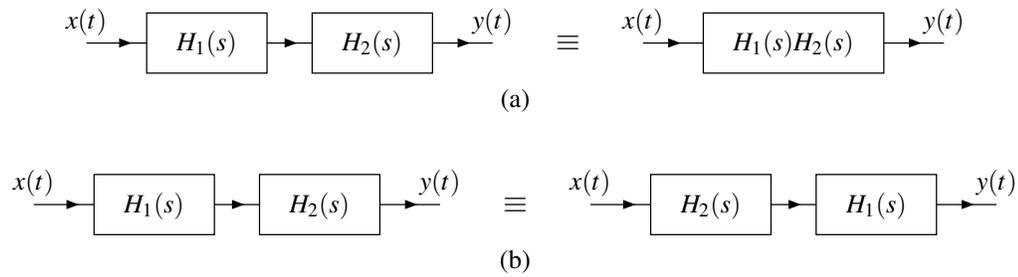
Figure 6.17: System function and series-interconnection of LTI systems.
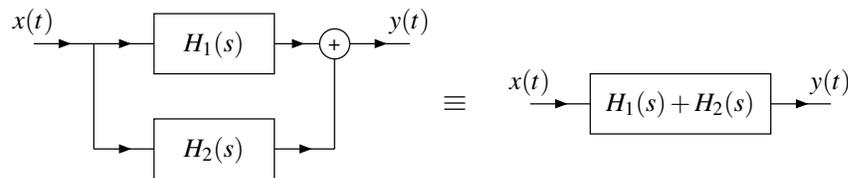
Figure 6.18: System function and parallel-interconnection of LTI systems.

6.13 Interconnection of LTI Systems

From the properties of the Laplace transform and the definition of the system function, we can derive a number of equivalences involving the system function and series- and parallel-interconnected systems.

Suppose that we have two LTI systems H_1 and H_2 with system functions $H_1(s)$ and $H_2(s)$, respectively, that are connected in a series configuration as shown in the left-hand side of Figure 6.17(a). Let $h_1(t)$ and $h_2(t)$ denote the impulse responses of H_1 and H_2, respectively. The impulse response $h(t)$ of the overall system is given by

$$h(t) = h_1(t) * h_2(t).$$

Taking the Laplace transform of both sides of this equation yields

$$\begin{aligned} H(s) &= \mathscr{L}\{h_1(t) * h_2(t)\} \\ &= \mathscr{L}\{h_1(t)\}\mathscr{L}\{h_2(t)\} \\ &= H_1(s)H_2(s). \end{aligned}$$

Thus, we have the equivalence shown in Figure 6.17(a). Also, since multiplication commutes, we also have the equivalence shown in Figure 6.17(b).

Suppose that we have two LTI systems H_1 and H_2 with system functions $H_1(s)$ and $H_2(s)$ that are connected in a parallel configuration as shown on the left-hand side of Figure 6.18. Let $h_1(t)$ and $h_2(t)$ denote the impulse responses of H_1 and H_2, respectively. The impulse response $h(t)$ of the overall system is given by

$$h(t) = h_1(t) + h_2(t).$$

Taking the Laplace transform of both sides of the equation yields

$$\begin{aligned} H(s) &= \mathscr{L}\{h_1(t) + h_2(t)\} \\ &= \mathscr{L}\{h_1(t)\} + \mathscr{L}\{h_2(t)\} \\ &= H_1(s) + H_2(s). \end{aligned}$$

Thus, we have the equivalence shown in Figure 6.18.
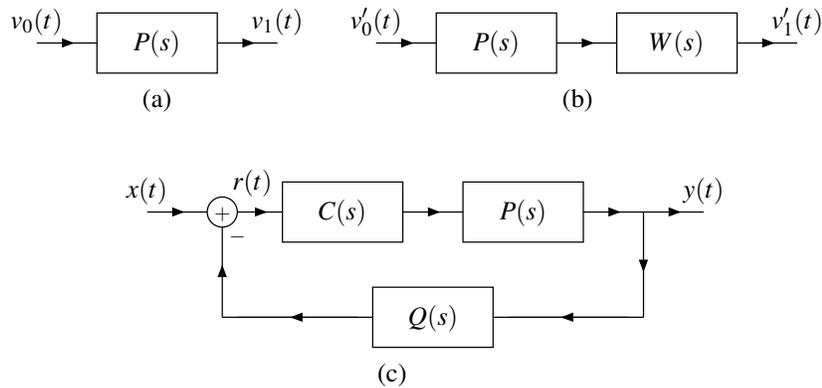
Figure 6.19: Feedback control system example. (a) Plant; (b) Modified system; (c) Feedback control system.

Example 6.36 (Feedback control system). In this problem, we introduce a very commonly occurring system config-
uration in control system applications. The particular configuration of interest is what is called a feedback control
system. A feedback control system consists of three interconnected subsystems: 1) a plant, which corresponds to
the entity/quantity being controlled; 2) a sensor, which is used to measure the actual value of the quantity being con-
trolled; and 3) a compensator, which is used to ensure that the overall system has certain desirable properties (e.g.,
stable). In what follows, we assume that the plant, sensor, and compensator are all causal LTI systems. We denote
the system functions of these systems as $P(s)$, $Q(s)$, and $C(s)$, respectively. The feedback configuration is shown in
Figure 6.19(c). The input $x(t)$ corresponds to the desired value for the quantity being controlled. The output $y(t)$
corresponds to the actual value of the quantity being controlled (which is measured by a sensor). The adder output
$r(t)$ corresponds to the difference between the desired and actual values of the controlled quantity (i.e., the error).

   (a) UNSTABLE PLANT. Consider the causal LTI system with input $v_0(t)$, output $v_1(t)$, and system function $P(s) =$
$\frac{1}{s-1}$, as depicted in Figure 6.19(a). Show that this system is not BIBO stable.

   (b) STABILIZATION OF UNSTABLE PLANT VIA POLE-ZERO CANCELLATION. Suppose that the system in Fig-
ure 6.19(a) is connected in series with another causal LTI system with system function $W(s) = s - 1$, in order to yield
a new system with input $v_0'(t)$ and output $v_1'(t)$, as shown in Figure 6.19(b). Show that this new system is BIBO stable.

   (c) STABILIZATION OF UNSTABLE PLANT VIA FEEDBACK. Suppose now that the system in Figure 6.19(a) is
interconnected with two other causal LTI systems (with system functions $C(s)$ and $Q(s)$), as shown in Figure 6.19(c),
in order to yield a new system with input $x(t)$, output $y(t)$, and system function $H(s)$. Suppose that $C(s) = \beta s$ and
$Q(s) = 3$, where β is a real constant. Show that, with an appropriate choice of β, the resulting system is BIBO stable.

   (d) PRACTICAL ISSUES. In effect, we have proposed two different schemes for stabilizing the unstable system
from Figure 6.19(a). As it turns out, a scheme like that shown in Figure 6.19(b) is rarely ever used in practice. Identify
the practical issue associated with this approach to stabilizing the system. Indicate whether the second scheme shown
in Figure 6.19(c) suffers from the same shortcoming.

Solution. (a) The system function $P(s)$ is rational and has a pole at 1. Since the system is causal, the ROC of $P(s)$
is the RHP given by $\text{Re}\{s\} > 1$. Clearly, the ROC does not include the imaginary axis. Therefore, the system is not
stable.

   (b) The system function of the overall system is $P(s)W(s) = 1$. The system function is rational and has no poles.
Thus, the corresponding ROC is the entire complex plane. Since the ROC includes the imaginary axis, the system is
BIBO stable.

   (c) From the block diagram in Figure 6.19(c), we can write

$$R(s) = X(s) - Q(s)Y(s) \quad \text{and}$$
$$Y(s) = C(s)P(s)R(s).$$

Combining these equations (by substituting the expression for $R(s)$ in the first equation into the second equation), we

obtain

$$Y(s) = C(s)P(s)[X(s) - Q(s)Y(s)] \Rightarrow$$
$$Y(s) = C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) \Rightarrow$$
$$[1 + C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s) \Rightarrow$$
$$\frac{Y(s)}{X(s)} = H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.$$

Substituting the given expressions for $P(s)$, $C(s)$, and $Q(s)$, we have

$$\begin{aligned}
H(s) &= \frac{(\beta s)(\frac{1}{s-1})}{1 + (\beta s)(\frac{1}{s-1})(3)} \\
&= \frac{\beta s}{s - 1 + 3\beta s} \\
&= \frac{\beta s}{(1 + 3\beta)s - 1} \\
&= \frac{\beta s}{(1 + 3\beta)(s - \frac{1}{1 + 3\beta})}.
\end{aligned}$$

The system function $H(s)$ is rational and has a single pole at $\frac{1}{1+3\beta}$. Since the system is causal, the ROC must be the RHP given by $\text{Re}\{s\} > \frac{1}{1+3\beta}$. For the system to be stable, we require that the ROC includes the imaginary axis. Thus, the system is stable if $\frac{1}{1+3\beta} < 0$ which implies $1 + 3\beta < 0$, or equivalently $\beta < -\frac{1}{3}$.

(d) The first scheme for stabilizing the unstable plant, shown in Figure 6.19(b), relies on pole-zero cancellation. Unfortunately, in practice, it is not usually possible to achieve exact pole-zero cancellation. This is due to the fact that we cannot usually build a system with such precision that it will have exactly some given system function. The system function will only be close to the desired one. Also, the system function of most systems will vary at least slightly with changes in the physical environment (e.g., changes in temperature and pressure, or changes in gravitational forces due to changes in the phase of the moon, etc.). For this reason, the pole-zero cancellation required for a stable system will not occur in practice.

The second scheme for stabilizing the unstable plant, shown in Figure 6.19(c), is based on feedback. With the feedback approach, the poles of the system function are not cancelled with zeros, instead the poles are completely changed/relocated. For this reason, we can place the poles such that, even if the poles are displaced slightly (due to the real-world system having a system function that differs slightly from the truly desired one), the stability of the system will not be compromised. Therefore, this second scheme does not suffer from the same practical problem that the first one does.   ☐

Example 6.37 (Feedback system). Suppose that we have the causal system with input $x(t)$ and output $y(t)$ formed by the interconnection of two causal LTI systems as shown in Figure 6.20. Let the system functions $H_1(s)$ and $H_2(s)$ be

$$H_1(s) = \frac{1}{s^2 + as + (a - 2)} \quad \text{and}$$
$$H_2(s) = -1.$$

where a is a real constant. (a) Find the system function $H(s)$ of the (overall) system. (b) Determine the values of the parameter a for which the system is stable.

Solution. (a) From the system diagram, we can write

$$V(s) = X(s) + H_2(s)Y(s) \quad \text{and}$$
$$Y(s) = H_1(s)V(s).$$

Combining these two equations and simplifying, we obtain

$$Y(s) = H_1(s)[X(s) + H_2(s)Y(s)]$$
$$\Rightarrow \quad Y(s) = H_1(s)X(s) + H_1(s)H_2(s)Y(s)$$
$$\Rightarrow \quad Y(s)[1 - H_1(s)H_2(s)] = H_1(s)X(s)$$
$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

Substituting the given expressions for $H_1(s)$ and $H_2(s)$, and simplifying, we can write

$$H(s) = \frac{\frac{1}{s^2 + as + (a-2)}}{1 + \left(\frac{1}{s^2 + as + (a-2)}\right)}$$
$$= \frac{1}{s^2 + as + (a-2) + 1}$$
$$= \frac{1}{s^2 + as + (a-1)}.$$

(b) In order to consider stability, we need to know the poles of the system function $H(s)$. So, we use the quadratic formula in order to factor the denominator of $H(s)$.

$$s = \frac{-a \pm \sqrt{a^2 - 4(a-1)}}{2}$$
$$= \frac{-a \pm \sqrt{a^2 - 4a - 4}}{2}$$
$$= \frac{-a \pm \sqrt{(a-2)^2}}{2}$$
$$= \frac{-a \pm (a-2)}{2}$$
$$= \{-1, 1-a\}.$$

So, $s^2 + as + (a-1) = (s+1)(s+a-1)$. Thus, we have

$$H(s) = \frac{1}{(s+1)(s+a-1)}.$$

Since the system is causal, the system is stable if and only if all of the poles are strictly to the left of the imaginary axis. The system has two poles, one at -1 and one at $1-a$. Thus, we know that

$$1 - a < 0 \Rightarrow a > 1.$$

Therefore, the system is stable if and only if $a > 1$. $\qquad\square$

6.14 Unilateral Laplace Transform

As mentioned earlier, two different versions of the Laplace transform are commonly employed, namely, the bilateral and unilateral versions. So far, we have considered only the bilateral Laplace transform. Now, we turn our attention to the unilateral Laplace transform. The **unilateral Laplace transform** of the signal $x(t)$ is denoted as $\mathscr{UL}\{x(t)\}$ or $X(s)$ and is defined as

$$\mathscr{UL}\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt. \tag{6.13}$$
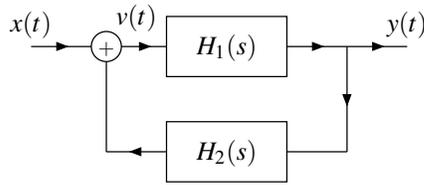
Figure 6.20: Feedback system.

The **inverse unilateral Laplace transform** has the same definition as in the case of the bilateral transform, namely (6.3).

Comparing the definitions of the unilateral and bilateral Laplace transforms given by (6.13) and (6.2), respectively, we can see that these definitions only differ in the lower limit of integration. Due to the similarity in these definitions, an important relationship exists between these two transforms, as we shall now demonstrate. Consider the bilateral Laplace transform of the signal $x(t)u(t)$ for an arbitrary signal $x(t)$. We have

$$
\begin{aligned}
\mathscr{L}\{x(t)u(t)\} &= \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt \\
&= \int_{0^-}^{\infty} x(t)e^{-st}dt \\
&= \mathscr{U}\mathscr{L}\{x(t)\}.
\end{aligned}
$$

In other words, the unilateral Laplace transform of the signal $x(t)$ is simply the bilateral Laplace transform of the signal $x(t)u(t)$. Since $\mathscr{U}\mathscr{L}\{x(t)\} = \mathscr{L}\{x(t)u(t)\}$ and $x(t)u(t)$ is always a right-sided signal, the ROC associated with $\mathscr{U}\mathscr{L}\{x(t)\}$ is always a right-half plane. For this reason, we often do not explicitly indicate the ROC when working with the unilateral Laplace transform.

From earlier in this chapter, we know that the bilateral Laplace transform is invertible. That is, if the signal $x(t)$ has the bilateral Laplace transform $X(s) = \mathscr{L}\{x(t)\}$, then $\mathscr{L}^{-1}\{X(s)\} = x(t)$. Now, let us consider the invertibility of the unilateral Laplace transform. To do this, we must consider the quantity $\mathscr{U}\mathscr{L}^{-1}\{\mathscr{U}\mathscr{L}\{x(t)\}\}$. Since $\mathscr{U}\mathscr{L}\{x(t)\} = \mathscr{L}\{x(t)u(t)\}$ and the inverse equations for the unilateral and bilateral Laplace transforms are identical, we can write

$$
\begin{aligned}
\mathscr{U}\mathscr{L}^{-1}\{\mathscr{U}\mathscr{L}\{x(t)\}\} &= \mathscr{U}\mathscr{L}^{-1}\{\mathscr{L}\{x(t)u(t)\}\} \\
&= \mathscr{L}^{-1}\{\mathscr{L}\{x(t)u(t)\}\} \\
&= x(t)u(t) \\
&= \begin{cases} x(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}
\end{aligned}
$$

Thus, we have that $\mathscr{U}\mathscr{L}^{-1}\{\mathscr{U}\mathscr{L}\{x(t)\}\} = x(t)$ only if $x(t)$ is causal. In other words, the unilateral Laplace transform is invertible only for causal signals. For noncausal signals, we can only recover $x(t)$ for $t > 0$. In essence, the unilateral Laplace transform discards all information about a signal $x(t)$ for $t < 0$. Since this information is discarded, it cannot be recovered by an inverse unilateral Laplace transform operation.

Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties. Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways. The properties of the unilateral Laplace transform are summarized in Table 6.3.

By comparing the properties of the unilateral and bilateral Laplace transform listed in Tables 6.3 and 6.1, respectively, we can see that the unilateral Laplace transform has some of the same properties as its bilateral counterpart, namely, the linearity, Laplace-domain shifting, conjugation, and Laplace-domain differentiation properties. The initial-value and final-value theorems also apply in the case of the unilateral Laplace transform.

Since the unilateral and bilateral Laplace transforms are defined differently, their properties also differ in some cases. These differences can be seen by comparing the bilateral Laplace transform properties listed in Table 6.1 with the unilateral Laplace transform properties listed in Table 6.3. In the unilateral case, we can see that

Table 6.3: Unilateral Laplace Transform Properties

Property	Time Domain	Laplace Domain
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(s) + a_2 X_2(s)$
Laplace-Domain Shifting	$e^{s_0 t} x(t)$	$X(s - s_0)$
Time/Frequency-Domain Scaling	$x(at), a > 0$	$\frac{1}{a} X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1(t) * x_2(t)$, $x_1(t)$ and $x_2(t)$ are causal	$X_1(s) X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt} x(t)$	$sX(s) - x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds} X(s)$
Time-Domain Integration	$\int_{0^-}^{t} x(\tau) d\tau$	$\frac{1}{s} X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s \to \infty} sX(s)$
Final Value Theorem	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$

Table 6.4: Unilateral Laplace Transform Pairs

Pair	$x(t), t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos \omega_0 t$	$\frac{s}{s^2 + \omega_0^2}$
7	$\sin \omega_0 t$	$\frac{\omega_0}{s^2 + \omega_0^2}$
8	$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$
9	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$

1. the time-domain convolution property has the additional requirement that the signals being convolved must be causal;

2. the time/frequency-domain scaling property has the additional constraint that the scaling factor must be positive;

3. the time-domain differentiation property has an extra term in the expression for $\mathcal{UL}\{\frac{d}{dt} x(t)\}$; and

4. the time-domain integration property has a different lower limit in the time-domain integral (namely, 0^- instead of $-\infty$).

Also, in the unilateral case, the time-domain shifting property does not hold (except in special circumstances).

Since $\mathcal{UL}\{x(t)\} = \mathcal{L}\{x(t)u(t)\}$, we can easily generate a table of unilateral Laplace transform pairs from a table of bilateral transform pairs. Using the bilateral Laplace transform pairs from Table 6.2, and the preceding relationship between the unilateral and bilateral Laplace transforms, we can trivially deduce the unilateral Laplace transform pairs in Table 6.4. Since, in the unilateral case, the ROC is always a right-half plane, we do not explicitly indicate the ROC in the table. That is, the ROC is implicitly assumed to be a right-half plane.

The inverse unilateral Laplace transform is computed through the same means used in the bilateral case (e.g., partial fraction expansions). The only difference is that the ROC is always assumed to be a right-half plane.

6.15 Solving Differential Equations Using the Unilateral Laplace Transform

Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations. As it turns out, a system that is described by such an equation need not be linear. In particular, the system will be linear only if the initial conditions for the differential equation are all zero. If one or more of the initial conditions is nonzero, then the system is what we refer to as **incrementally linear**. For our purposes here, incrementally linear systems can be thought of as a generalization of linear systems. The unilateral Laplace transform is sometimes quite useful due to its ability to easily handle nonzero initial conditions. For example, one common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions. In what follows, we consider some examples that exploit the unilateral Laplace transform to this end.

Example 6.38. Suppose that we have a causal incrementally-linear TI system with input $x(t)$ and output $y(t)$ characterized by the differential equation

$$\frac{d}{dt}y(t) + 3y(t) = x(t).$$

If $x(t) = e^{-t}u(t)$ and $y(0^-) = 1$, find $y(t)$.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\mathcal{UL}\left\{\frac{d}{dt}y(t)\right\} + 3\mathcal{UL}\{y(t)\} = \mathcal{UL}\{x(t)\}$$

$$\Rightarrow \quad sY(s) - y(0^-) + 3Y(s) = X(s)$$

$$\Rightarrow \quad (s+3)Y(s) = X(s) + y(0^-)$$

$$\Rightarrow \quad Y(s) = \frac{X(s) + y(0^-)}{s+3}.$$

Since $x(t) = e^{-t}u(t)$, we have

$$X(s) = \mathcal{UL}\{e^{-t}\} = \frac{1}{s+1}.$$

Substituting this expression for $X(s)$ and the given initial conditions (i.e., $y(0^-) = 1$) into the above equation for $Y(s)$, we obtain

$$Y(s) = \frac{\left(\frac{1}{s+1}\right) + 1}{s+3} = \frac{\left(\frac{s+2}{s+1}\right)}{s+3} = \frac{s+2}{(s+1)(s+3)}.$$

Now, we find a partial fraction expansion for $Y(s)$. Such an expansion has the form

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3}.$$

Calculating the expansion coefficients, we obtain

$$
\begin{aligned}
A_1 &= (s+1)Y(s)|_{s=-1} \\
&= \left.\frac{s+2}{s+3}\right|_{s=-1} \\
&= \frac{1}{2} \quad \text{and} \\
A_2 &= (s+3)Y(s)|_{s=-3} \\
&= \left.\frac{s+2}{s+1}\right|_{s=-3} \\
&= \frac{1}{2}.
\end{aligned}
$$

So, we can rewrite $Y(s)$ as

$$
Y(s) = \frac{1/2}{s+1} + \frac{1/2}{s+3}.
$$

Taking the inverse unilateral Laplace transform of $Y(s)$, we obtain

$$
\begin{aligned}
y(t) &= \mathscr{U}\mathscr{L}^{-1}\{Y(s)\} \\
&= \tfrac{1}{2}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \tfrac{1}{2}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} \\
&= \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t} \quad \text{for } t > 0^-.
\end{aligned}
$$

$\square$

Example 6.39 (Unilateral Laplace transform of second-order derivative)**.** Find the unilateral Laplace transform of $\left(\frac{d}{dt}\right)^2 y(t)$.

Solution. We need to apply the time-domain differentiation property of the unilateral Laplace transform twice. We have

$$
\mathscr{U}\mathscr{L}\left\{\tfrac{d}{dt}y(t)\right\} = sY(s) - y(0^-).
$$

Now, we differentiate in the time domain a second time to obtain

$$
\begin{aligned}
\mathscr{U}\mathscr{L}\left\{\left(\tfrac{d}{dt}\right)^2 y(t)\right\} &= \mathscr{U}\mathscr{L}\left\{\tfrac{d}{dt}\left[\tfrac{d}{dt}y(t)\right]\right\} \\
&= s\left(sY(s) - y(0^-)\right) - y'(0^-) \\
&= s^2 Y(s) - sy(0^-) - y'(0^-).
\end{aligned}
$$

Thus, we have that $\mathscr{U}\mathscr{L}\left\{\left(\tfrac{d}{dt}\right)^2 y(t)\right\} = s^2 Y(s) - sy(0^-) - y'(0^-)$. Clearly, the above process can be repeated to find the unilateral Laplace transform of higher-order derivatives of $y(t)$. $\square$

Example 6.40. Suppose that we have a causal incrementally-linear TI system with input $x(t)$ and output $y(t)$ characterized by the differential equation

$$
\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t).
$$

If $x(t) = 5u(t)$, $y(0^-) = 1$, and $y'(0^-) = -1$, find $y(t)$.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\mathscr{U}\mathscr{L}\left\{\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+2y(t)\right\}=\mathscr{U}\mathscr{L}\left\{x(t)\right\}$$

$$\Rightarrow \quad \mathscr{U}\mathscr{L}\left\{\frac{d^2}{dt^2}y(t)\right\}+3\mathscr{U}\mathscr{L}\left\{\frac{d}{dt}y(t)\right\}+2\mathscr{U}\mathscr{L}\left\{y(t)\right\}=\mathscr{U}\mathscr{L}\left\{x(t)\right\}$$

$$\Rightarrow \quad \left[s^2Y(s)-sy(0^-)-y'(0^-)\right]+3\left[sY(s)-y(0^-)\right]+2Y(s)=X(s)$$

$$\Rightarrow \quad s^2Y(s)-sy(0^-)-y'(0^-)+3sY(s)-3y(0^-)+2Y(s)=X(s)$$

$$\Rightarrow \quad \left[s^2+3s+2\right]Y(s)=X(s)+sy(0^-)+y'(0^-)+3y(0^-)$$

$$\Rightarrow \quad Y(s)=\frac{X(s)+sy(0^-)+y'(0^-)+3y(0^-)}{s^2+3s+2}.$$

Since $x(t)=5u(t)$, we have

$$X(s)=\mathscr{U}\mathscr{L}\{5u(t)\}=\tfrac{5}{s}.$$

Substituting this expression for $X(s)$ and the given initial conditions into the above equation yields

$$Y(s)=\frac{\left(\frac{5}{s}\right)+s-1+3}{s^2+3s+2}=\frac{s^2+2s+5}{s(s+1)(s+2)}.$$

Now, we must find a partial fraction expansion of $Y(s)$. Such an expansion is of the form

$$Y(s)=\frac{A_1}{s}+\frac{A_2}{s+1}+\frac{A_3}{s+2}.$$

Calculating the expansion coefficients, we obtain

$$A_1 = sY(s)|_{s=0}$$

$$= \left.\frac{s^2+2s+5}{(s+1)(s+2)}\right|_{s=0}$$

$$= \tfrac{5}{2},$$

$$A_2 = (s+1)Y(s)|_{s=-1}$$

$$= \left.\frac{s^2+2s+5}{s(s+2)}\right|_{s=-1}$$

$$= -4, \quad \text{and}$$

$$A_3 = (s+2)Y(s)|_{s=-2}$$

$$= \left.\frac{s^2+2s+5}{s(s+1)}\right|_{s=-2}$$

$$= \tfrac{5}{2}.$$

So, we can rewrite $Y(s)$ as

$$Y(s)=\frac{5/2}{s}-\frac{4}{s+1}+\frac{5/2}{s+2}.$$

Taking the inverse unilateral Laplace transform of $Y(s)$ yields

$$y(t)=\mathscr{U}\mathscr{L}^{-1}\{Y(s)\}$$

$$= \tfrac{5}{2}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-4\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}+\tfrac{5}{2}\mathscr{U}\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \tfrac{5}{2}-4e^{-t}+\tfrac{5}{2}e^{-2t} \quad \text{for } t>0^-.$$

$\square$

Example 6.41 (RC Network). Consider the RC network shown in Figure 6.21 with input $v_0(t)$ and output $v_1(t)$. If $R = 100$, $C = \frac{1}{100}$, $v_0(t) = 3e^{-2t}u(t)$, and $v_1(0^-) = 1$, find $v_1(t)$.

Solution. From basic circuit analysis, we have

$$v_0(t) = Ri(t) + v_1(t) \quad \text{and}$$
$$i(t) = C\tfrac{d}{dt}v_1(t).$$

Combining the preceding two equations, we obtain

$$v_0(t) = R\left[C\frac{d}{dt}v_1(t)\right] + v_1(t)$$
$$= RC\frac{d}{dt}v_1(t) + v_1(t).$$

Taking the unilateral Laplace transform of both sides of this equation yields

$$\mathscr{UL}\{v_0(t)\} = \mathscr{UL}\left\{RC\frac{d}{dt}v_1(t) + v_1(t)\right\}$$
$$\Rightarrow \quad \mathscr{UL}\{v_0(t)\} = RC\,\mathscr{UL}\left\{\frac{d}{dt}v_1(t)\right\} + \mathscr{UL}\{v_1(t)\}$$
$$\Rightarrow \quad V_0(s) = RC\left[sV_1(s) - v_1(0^-)\right] + V_1(s)$$
$$\Rightarrow \quad V_0(s) = RCsV_1(s) - RCv_1(0^-) + V_1(s)$$
$$\Rightarrow \quad V_0(s) + RCv_1(0^-) = RCsV_1(s) + V_1(s)$$
$$\Rightarrow \quad V_1(s) = \frac{V_0(s) + RCv_1(0^-)}{RCs + 1}.$$

Since $v_0(t) = 3e^{-2t}u(t)$, we have

$$V_0(s) = \mathscr{UL}\{v_0(t)\} = \frac{3}{s+2}.$$

Substituting this expression for $V_0(s)$ into the above equation for $V_1(s)$, we obtain

$$V_1(s) = \frac{\left(\frac{3}{s+2}\right) + 1}{s + 1}$$
$$= \frac{3}{(s+1)(s+2)} + \frac{1}{s+1}$$
$$= \frac{s+5}{(s+1)(s+2)}.$$

Now, we find a partial fraction expansion of $V_1(s)$. Such an expansion is of the form

$$V_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Calculating the expansion coefficients yields

$$A_1 = (s+1)V_1(s)\big|_{s=-1}$$
$$= \frac{s+5}{s+2}\bigg|_{s=-1}$$
$$= 4 \quad \text{and}$$
$$A_2 = (s+2)V_1(s)\big|_{s=-2}$$
$$= \frac{s+5}{s+1}\bigg|_{s=-2}$$
$$= -3.$$
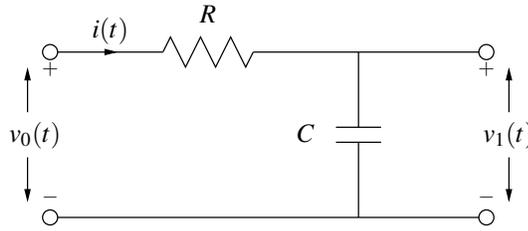
Figure 6.21: RC network.

Thus, we can rewrite $V_1(s)$ as

$$V_1(s) = \frac{4}{s+1} - \frac{3}{s+2}.$$

Taking the inverse unilateral Laplace transform of $V_1(s)$, we obtain

$$
\begin{aligned}
v_1(t) &= \mathcal{UL}^{-1}\{V_1(s)\} \\
&= \mathcal{UL}^{-1}\left\{\frac{4}{s+1} - \frac{3}{s+2}\right\} \\
&= 4\mathcal{UL}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathcal{UL}^{-1}\left\{\frac{1}{s+2}\right\} \\
&= 4e^{-t} - 3e^{-2t} \quad \text{for } t > 0^-.
\end{aligned}
$$

$\square$

Example 6.42 (RLC Network). Consider the RLC network shown in Figure 6.22 with input $v_0(t)$ and output $v_1(t)$. If $R = 2$, $L = 1$, $C = 1$, $v_0(t) = u(t)$, $v_1(0^-) = 0$, and $v_1'(0^-) = 1$, find $v_1(t)$.

Solution. From basic circuit analysis, we can write

$$v_0(t) = Ri(t) + L\frac{d}{dt}i(t) + v_1(t) \quad \text{and}$$

$$v_1(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \quad \Rightarrow \quad i(t) = C\frac{d}{dt}v_1(t).$$

Combining the above equations, we obtain

$$
\begin{aligned}
v_0(t) &= R\left[C\frac{d}{dt}v_1(t)\right] + L\frac{d}{dt}\left[C\frac{d}{dt}v_1(t)\right] + v_1(t) \\
&= RC\frac{d}{dt}v_1(t) + LC\frac{d^2}{dt^2}v_1(t) + v_1(t).
\end{aligned}
$$

Taking the unilateral Laplace transform of both sides of the preceding equation yields

$$\mathcal{UL}\{v_0(t)\} = RC\mathcal{UL}\left\{\frac{d}{dt}v_1(t)\right\} + LC\mathcal{UL}\left\{\frac{d^2}{dt^2}v_1(t)\right\} + \mathcal{UL}\{v_1(t)\}$$

$$\Rightarrow \quad V_0(s) = RC\left[sV_1(s) - v_1(0^-)\right] + LC\left[s^2V_1(s) - sv_1(0^-) - v_1'(0^-)\right] + V_1(s)$$

$$\Rightarrow \quad V_0(s) = RCsV_1(s) - RCv(0^-) + LCs^2V_1(s) - LCsv_1(0^-) - LCv_1'(0^-) + V_1(s)$$

$$\Rightarrow \quad \left[LCs^2 + RCs + 1\right]V_1(s) = V_0(s) + RCv(0^-) + LCsv_1(0^-) + LCv_1'(0^-)$$

$$\Rightarrow \quad V_1(s) = \frac{V_0(s) + [RC + LCs]v_1(0^-) + LCv_1'(0^-)}{LCs^2 + RCs + 1}.$$
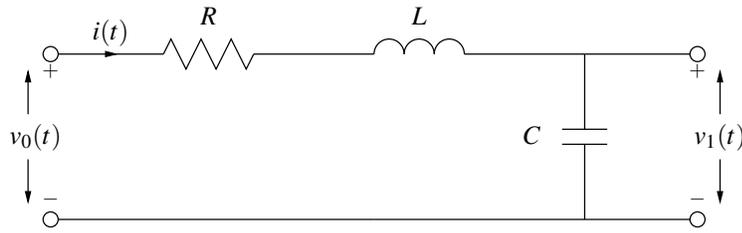
Figure 6.22: RLC network.

Since $v_0(t) = u(t)$, we have $V_0(s) = \frac{1}{s}$. Substituting this expression for $V_0(s)$ into the preceding equation for $V_1(s)$, we obtain

$$
\begin{aligned}
V_1(s) &= \frac{\left(\frac{1}{s}\right) + 1}{s^2 + 2s + 1} \\
&= \frac{\left(\frac{s+1}{s}\right)}{(s+1)^2} \\
&= \frac{1}{s(s+1)}.
\end{aligned}
$$

Now, we find a partial fraction expansion of $V_1(s)$. Such an expansion is of the form

$$
V_1(s) = \frac{A_1}{s} + \frac{A_2}{s+1}.
$$

Solving for the expansion coefficients, we obtain

$$
\begin{aligned}
A_1 &= sV_1(s)|_{s=0} \\
&= \left.\frac{1}{s+1}\right|_{s=0} \\
&= 1 \quad \text{and} \\
A_2 &= (s+1)V_1(s)|_{s=-1} \\
&= \left.\frac{1}{s}\right|_{s=-1} \\
&= -1.
\end{aligned}
$$

Thus, we can rewrite $V_1(s)$ as

$$
V_1(s) = \frac{1}{s} - \frac{1}{s+1}.
$$

Taking the inverse unilateral Laplace transform of $V_1(s)$, we obtain

$$
\begin{aligned}
v_1(t) &= \mathscr{UL}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{UL}^{-1}\left\{\frac{1}{s+1}\right\} \\
&= 1 - e^{-t} \quad \text{for } t > 0^-.
\end{aligned}
$$

$\square$

6.16 Problems

6.1 Using the definition of the Laplace transform, find the Laplace transform of each of the following signals (including the corresponding ROC):
(a) $x(t) = e^{-at}u(t)$;
(b) $x(t) = e^{-a|t|}$; and
(c) $x(t) = [\cos \omega_0 t]u(t)$. [Note: $\int e^{ax} \cos bx\, dx = \frac{1}{a^2+b^2}(e^{ax}[a\cos bx + b\sin bx])$.]

6.2 Find the Laplace transform (including the ROC) of the following signals:
(a) $x(t) = e^{-2t}u(t)$;
(b) $x(t) = 3e^{-2t}u(t) + 2e^{5t}u(-t)$;
(c) $x(t) = e^{-2t}u(t+4)$;
(d) $x(t) = \int_{-\infty}^{t} e^{-2\tau}u(\tau)d\tau$;
(e) $x(t) = -e^{at}u(-t+b)$ where a is a positive real constant and b is a real constant;
(f) $x(t) = te^{-3t}u(t+1)$; and
(g) $x(t) = tu(t+2)$.

6.3 Suppose that $x(t) \xleftrightarrow{\mathscr{L}} X(s)$ with ROC R_X and $y(t) \xleftrightarrow{\mathscr{L}} Y(s)$ with ROC R_Y. Then, express $Y(s)$ in terms of $X(s)$, and express R_Y in terms of R_X, in each of the cases below.
(a) $y(t) = x(at - b)$ where a and b are real constants and $a \neq 0$;
(b) $y(t) = e^{-3t}[x(\lambda) * x(\lambda)]|_{\lambda=t-1}$;
(c) $y(t) = tx(3t - 2)$;
(d) $y(t) = \frac{d}{dt}[x^*(t - 3)]$;
(e) $y(t) = e^{-5t}x(3t + 7)$; and
(f) $y(t) = e^{-j5t}x(t + 3)$.

6.4 The Laplace transform $X(s)$ of a signal $x(t)$ is given by

$$X(s) = \frac{(s + \frac{1}{2})(s - \frac{1}{2})}{s(s+1)(s-1)}.$$

Determine whether $x(t)$ is left sided, right sided, two sided, or finite duration for each of the following ROCs of $X(s)$: (a) $\operatorname{Re}\{s\} < -1$; (b) $-1 < \operatorname{Re}\{s\} < 0$; (c) $0 < \operatorname{Re}\{s\} < 1$; and (d) $\operatorname{Re}\{s\} > 1$.

6.5 Let $X(s)$ denote the Laplace transform of $x(t)$. Suppose that $X(s)$ is given by

$$X(s) = \frac{s + \frac{1}{2}}{(s+1-j)(s+1+j)(s+2)}.$$

Plot the ROC of $X(s)$ if $x(t)$ is (a) left sided; (b) right sided; (c) two sided; (d) causal.

6.6 The Laplace transform $X(s)$ of a signal $x(t)$ is given by

$$X(s) = \frac{s + \frac{1}{2}}{(s-1)(s+1-j)(s+1+j)(s+2)}.$$

Determine how many distinct possibilities exist for the signal $x(t)$.

6.7 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform $X(s)$ of the signal $x(t)$ for each of the cases below.
(a) $x(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ t-2 & \text{for } 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$

6.8 The Laplace transform $X(s)$ of the causal signal $x(t)$ is given by

$$X(s) = \frac{-2s}{s^2 + 3s + 2}.$$

(a) Assuming that $x(t)$ has no singularities at $t = 0$, find $x(0^+)$.
(b) Assuming that $x(t)$ has a finite limit as $t \to \infty$, find this limit.

6.9 Find the inverse Laplace transform of each of the following:
(a) $X(s) = \dfrac{s-5}{s^2-1}$ for $-1 < \mathrm{Re}\{s\} < 1$;
(b) $X(s) = \dfrac{2s^2 + 4s + 5}{(s+1)(s+2)}$ for $\mathrm{Re}\{s\} > -1$;
(c) $X(s) = \dfrac{3s+1}{s^2 + 3s + 2}$ for $-2 < \mathrm{Re}\{s\} < -1$.

6.10 Find the causal inverse Laplace transform of each of the following:
(a) $X(s) = \dfrac{s^2 + 4s + 5}{s^2 + 2s + 1}$; and
(b) $X(s) = \dfrac{-3s^2 - 6s - 2}{(s+1)^2(s+2)}$.

6.11 Find all possible inverse Laplace transforms of

$$H(s) = \frac{7s-1}{s^2-1} = \frac{4}{s+1} + \frac{3}{s-1}.$$

6.12 Suppose that we have a LTI system with input $x(t)$, output $y(t)$, and system function $H(s)$, where

$$H(s) = \frac{s+1}{s^2 + 2s + 2}.$$

Find the differential equation that characterizes the behavior of the system.

6.13 Suppose that we have a causal LTI system characterized by the differential equation

$$\tfrac{d^2}{dt^2} y(t) + 4 \tfrac{d}{dt} y(t) + 3y(t) = 2 \tfrac{d}{dt} x(t) + x(t),$$

where $x(t)$ and $y(t)$ denote the input and output of the system, respectively. Find the system function $H(s)$.

6.14 Consider the LTI system with input $x(t)$, output $y(t)$, and system function $H(s)$, as shown in the figure below. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are causal LTI systems with system functions $H_1(s)$ and $H_2(s)$, respectively, given by

$$H_1(s) = \frac{1}{s-1} \quad \text{and} \quad H_2(s) = A,$$

where A is a real constant.
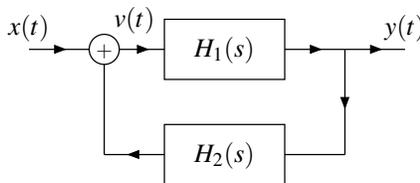
(a) Find an expression for $H(s)$ in terms of $H_1(s)$ and $H_2(s)$.
(b) Determine for what values of A the system is BIBO stable.

6.15 Suppose that we have the RL network with input $v_1(t)$ and output $v_2(t)$ as shown in the figure below. This system is LTI and can be characterized by a linear differential equation with constant coefficients.
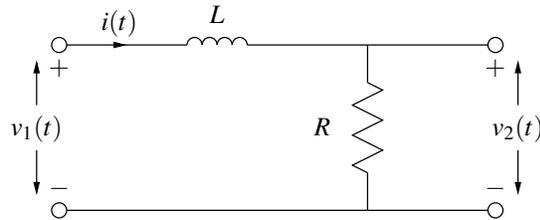
(a) Find the system function $H(s)$ of the system.
(b) Determine whether the system is stable.
(c) Find the step response $s(t)$ of the system.

6.16 Suppose that we have a causal (incrementally-linear TI) system with input $x(t)$ and output $y(t)$ characterized by the differential equation

$$\frac{d^2}{dt^2}y(t) + 7\frac{d}{dt}y(t) + 12y(t) = x(t).$$

If $y(0^-) = -1$, $y'(0^-) = 0$, and $x(t) = u(t)$, find $y(t)$.

6.17 Consider the RC network shown in the figure below, where $R = 1000$ and $C = \frac{1}{1000}$.
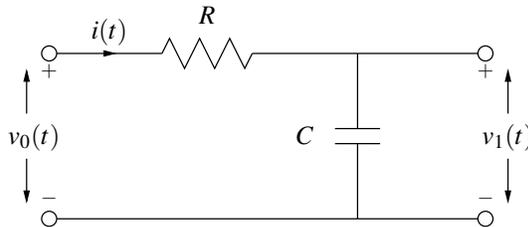
(a) Find the differential equation that characterizes the relationship between the input $v_0(t)$ and output $v_1(t)$.
(b) If $v_1(0^-) = 2$, and $v_0(t) = 2e^{-3t}$, find $v_1(t)$.

6.18 Using properties of the Laplace transform and a Laplace transform table, find the Laplace transform $X(s)$ of the signal $x(t)$ shown in the figure below.
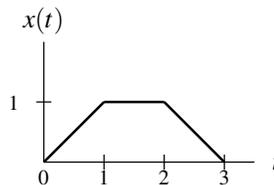
6.19 Suppose that we have a LTI system with system function $H(s)$ given by

$$H(s) = \frac{s^2 + 7s + 12}{s^2 + 3s + 12}.$$

Find all possible inverses of this system. For each inverse, identify its system function and the corresponding ROC. Also, indicate whether the inverse is causal and/or stable. (Note: You do not need to find the impulse responses of these inverse systems.)

6.20 Consider the causal LTI system characterized by the differential equation

$$\left(\frac{d}{dt}\right)^2 y(t) - y(t) = \frac{d}{dt}x(t) + ax(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

6.17 MATLAB Problems

6.101 Suppose that we have a causal LTI system with the system function

$$H(s) = \frac{1}{-2s^7 - s^6 - 3s^5 + 2s^3 + s - 3}.$$

(a) Use MATLAB to find and plot the poles of $H(s)$.
(b) Is this system stable?

6.102 Suppose that we have a LTI system with the system function

$$H(s) = \frac{1}{1.0000s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1.0000}.$$

(This corresponds to a fourth-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Plot the responses of the system to each of the inputs given below. In each case, plot the response over the interval $[0, 20]$.
(a) Dirac delta function;
(b) unit-step function.
(Hint: The `tf`, `impulse`, and `step` functions may be helpful.)

Appendix A

Complex Analysis

A.1 Introduction

Complex analysis is an essential tool in the study of signals and systems. For this reason, a brief review of complex analysis is provided in this appendix.

A.2 Complex Numbers

A **complex number** is a number of the form

$$z = x + jy$$

where x and y are real and j is the constant defined by

$$j^2 = -1$$

(i.e., $j = \sqrt{-1}$). The **real part**, **imaginary part**, **magnitude**, and **argument** of the complex number z are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, $|z|$, and $\arg z$, respectively, and defined as

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y, \quad |z| = \sqrt{x^2 + y^2}, \quad \text{and}$$
$$\arg z = \operatorname{atan2}(y, x) + 2\pi k,$$

where k is an arbitrary integer, and

$$\operatorname{atan2}(y,x) \triangleq \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ -\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & \text{for } x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & \text{for } x < 0 \text{ and } y < 0. \end{cases} \tag{A.1}$$

(The notation $\angle z$ is sometimes also used to denote the quantity $\arg z$.) The complex number z can be represented by a point (x, y) in the complex plane, as illustrated in Figure A.1. This figure also shows the relationship between the real part, imaginary part, magnitude, and argument of a complex number.

For any given complex number z, the quantity $\arg z$ is not unique. This follows from the fact that, for any integer k, the quantities θ and $\theta + 2\pi k$ physically represent the same overall angular displacement. The value θ of $\arg z$ that lies in the range $-\pi < \theta \leq \pi$ is called the **principal argument** of z and is denoted as $\operatorname{Arg} z$. For a particular z, this quantity is uniquely specified. In particular, $\operatorname{Arg} z = \operatorname{atan2}(y, x)$.

As an aside, we note that the function $\operatorname{atan2}(y, x)$ computes the angle that the directed line segment from $(0,0)$ to (x, y) forms with the real axis, and is defined such that $-\pi < \operatorname{atan2}(y, x) \leq \pi$.
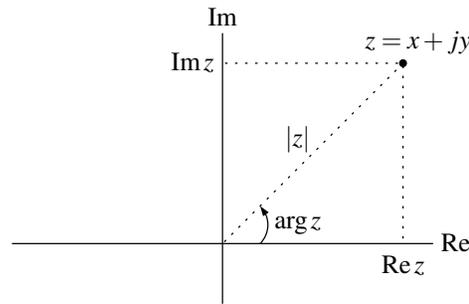
Figure A.1: Graphical representation of a complex number.

Example A.1. Compute the real part, imaginary part, magnitude, and principal argument of the following complex quantities: $z_1 = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ and $z_2 = 1 - j$.

Solution. First, let us consider z_1. We have:

$$\operatorname{Re} z_1 = \tfrac{\sqrt{3}}{2}, \ \operatorname{Im} z_1 = \tfrac{1}{2},$$

$$|z_1| = \sqrt{\left(\tfrac{\sqrt{3}}{2}\right)^2 + (\tfrac{1}{2})^2} = 1, \text{ and}$$

$$\operatorname{Arg} z_1 = \operatorname{atan2}\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right) = \arctan\left(\tfrac{1}{\sqrt{3}}\right) = \tfrac{\pi}{6}.$$

Second, let us consider z_2. We can write:

$$\operatorname{Re} z_2 = 1, \ \operatorname{Im} z_2 = -1,$$

$$|z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \text{ and}$$

$$\operatorname{Arg} z_2 = \operatorname{atan2}(-1, 1) = \arctan\left(\tfrac{-1}{1}\right) = -\pi/4.$$

$\square$

A.3   Representations of Complex Numbers

Two different representations are commonly used for complex numbers: the Cartesian and polar forms. The Cartesian form is also sometimes referred to as rectangular form. Depending on the particular situation, one form may be more convenient to use than the other.

In the case of the **Cartesian form**, a complex number z is represented as

$$z = x + jy$$

where x and y are real. That is, z is expressed directly in terms of its real and imaginary parts. The quantity z can also be treated as a point (x, y) in a Cartesian coordinate system as shown in Figure A.2(a).

In the case of the **polar form**, a complex number z is represented as

$$z = r(\cos\theta + j\sin\theta)$$

where r and θ are real and $r \geq 0$. One can show through simple geometry that $r = |z|$ and $\theta = \arg z$. Thus, in the polar case, a complex number is expressed directly in terms of its magnitude and argument. In this way, we can treat the quantity z as a point (r, θ) in a polar coordinate system, as shown in Figure A.2(b). As we note later, $\cos\theta + j\sin\theta = e^{j\theta}$. Therefore, the polar form can equivalently be expressed as

$$z = re^{j\theta}.$$

This exponential notation is often used, due to its relative compactness.
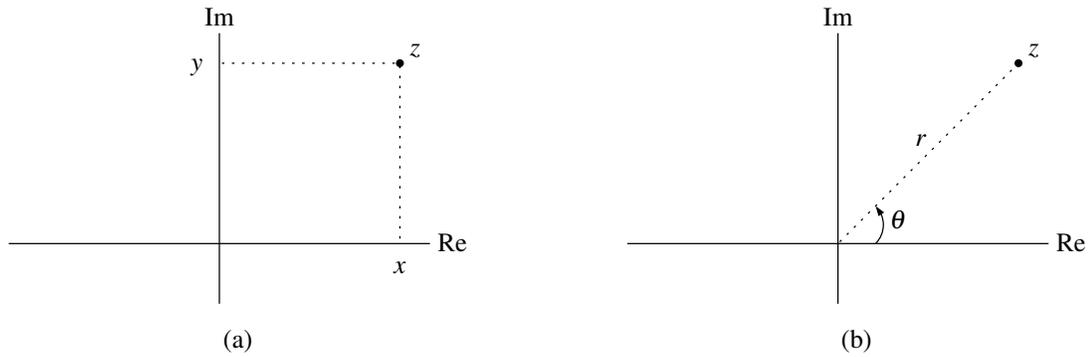
Figure A.2: Representations of complex numbers. The (a) Cartesian and (b) polar forms.

A.4   Arithmetic Operations

In what follows, we consider a few basic arithmetic operations on complex numbers.

A.4.1   Conjugation

The **conjugate** of the complex number $z = x + jy$ is denoted as z^* and defined as

$$z^* = x - jy.$$

Geometrically, the conjugation operation reflects a point in the complex plane about the real axis, as illustrated in Figure A.3. One can easily verify that for any complex numbers z, z_1, and z_2, the following identities hold:

$$zz^* = |z|^2,$$
$$\operatorname{Re} z = \tfrac{1}{2}(z + z^*),$$
$$\operatorname{Im} z = \tfrac{1}{2j}(z - z^*),$$
$$(z_1 + z_2)^* = z_1^* + z_2^*,$$
$$(z_1 z_2)^* = z_1^* z_2^*, \text{ and}$$
$$(z_1/z_2)^* = z_1^*/z_2^*.$$

Trivially, we also have that

$$|z^*| = |z| \quad \text{and}$$
$$\arg z^* = -\arg z.$$

A.4.2   Addition

Suppose that we have two complex numbers $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$. Then, the sum of these numbers is given by

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$
$$= (x_1 + x_2) + j(y_1 + y_2),$$

or

$$z_1 + z_2 = r_1 e^{j\theta_1} + r_2 e^{j\theta_2}$$
$$= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2).$$
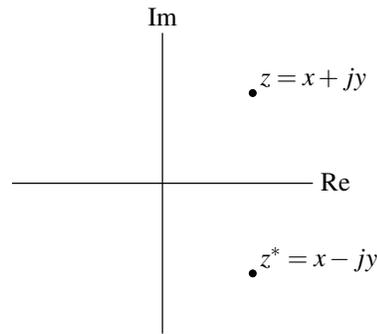
Figure A.3: Conjugate of complex number.

Example A.2 (Addition with Cartesian form). Compute $z_1 + z_2$ where $z_1 = 3 + j4$ and $z_2 = -2 - j3$.

Solution. This is simply computed as follows:

$$z_1 + z_2 = (3 + j4) + (-2 - j3) = 1 + j.$$

$\square$

Example A.3 (Addition involving the polar form). Given that $z_1 = 3 + j4$ and $z_2 = \sqrt{2}e^{j5\pi/4}$, compute the quantity $z_1 + z_2$.

Solution. In order to compute this sum, we must first convert z_2 to Cartesian form:

$$\operatorname{Re} z_2 = \sqrt{2}\cos 5\pi/4 = -1 \quad \text{and}$$
$$\operatorname{Im} z_2 = \sqrt{2}\sin 5\pi/4 = -1.$$

Then, we have

$$z_1 + z_2 = (3 + j4) + (-1 - j) = 2 + j3.$$

$\square$

A.4.3 Multiplication

Suppose that we have two complex numbers $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$. Then, the product of these numbers is given by

$$\begin{aligned}
z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\
&= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2 \\
&= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1).
\end{aligned}$$

In the case of the polar form, we have

$$\begin{aligned}
z_1 z_2 &= \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) \\
&= r_1 r_2 e^{j(\theta_1 + \theta_2)}.
\end{aligned}$$

Example A.4 (Multiplication with the Cartesian form). Given that $z_1 = 2 + j3$ and $z_2 = 3 - j4$, compute the quantity $z_1 z_2$.

Solution. Using straightforward algebraic manipulation, we can write:

$$z_1 z_2 = (2 + j3)(3 - j4) = 6 - j8 + j9 + 12 = 18 + j.$$

$\square$

Example A.5 (Multiplication with the polar form). Given that $z_1 = \sqrt{2}e^{j\pi/4}$ and $z_2 = 3e^{j\pi/6}$, compute the quantity $z_1 z_2$.

Solution. Using straightforward algebraic manipulation, we obtain:

$$z_1 z_2 = (\sqrt{2}e^{j\pi/4})(3e^{j\pi/6}) = 3\sqrt{2}e^{j5\pi/12}.$$

$\square$

A.4.4 Division

Suppose that we have two complex numbers $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$ and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$. In the case of Cartesian form, the quotient of these numbers is given by

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1 z_2^*}{z_2 z_2^*} \\
&= \frac{z_1 z_2^*}{|z_2|^2} \\
&= \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2} \\
&= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} \\
&= \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}.
\end{aligned}$$

In the case of polar form, we have

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \\
&= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}.
\end{aligned}$$

Example A.6 (Division with the Cartesian form). Given that $z_1 = 1 + j$ and $z_2 = 2 - j$, compute the quantity z_1/z_2.

Solution. Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \left(\frac{1+j}{2-j}\right)\left(\frac{2+j}{2+j}\right) = \frac{2 + 2j + j - 1}{2^2 + 1^2} = \frac{1 + 3j}{5}.$$

$\square$

Example A.7 (Division with the polar form). Given that $z_1 = 2e^{j\pi/3}$ and $z_2 = 3e^{j\pi/4}$, compute the quantity z_1/z_2.

Solution. Using straightforward algebraic manipulation, we have

$$\frac{z_1}{z_2} = \frac{2e^{j\pi/3}}{3e^{j\pi/4}} = \frac{2}{3}e^{j\left(\frac{\pi}{3} - \frac{\pi}{4}\right)} = \frac{2}{3}e^{j\pi/12}.$$

$\square$

A.4.5 Miscellany

For arbitrary complex numbers z_1 and z_2, the following identities hold:

$$|z_1 z_2| = |z_1| \, |z_2|,$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0,$$

$$\arg z_1 z_2 = \arg z_1 + \arg z_2, \quad \text{and}$$

$$\arg \left(\frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0.$$

A.5 Arithmetic Properties of Complex Numbers

In what follows, we consider some of the properties of arithmetic over the complex numbers.

A.5.1 Commutative Property

For complex numbers, addition and multiplication are commutative operations. That is, for any two complex numbers z_1 and z_2, we have

$$z_1 + z_2 = z_2 + z_1 \quad \text{and}$$

$$z_1 z_2 = z_2 z_1.$$

A.5.2 Associative Property

For complex numbers, addition and multiplication are associative operations. That is, for any two complex numbers z_1 and z_2, we have

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

A.5.3 Distributive Property

The distributive property also holds for complex numbers. That is, for any three complex numbers z_1, z_2, and z_3, the following identity holds:

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

A.6 Roots of Complex Numbers

Every complex number z has n distinct nth roots in the complex plane. In particular, the nth roots of $z = re^{j\theta}$ (where $r = |z|$ and $\theta = \arg z$) are given by

$$\sqrt[n]{r} e^{j(\theta + 2\pi k)/n} \quad \text{for } k = 0, 1, \ldots, n-1. \tag{A.2}$$

Example A.8. Find the four fourth roots of 2.

Solution. Let z_k for $k = 0, 1, 2, 3$ denote each of the four fourth roots of 2. From (A.2), we have

$$z_k = \sqrt[4]{2} e^{j(0 + 2\pi k)/4} = \sqrt[4]{2} e^{jk\pi/2}.$$

Thus, we obtain

$$z_0 = \sqrt[4]{2}e^{j0} = \sqrt[4]{2},$$
$$z_1 = \sqrt[4]{2}e^{j\pi/2} = j\sqrt[4]{2},$$
$$z_2 = \sqrt[4]{2}e^{j\pi} = -\sqrt[4]{2}, \quad \text{and}$$
$$z_3 = \sqrt[4]{2}e^{j3\pi/2} = -j\sqrt[4]{2}.$$

So, we conclude that the four fourth roots of 2 are

$$\sqrt[4]{2}, \quad j\sqrt[4]{2}, \quad -\sqrt[4]{2}, \quad \text{and} \quad -j\sqrt[4]{2}.$$

$\square$

A.7 Euler's Relation and De Moivre's Theorem

Recall that the Maclaurin series expansions of e^x, $\cos x$, and $\sin x$ are given by:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{A.3}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \quad \text{and} \tag{A.4}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{A.5}$$

Using (A.3), we can write $e^{j\theta}$ as the following series:

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots.$$

By regrouping terms and using (A.4) and (A.5), we obtain

$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$
$$= \cos\theta + j\sin\theta.$$

Thus, we have shown that the following identity holds:

$$e^{j\theta} = \cos\theta + j\sin\theta. \tag{A.6}$$

This identity is known as **Euler's relation**. From Euler's relation, we can deduce the following useful identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and} \tag{A.7a}$$
$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}). \tag{A.7b}$$

We can also show that the following useful result holds:

$$e^{jn\theta} = \left(e^{j\theta}\right)^n \quad \text{for integer } n \text{ (and real } \theta\text{)}.$$

This result is known as **De Moivre's theorem**. Note that n must be an integer. The above result does not necessarily hold if n is not an integer. For example, consider $\theta = -\pi$ and $n = \frac{1}{2}$ (where n is clearly not an integer). We have that

$$e^{jn\theta} = e^{j(1/2)(-\pi)} = e^{-j\pi/2} = -j \quad \text{and}$$
$$\left(e^{j\theta}\right)^n = \left(e^{-j\pi}\right)^{1/2} = (-1)^{1/2} = j.$$

Clearly, in this case $e^{jn\theta} \neq \left(e^{j\theta}\right)^n$.

A.8 Conversion Between Cartesian and Polar Form

Suppose that we have a complex number $z = x + jy = re^{j\theta}$. Using Euler's relation, we can derive the following expressions for converting from polar to Cartesian form:

$$x = r\cos\theta \quad \text{and} \tag{A.8a}$$
$$y = r\sin\theta. \tag{A.8b}$$

Similarly, we can deduce the following expressions for converting from Cartesian to polar form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \tag{A.9a}$$
$$\theta = \text{atan2}(y, x), \tag{A.9b}$$

where the atan2 function is as defined in (A.1).

If we choose to use the arctan function directly in order to compute θ (instead of using the atan2 function), we must be careful to consider the quadrant in which the point (x, y) lies. This complication is due to the fact that the principal value of arctan θ is defined such that $-\pi/2 \le \theta \le \pi/2$. Consequently, if the point does not lie in the first or fourth quadrant of the complex plane, the arctan function will not yield the desired angle.

Example A.9. Express the quantities $z_1 = 1 + j\sqrt{2}$ and $z_2 = -1 - j$ in polar form.

Solution. First, we express z_1 in polar form. The real and imaginary parts of z_1 are given by

$$r_1 = \sqrt{1^2 + \left(\sqrt{2}\right)^2} = \sqrt{3} \quad \text{and}$$
$$\theta_1 = \text{atan2}(\sqrt{2}, 1) = \arctan(\sqrt{2}/1).$$

Thus, we have

$$z_1 = r_1 e^{j\theta_1} = \sqrt{3} e^{j(\arctan\sqrt{2})}.$$

The result is plotted in Figure A.4(a).

Second, we express z_2 in polar form. The real and imaginary parts of z_2 are given by

$$r_2 = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} \quad \text{and}$$
$$\theta_2 = \text{atan2}(-1, -1) = \arctan(\tfrac{-1}{-1}) - \pi = -\tfrac{3\pi}{4}.$$

Thus, we have

$$z_2 = r_2 e^{j\theta_2} = \sqrt{2} e^{-j3\pi/4}.$$

The result is plotted in Figure A.4(b). □

A.9 Complex Functions

A complex function maps complex numbers to complex numbers. For example, the function

$$F(z) = z^2 + 2z + 1 \quad \text{where } z \text{ is complex}$$

is a complex function.

A **complex polynomial function** is mapping of the form
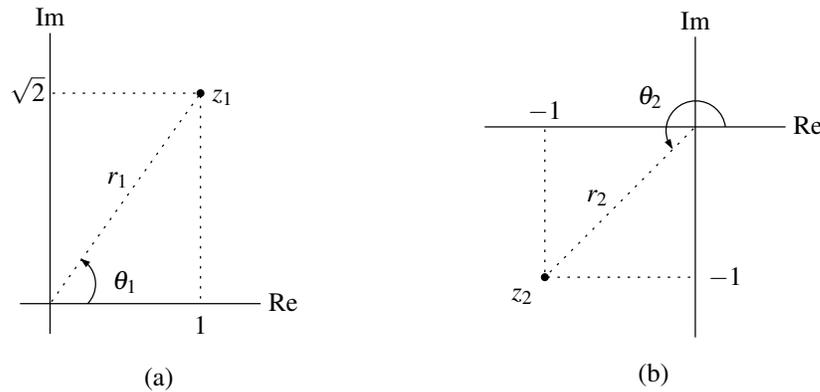
$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

Figure A.4: Examples of converting complex numbers from Cartesian to polar forms.

where z, a_0, a_1, ..., a_n are complex. A **complex rational function** is a mapping of the form

$$F(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m}$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex. In the context of this course, we will mostly focus our attention on polynomial and rational functions.

Given any complex function $F(z)$, we can always write $F(z) = \text{Re}\{F(z)\} + j\,\text{Im}\{F(z)\}$. If $z = x + jy$, then we have that $F(z) = \text{Re}\{F(x+jy)\} + j\,\text{Im}\{F(x+jy)\}$. Now, we can express $\text{Re}\{F(x+jy)\}$ as a real-valued function, say $U(x,y)$, of the two real variables x and y. Similarly, we can express $\text{Im}\{F(x+jy)\}$ as a real-valued function, say $V(x,y)$, of the two real variables x and y. Thus, we can always express a complex function $F(z)$ in the form

$$F(z) = F(x,y) = U(x,y) + jV(x,y), \tag{A.10}$$

where $U(x,y)$ and $V(x,y)$ and both real-valued functions of the two real variables x and y, and $z = x + jy$.

A.10 Circles, Disks, and Annuli

A **circle** in the complex plane with center z_0 and radius r is given by the equation

$$|z - z_0| = r.$$

A plot of a circle is shown in Figure A.5.

A **disk** is the set of points inside of a circle, possibly including the points on the circle itself. If the points on the circle are not included in the set, the disk is said to be open. Otherwise, the disk is said to be closed. An **open disk** with center z_0 and radius r is defined by the inequality

$$|z - z_0| < r.$$

A plot of an open disk is shown in Figure A.6. Similarly, a **closed disk** is defined by the inequality

$$|z - z_0| \leq r.$$

A plot of a closed disk is shown in Figure A.7.

An **annulus** (i.e., a ring) is the set of points between the concentric circles, possibly including the points on one or both circles. If an annulus does not include the points on its two defining circles, it is said to be open. An **open annulus** with center z_0, inner radius r_1, and outer radius r_2 is defined by the inequality

$$r_1 < |z - z_0| < r_2.$$

A plot of an annulus is shown in Figure A.8.
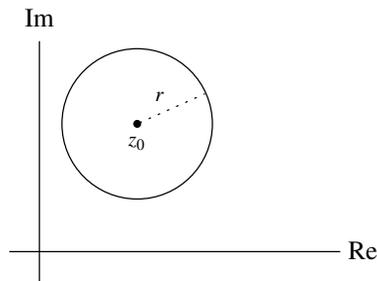
Figure A.5: Circle about z_0 with radius r.
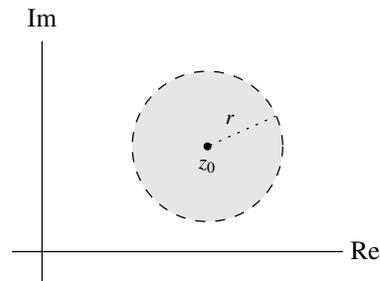
Figure A.6: Open disk of radius r about z_0.
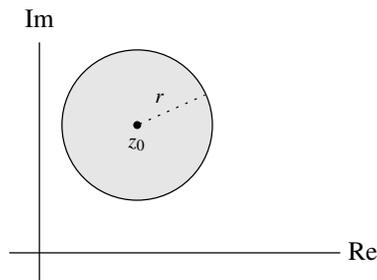
Figure A.7: Closed disk of radius r about z_0.
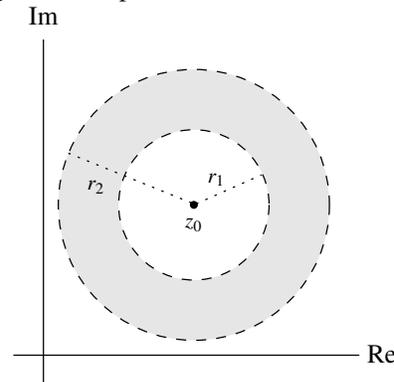
Figure A.8: Open annulus about z_0 with inner radius r_1 and outer radius r_2.

A.11 Limit

Let $F(z)$ be a complex function and z_0 a complex number. We want to define the limit of $F(z)$ as z approaches z_0. Unlike in the case of real functions, the value z_0 can be approached from infinitely many directions in the complex plane. In order for the limit to be useful, however, we want it to be defined in such a way that it is independent of the direction from which z_0 is approached. With this in mind, we define the notion of a limit below.

A function $F(z)$ is said to have the limit L as z approaches z_0 if

1. $F(z)$ is defined in some open disk about z_0, except possibly at the point z_0; and

2. for every positive real number ε, there exists a positive real number δ such that $|F(z) - L| < \varepsilon$ for all values of z in the disk $|z - z_0| < \delta$ except $z = z_0$.

This limit can be expressed as

$$\lim_{z \to z_0} F(z) = L.$$

A.12 Continuity

A function $F(z)$ is said to be **continuous** at the point z_0 if $F(z_0)$ is defined and given by

$$F(z_0) = \lim_{z \to z_0} F(z).$$

A **continuous function** is one that is continuous at every point where it is defined. Polynomial functions are continuous everywhere. For example, the function $F(z) = 3z^3 + z^2 - z + 1$ is continuous for all z. Rational functions (i.e., quotients of polynomials) are continuous everywhere except at points where the denominator polynomial becomes zero. For example, $F(z) = \frac{(z+j)(z-j)}{(z-1)(z+1)}$ is continuous everywhere except at $z = 1$ and $z = -1$.

A.13 Differentiability

A function $F(z)$ is said to be **differentiable** at a point $z = z_0$ if the limit

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$

exists. This limit is called the **derivative** of $F(z)$ at the point $z = z_0$.

In general, the rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. If $F'(z_0)$ and $G'(z_0)$ exist, then

1. $(aF)'(z_0) = aF'(z_0)$ for any complex constant a;

2. $(F + G)'(z_0) = F'(z_0) + G'(z_0)$;

3. $(FG)'(z_0) = F'(z_0)G(z_0) + F(z_0)G'(z_0)$;

4. $(F/G)'(z_0) = \frac{G(z_0)F'(z_0) - F(z_0)G'(z_0)}{G(z_0)^2}$; and

5. if $z_0 = G(w_0)$ and $G'(w_0)$ exists, then the derivative of $F(G(z))$ at w_0 is $F'(z_0)G'(w_0)$ (i.e., the chain rule).

Polynomial functions are differentiable everywhere. Rational functions are differentiable everywhere except at points where the denominator polynomial becomes zero.

A.14 Analyticity

A function $F(z)$ is said to be **analytic** at a point z_0 if it is differentiable at every point in some open disk about z_0. A function $F(z)$ is said to be analytic in a domain D if it is analytic at every point in D.

One extremely useful test for the analyticity of a function is given by the theorem below.

Theorem A.1 (Cauchy-Riemann Equations). *Let $F(z)$ be a complex function expressed in the form of (A.10). So, we have*

$$F(z) = U(x, y) + jV(x, y).$$

If $F(z)$ is analytic in the domain D, then $U(x, y)$ and $V(x, y)$ satisfy the following equations at all points in D:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad and \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.$$

*These equations are known as the **Cauchy-Riemann equations**.*

If $U(x, y)$ and $V(x, y)$ are continuous with continuous first partial derivatives satisfying the Cauchy-Riemann equations at all points in the domain D, then $F(z)$ is analytic in D.

Polynomial functions are both continuous and differentiable everywhere. Therefore, such functions are analytic everywhere. Rational functions are both continuous and differentiable everywhere, except at points where the denominator polynomial becomes zero. Consequently, rational functions are analytic at all but these points.

Example A.10. Determine for what values of z the function $F(z) = z^2$ is analytic.

Solution. First, we observe that $F(z)$ is a polynomial function. Then, we recall that polynomial functions are analytic everywhere. Therefore, $F(z)$ is analytic everywhere. $\square$

Alternate Solution. We can demonstrate the analyticity of $F(z)$ using Theorem A.1. We express z in Cartesian form as $z = x + jy$. We rewrite $F(z)$ in the form of $F(x,y) = U(x,y) + jV(x,y)$ as follows:

$$F(z) = z^2 = F(x,y) = (x + jy)^2 = x^2 + j2xy - y^2 = (x^2 - y^2) + j(2xy).$$

Thus, we have that $F(z) = U(x,y) + jV(x,y)$ where

$$U(x,y) = x^2 - y^2 \quad \text{and}$$
$$V(x,y) = 2xy.$$

Now, we compute the partial derivatives:

$$\frac{\partial U}{\partial x} = 2x, \quad \frac{\partial V}{\partial y} = 2x, \quad \frac{\partial U}{\partial y} = -2y, \quad \text{and} \quad \frac{\partial V}{\partial x} = 2y.$$

From this, we can see that $\partial U/\partial x = \partial V/\partial y$ and $\partial U/\partial y = -\partial V/\partial x$. Therefore, the Cauchy-Riemann equations are satisfied. Since $U(x,y)$ and $V(x,y)$ are also continuous with continuous first partial derivatives, $F(z)$ is analytic.   □

Example A.11. Determine for what values of z the function $F(z) = 1/z$ is analytic.

Solution. We can deduce the analyticity properties of $F(z)$ as follows. First, we observe that $F(z)$ is a rational function. Then, we recall that a rational function is analytic everywhere except at points where its denominator polynomial becomes zero. Since the denominator polynomial of $F(z)$ only becomes zero at $z = 0$, $F(z)$ is analytic everywhere except at $z = 0$.   □

Alternate Solution. To study of the analyticity of $F(z)$, we use Theorem A.1. We express z in Cartesian form as $z = x + jy$. We rewrite $F(z)$ in the form $F(x,y) = U(x,y) + jV(x,y)$ as follows:

$$F(z) = 1/z = F(x,y) = \frac{1}{x + jy} = \left(\frac{1}{x + jy}\right)\left(\frac{x - jy}{x - jy}\right) = \frac{x - jy}{x^2 + y^2}.$$

Thus, we have that $F(x,y) = U(x,y) + jV(x,y)$ where

$$U(x,y) = \frac{x}{x^2 + y^2} = x(x^2 + y^2)^{-1} \quad \text{and}$$
$$V(x,y) = \frac{-y}{x^2 + y^2} = -y(x^2 + y^2)^{-1}.$$

Now, we compute the partial derivatives:

$$\frac{\partial U}{\partial x} = (x^2 + y^2)^{-1} + (-1)(x^2 + y^2)^{-2}(2x^2) = \frac{-2x^2 + (x^2 + y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$
$$\frac{\partial V}{\partial y} = (-1)(x^2 + y^2)^{-1} + (-1)(x^2 + y^2)^{-2}(2y)(-y) = \frac{2y^2 - (x^2 + y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$
$$\frac{\partial U}{\partial y} = (-1)(x^2 + y^2)^{-2}(2y)x = \frac{-2xy}{(x^2 + y^2)^2}, \quad \text{and}$$
$$\frac{\partial V}{\partial x} = (-1)(x^2 + y^2)^{-2}(2x)(-y) = \frac{2xy}{(x^2 + y^2)^2}.$$

So, we have that, for $z \neq 0$ (i.e., x and y not both zero), $\partial U/\partial x = \partial V/\partial y$ and $\partial U/\partial y = -\partial V/\partial x$ (i.e., the Cauchy-Riemann equations are satisfied). Also, we know that for $z \neq 0$, $U(x,y)$ and $V(x,y)$ are continuous with continuous first partial derivatives. Therefore, $F(z)$ is analytic everywhere except at $z = 0$.   □

A.15 Zeros and Singularities

If a function $F(z)$ is analytic in a domain D and is zero at a point z_0 in D, the function is said to have a **zero** at z_0. If, in addition, the derivatives $F^{(1)}(z)$, $F^{(2)}(z)$, $F^{(n-1)}(z)$ are also zero, the function is said to have a zero of order n at z_0. An analytic function $F(z)$ is said to have an nth order zero at infinity if $F(1/z)$ has an nth order zero at $z = 0$.

A point at which a function $F(z)$ fails to be analytic is called a **singularity**. A singularity may be isolated or nonisolated. If $F(z)$ is analytic in an annulus $0 < |z - z_0| < r$ but not at z_0, $F(z)$ is said to have an isolated singularity at z_0. There are three types of isolated singularities: a removable singularity, essential singularity, and pole.

In this course, we are primarily interested in polynomial and rational functions. Polynomial functions do not have singularities, since such functions are analytic everywhere. In contrast, rational functions can have singularities. In the case of rational functions, we are normally interested in poles.

Suppose that we have a rational function $F(z)$. We can always express such a function in factored form as

$$F(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

where K is complex, $a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_N$ are distinct complex values, and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ are positive integers. One can show that $F(z)$ has poles at $b_1, b_2, \ldots, b_N$ and zeros at $a_1, a_2, \ldots, a_M$. Furthermore, the kth pole (i.e., b_k) is of order β_k, and the kth zero (i.e., a_k) is of order α_k. A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

Example A.12 (Poles and zeros of a rational function). Find and plot the poles and (finite) zeros of the function

$$F(z) = \frac{z^2(z^2+1)(z-1)}{(z+1)(z^2+3z+2)(z^2+2z+2)}.$$

Solution. We observe that $F(z)$ is a rational function, so we can easily determine the poles and zeros of $F(z)$ from its factored form. We now proceed to factor $F(z)$. First, we factor $z^2 + 3z + 2$. To do this, we solve for the roots of $z^2 + 3z + 2 = 0$ as follows:

$$z = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} = -\frac{3}{2} \pm \frac{1}{2} = \{-1, -2\}.$$

(For additional information on how to find the roots of a quadratic equation, see Section A.16.) So, we have $z^2 + 3z + 2 = (z+1)(z+2)$. Second, we factor $z^2 + 2z + 2$. To do this, we solve for the roots of $z^2 + 2z + 2 = 0$ as follows:

$$z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = -1 \pm j = \{-1+j, -1-j\}.$$

So, we have $z^2 + 2z + 2 = (z+1-j)(z+1+j)$. Next, we observe that $z^2 + 1 = (z+j)(z-j)$. Using the above results, we can rewrite $F(z)$ as

$$F(z) = \frac{z^2(z+j)(z-j)(z-1)}{(z+1)^2(z+2)(z+1-j)(z+1+j)}.$$

From this expression, we can trivially deduce that $F(z)$ has first order zeros at 1, j, and $-j$, a second order zero at 0, first order poles at $-1+j$, $-1-j$, -2, and a second order pole at -1. The zeros and poles of this function are plotted in Figure A.9. In such plots, the poles and zeros are typically denoted by the symbols "x" and "o", respectively. □

Example A.13. Find the zeros and poles of the function $F(s) = \frac{s^3 - 2}{s^5 + 4}$.
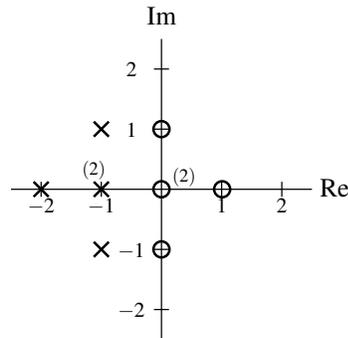
Figure A.9: Plot of poles and zeros for $F(z)$.

Solution. The zeros of $F(s)$ are given by the roots of $s^3 - 2 = 0$, which is equivalent to $s^3 = 2 = 2e^{j0}$. This equation has three distinct solutions (i.e., the three third roots of 2), namely,

$$\sqrt[3]{2}, \quad \sqrt[3]{2}e^{j2\pi/3}, \quad \text{and} \quad \sqrt[3]{2}e^{j4\pi/3}.$$

The poles of $F(s)$ are given by the roots of $s^5 + 4 = 0$, which is equivalent to $s^5 = -4 = 4e^{-j\pi}$. This equation has five distinct solutions (i.e., the five fifth roots of -4), namely,

$$\sqrt[5]{4}e^{j(-\pi+0)/5} = \sqrt[5]{4}e^{-j\pi/5},$$
$$\sqrt[5]{4}e^{j(-\pi+2\pi)/5} = \sqrt[5]{4}e^{j\pi/5},$$
$$\sqrt[5]{4}e^{j(-\pi+4\pi)/5} = \sqrt[5]{4}e^{j3\pi/5},$$
$$\sqrt[5]{4}e^{j(-\pi+6\pi)/5} = \sqrt[5]{4}e^{j\pi} = -\sqrt[5]{4}, \quad \text{and}$$
$$\sqrt[5]{4}e^{j(-\pi+8\pi)/5} = \sqrt[5]{4}e^{j7\pi/5}.$$

$\square$

A.16   Quadratic Formula

Consider the equation $az^2 + bz + c = 0$ where a, b, and c are real, z is complex, and $a \neq 0$. The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{A.11}$$

This formula is often useful in factoring quadratic polynomials with complex conjugate roots. For example, the general quadratic $az^2 + bz + c$ has the factors $z - z_0$ and $z - z_1$ where $z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$. In particular, $az^2 + bz + c = a(z - z_0)(z - z_1)$.

A.17 Problems

A.1 Express each of the following complex numbers in Cartesian form: (a) $2e^{j2\pi/3}$; (b) $\sqrt{2}e^{j\pi/4}$; (c) $2e^{j7\pi/6}$; and (d) $3e^{j\pi/2}$.

A.2 Express each of the following complex numbers in polar form: (a) $-\sqrt{3}+j$; (b) $-\frac{1}{2}-j\frac{\sqrt{3}}{2}$; (c) $\sqrt{2}-j\sqrt{2}$; (d) $1+j\sqrt{3}$; (e) $-1-j\sqrt{3}$; and (f) $-3+4j$. In each case, plot the value in the complex plane, clearly indicating its magnitude and argument. State the principal value for the argument (i.e., the value θ of the argument that lies in the range $-\pi < \theta \le \pi$).

A.3 Evaluate each of the expressions below, stating the final result in the specified form. When giving a final result in polar form, state the principal value of the argument (i.e., choose the argument θ such that $-\pi < \theta \le \pi$).
(a) $2\left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)+j\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ (in Cartesian form);
(b) $\left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}e^{j(-3\pi/4)}\right)$ (in polar form);
(c) $\left(\frac{\sqrt{3}}{2}-j\frac{1}{2}\right)/(1+j)$ (in polar form);
(d) $e^{1+j\pi/4}$ (in Cartesian form);
(e) $\left(\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^*\right)^8$ (in polar form);
(f) $(1+j)^{10}$ (in Cartesian form); and
(g) $\dfrac{1+j}{1-j}$ (in polar form).

A.4 Show that the following identities hold, where z, z_1, and z_2 are arbitrary complex numbers:
(a) $|z_1/z_2| = |z_1|/|z_2|$ for $z_2 \ne 0$;
(b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ for $z_2 \ne 0$;
(c) $z + z^* = 2\operatorname{Re}\{z\}$;
(d) $zz^* = |z|^2$; and
(e) $(z_1 z_2)^* = z_1^* z_2^*$.

A.5 Use Euler's relation to prove the following identities:
(a) $\cos\theta = \frac{1}{2}\left[e^{j\theta}+e^{-j\theta}\right]$;
(b) $\sin\theta = \frac{1}{2j}\left[e^{j\theta}-e^{-j\theta}\right]$; and
(c) $\cos^2\theta = \frac{1}{2}[1+\cos 2\theta]$.

A.6 Consider the rational functions given below, where z is a complex variable. For each function, find the value and order of its poles and zeros. Also, plot the poles and zeros in the complex plane.
(a) $F(z) = z^2 + jz + 3$;
(b) $F(z) = z + 3 + 2z^{-1}$;
(c) $F(z) = \dfrac{(z^2+2z+5)(z^2+1)}{(z^2+2z+2)(z^2+3z+2)}$;
(d) $F(z) = \dfrac{z^3 - z}{z^2 - 4}$;
(e) $F(z) = \dfrac{z+\frac{1}{2}}{(z^2+2z+2)(z^2-1)}$; and
(f) $F(z) = \dfrac{z^2(z^2-1)}{(z^2+4z+\frac{17}{4})^2(z^2+2z+2)}$.

A.7 Determine the values of z for which each of the functions given below is: i) continuous, ii) differentiable, and iii) analytic. Use your knowledge about polynomial and rational functions to deduce the answer. Simply state the final answer along with a short justification (i.e., two or three sentences). (This problem does not require a rigorous proof. In other words, do not use the Cauchy-Riemann equations for this problem.)

(a) $F(z) = 3z^3 - jz^2 + z - \pi$ and

(b) $F(z) = \dfrac{z-1}{(z^2+3)(z^2+z+1)}$.

A.8 Use the Cauchy-Riemann equations to show that the function $F(z) = e^{az}$ is analytic for all z, where a is a real constant and z is complex.

A.9 Let $H(\omega)$ be a complex-valued function of the real variable ω. For each of the cases below, find $|H(\omega)|$ and $\arg H(\omega)$.

(a) $H(\omega) = \dfrac{1}{(1+j\omega)^{10}}$; and

(b) $H(\omega) = \dfrac{-2-j\omega}{(3+j\omega)^2}$.

A.18   MATLAB Problems

A.101 Use the `sym` and `factor` functions of the Symbolic Math Toolbox in order to find the poles and zeros of the rational function

$$F(z) = \frac{64z^4 - 48z^3 - 36z^2 + 27z}{64z^6 - 128z^5 - 112z^4 + 320z^3 - 84z^2 - 72z + 27}.$$

A.102 Use the roots function to find the poles and zeros of the rational function

$$F(z) = \frac{z^4 + 6z^3 + 10z^2 + 8z}{z^9 + 21z^8 + 199z^7 + 1111z^6 + 4007z^5 + 9639z^4 + 15401z^3 + 15689z^2 + 9192z + 2340}.$$

Plot these poles and zeros using the plot function.

Appendix B

Partial Fraction Expansions

Sometimes we find it beneficial to be able to express a rational function as a sum of lower-order rational functions. This type of decomposition is known as a partial fraction expansion. Partial fraction expansions are often useful in the calculation of inverse Laplace and inverse Fourier transforms.

Suppose that we have a rational function

$$F(v) = \frac{\alpha_m v^m + \alpha_{m-1} v^{m-1} + \ldots + \alpha_1 v + \alpha_0}{\beta_n v^n + \beta_{n-1} v^{n-1} + \ldots + \beta_1 v + \beta_0}.$$

Such a function is said to be **strictly proper** if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial). We can always write a rational function as the sum of a polynomial function and a strictly proper rational function. This can be accomplished through polynomial long division. In what follows, we consider partial fraction expansions of strictly proper rational functions.

Suppose that we have a rational function of the form

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_1 v + a_0}{v^n + b_{m-1} v^{m-1} + \ldots + b_1 v + b_0}$$

and $m < n$ (i.e., $F(v)$ is strictly proper). Let us denote the polynomial in the denominator of the above expression for $F(v)$ as $D(v)$. We can always factor $D(v)$ as

$$D(v) = (v - p_1)(v - p_2) \cdots (v - p_n),$$

where the p_k are the roots of $D(v)$.

First, let us suppose that the roots of $D(v)$ are distinct (i.e., the p_k are distinct). In this case, $F(v)$ can be expanded as

$$F(v) = \frac{A_1}{v - p_1} + \frac{A_2}{v - p_2} + \ldots + \frac{A_{n-1}}{v - p_{n-1}} + \frac{A_n}{v - p_n},$$

where

$$A_k = (v - p_k)F(v)|_{v=p_k}.$$

To see why the preceding formula for A_k is correct, we simply evaluate $(v - p_k)F(v)|_{v=p_k}$. We have that

$$(v - p_k)F(v)|_{v=p_k} = \left[\frac{A_1(v - p_k)}{v - p_1} + \frac{A_2(v - p_k)}{v - p_2} + \ldots + \frac{A_{k-1}(v - p_k)}{v - p_{k-1}} + A_k + \right.$$
$$\left. \frac{A_{k+1}(v - p_k)}{v - p_{k+1}} + \ldots + \frac{A_n(v - p_k)}{v - p_n} \right] \Bigg|_{v=p_k}$$
$$= A_k$$

Now, let us suppose that the roots of $D(v)$ are not distinct. In this case, we can factor $D(v)$ as

$$D(v) = (v - p_1)^{q_1} (v - p_2)^{q_2} \cdots (v - p_P)^{q_P}.$$

One can show that, in this case, $F(v)$ has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{11}}{v - p_1} + \frac{A_{12}}{(v - p_1)^2} + \ldots + \frac{A_{1q_1}}{(v - p_1)^{q_1}} \right]$$
$$+ \left[\frac{A_{21}}{v - p_2} + \ldots + \frac{A_{2q_2}}{(v - p_2)^{q_2}} \right]$$
$$+ \ldots + \left[\frac{A_{P1}}{v - p_P} + \ldots + \frac{A_{Pq_P}}{(v - p_P)^{q_P}} \right],$$

where

$$A_{kl} = \frac{1}{(q_k - l)!} \left[\frac{d^{q_k - l}}{dv^{q_k - l}} [(v - p_k)^{q_k} F(v)] \right] \bigg|_{v = p_k}.$$

Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.

Example B.1 (Simple pole). Find the partial fraction expansion of the function $H(s)$ given by

$$H(s) = \frac{3}{s^2 + 3s + 2}.$$

Solution. First, we rewrite $H(s)$ with the denominator polynomial factored to obtain

$$H(s) = \frac{3}{(s + 1)(s + 2)}.$$

From this, we know that $H(s)$ has a partial fraction expansion of the form

$$H(s) = \frac{A_1}{s + 1} + \frac{A_2}{s + 2},$$

where A_1 and A_2 are constants to be determined. Now, we calculate A_1 and A_2 as follows:

$$A_1 = (s + 1)H(s)|_{s=-1}$$
$$= \frac{3}{s + 2}\bigg|_{s=-1}$$
$$= 3 \quad \text{and}$$
$$A_2 = (s + 2)H(s)|_{s=-2}$$
$$= \frac{3}{s + 1}\bigg|_{s=-2}$$
$$= -3.$$

Thus, the partial fraction expansion of $H(s)$ is given by

$$H(s) = \frac{3}{s + 1} - \frac{3}{s + 2}.$$

$\square$

Example B.2 (Repeated pole). Find the partial fraction expansion of the function $H(s)$ given by

$$H(s) = \frac{4s + 8}{(s + 1)^2 (s + 3)}.$$

Solution. Since $H(s)$ has a repeated pole, we know that $H(s)$ has a partial fraction expansion of the form

$$H(s) = \frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{s+3}.$$

where A_{11}, A_{12}, and A_{21} are constants to be determined. To calculate these constants, we proceed as follows:

$$
\begin{aligned}
A_{11} &= \frac{1}{(2-1)!} \left[\left(\frac{d}{ds} \right)^{2-1} \left[(s+1)^2 H(s) \right] \right] \bigg|_{s=-1} \\
&= \frac{1}{1!} \left[\frac{d}{ds} \left[(s+1)^2 H(s) \right] \right] \bigg|_{s=-1} \\
&= \left[\frac{d}{ds} \left[\frac{4s+8}{s+3} \right] \right] \bigg|_{s=-1} \\
&= \left[4(s+3)^{-1} + (-1)(s+3)^{-2}(4s+8) \right] \big|_{s=-1} \\
&= \left[\frac{4}{(s+3)^2} \right] \bigg|_{s=-1} \\
&= \frac{4}{4} \\
&= 1,
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= \frac{1}{(2-2)!} \left[\left(\frac{d}{ds} \right)^{2-2} \left[(s+1)^2 H(s) \right] \right] \bigg|_{s=-1} \\
&= \frac{1}{0!} \left[(s+1)^2 H(s) \right] \big|_{s=-1} \\
&= \left[\frac{4s+8}{s+3} \right] \bigg|_{s=-1} \\
&= \frac{4}{2} \\
&= 2, \quad \text{and} \\
A_{21} &= (s+3)H(s) \big|_{s=-3} \\
&= \frac{4s+8}{(s+1)^2} \bigg|_{s=-3} \\
&= \frac{-4}{4} \\
&= -1.
\end{aligned}
$$

Thus, the partial fraction expansion of $H(s)$ is given by

$$H(s) = \frac{1}{s+1} + \frac{2}{(s+1)^2} - \frac{1}{s+3}.$$

$\square$

Example B.3 (Improper rational function). Find the partial fraction expansion of

$$X(s) = \frac{2s^3 + 9s^2 - s + 2}{s^2 + 3s + 2}.$$

Solution. Since $X(s)$ is not strictly proper, we must rewrite $X(s)$ as the sum of a polynomial function and a strictly

proper rational function. Using polynomial long division, we have

$$
\begin{array}{r}
2s \;\; + 3 \\
s^2 + 3s + 2 \overline{\big)\; 2s^3 \;+\; 9s^2 \;-\; s \;\;\; + 2} \\
\underline{2s^3 \;+\; 6s^2 \;+\; 4s} \\
3s^2 \;-\; 5s \;\;+ 2 \\
\underline{3s^2 \;+\; 9s \;\;+ 6} \\
-\,14s \;-\; 4.
\end{array}
$$

Thus, we have

$$X(s) = 2s + 3 + V(s),$$

where

$$V(s) = \frac{-14s - 4}{s^2 + 3s + 2} = \frac{-14s - 4}{(s+2)(s+1)}.$$

Now, we find a partial fraction expansion of $V(s)$. Such an expansion is of the form

$$V(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}.$$

Solving for the expansion coefficients, we have

$$
\begin{aligned}
A_1 &= (s+1)V(s)\big|_{s=-1} \\
&= \left.\frac{-14s - 4}{s+2}\right|_{s=-1} \\
&= 10 \quad \text{and} \\
A_2 &= (s+2)V(s)\big|_{s=-2} \\
&= \left.\frac{-14s - 4}{s+1}\right|_{s=-2} \\
&= -24.
\end{aligned}
$$

Thus, $V(s)$ has the expansion

$$V(s) = \frac{10}{s+1} - \frac{24}{s+2}.$$

Thus, we can decompose $X(s)$ using a partial fraction expansion as

$$X(s) = 2s + 3 + \frac{10}{s+1} - \frac{24}{s+2}.$$

□

B.1 Problems

B.1 Find the partial fraction expansion of the function $F(v)$ in each of the following cases:

(a) $F(v) = \dfrac{-v^2 + 2v + 7}{4v^3 + 24v^2 + 44v + 24}$;

(b) $F(v) = \dfrac{-16v - 10}{8v^2 + 6v + 1}$; and

(c) $F(v) = \dfrac{7v + 26}{v^2 + 7v + 10}$.

B.2 Find the partial fraction expansion of the function $F(v)$ in each of the following cases:

(a) $F(v) = \dfrac{-2v^2 + 5}{v^3 + 4v^2 + 5v + 2}$;

(b) $F(v) = \dfrac{2v^2 + 15v + 21}{v^2 + 4v + 3}$; and

(c) $F(v) = \dfrac{4v^3 + 36v^2 + 103v + 95}{(v+1)(v+3)^3}$.

Appendix C

Solution of Constant-Coefficient Linear Differential Equations

C.1   Overview

Many systems of practical interest can be represented using linear differential equations with constant coefficients. For this reason, we are interested in solution techniques for such equations. This appendix briefly introduces time-domain methods for solving constant-coefficient linear differential equations.

C.2   Constant-Coefficient Linear Differential Equations

An Nth-order linear differential equation with constant coefficients has the general form

$$\sum_{k=0}^{N} b_k \left(\frac{d}{dt} \right)^k y(t) = \sum_{k=0}^{M} a_k \left(\frac{d}{dt} \right)^k x(t),$$

where the a_k and b_k are constants. If the right-hand side of the above equation is identically equal to zero, the equation is said to be **homogeneous**. Otherwise, the equation is said to be **nonhomogeneous**. Depending on whether the above equation is homogeneous or nonhomogeneous, the solution method differs slightly.

C.3   Solution of Homogeneous Equations

First, we consider the solution of homogeneous equations. In this case, we have an equation of the form

$$\sum_{k=0}^{N} b_k \left(\frac{d}{dt} \right)^k y(t) = 0. \tag{C.1}$$

Let us define the quantity

$$\phi(s) \triangleq \sum_{k=0}^{N} b_k s^k.$$

Then, we refer to

$$\phi(s) = 0$$

as the characteristic (or auxiliary) equation of (C.1). The solution of (C.1) depends on the roots of the characteristic equation, as specified by the theorem below.

Theorem C.1. *Suppose that $\phi(s) = 0$ is the characteristic equation associated with the homogeneous linear differential equation*

$$\sum_{k=0}^{N} b_k \left[\tfrac{d}{dt}\right]^k y(t) = 0.$$

If $\phi(s) = 0$ has a real root p of multiplicity k, then a solution of the differential equation is

$$(a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}) e^{pt}.$$

If $\phi(s) = 0$ has a pair of complex conjugate roots $\sigma \pm j\omega$, each root being of multiplicity k, then a solution of the differential equation is

$$e^{\sigma t} \left[(a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}) \cos \omega t + (b_0 + b_1 t + \ldots + b_{k-1} t^{k-1}) \sin \omega t \right].$$

A general solution of the differential equation is obtained by taking a linear combination of the solutions obtained by considering all roots of $\phi(s) = 0$.

From the above theorem, we can see that, in order to solve an equation of the form (C.1), we begin by finding the roots of the corresponding characteristic equation. Then, we find a solution associated with each distinct root (or pair of complex conjugate roots). Finally, the general solution is formed by taking a linear combination of all of these individual solutions.

Example C.1. Find the general solution to the differential equation

$$y''(t) + 4y'(t) + 5y(t) = 0.$$

Solution. The given differential equation has the characteristic equation

$$s^2 + 4s + 5 = 0.$$

Solving for the roots of the characteristic equation yields

$$\begin{aligned} s &= \frac{-4 \pm \sqrt{4^2 - 4(5)}}{2} \\ &= -2 \pm \tfrac{1}{2}\sqrt{-4} \\ &= -2 \pm j. \end{aligned}$$

Thus, we have one pair of complex conjugate roots (i.e., $-2 \pm j$), each root being of multiplicity 1. Therefore, the general solution to the given equation is of the form

$$y(t) = e^{-2t} \left[a_1 \cos t + b_1 \sin t \right].$$

$\square$

Example C.2. Find the general solution to the differential equation

$$y''(t) + 5y'(t) + 6y(t) = 0.$$

Solution. The given differential equation has the characteristic equation

$$s^2 + 5s + 6 = 0$$

which can be factored as

$$(s + 2)(s + 3) = 0.$$

Clearly, the characteristic equation has the roots -2 and -3, each of multiplicity 1. Therefore, the general solution of the given equation is of the form

$$y(t) = a_1 e^{-3t} + a_2 e^{-2t}.$$

$\square$

Example C.3. Find the general solution to the differential equation

$$y''(t) + 2y'(t) + y(t) = 0.$$

Solution. The given differential equation has the characteristic equation

$$s^2 + 2s + 1 = 0$$

which can be factored as

$$(s+1)^2 = 0.$$

Clearly, the characteristic equation has the root -1 of multiplicity 2. Therefore, the general solution to the given equation is of the form

$$y(t) = (a_0 + a_1 t)e^{-t}.$$

$\square$

C.4 Particular Solution of Nonhomogeneous Equations

So far, we have only considered the solution of homogeneous equations. Now, we consider the nonhomogeneous case. In the nonhomogeneous case, we have an equation of the form

$$\sum_{k=0}^{N} b_k \left(\frac{d}{dt}\right)^k y(t) = f(t). \tag{C.2}$$

As it turns out, in order to find a general solution to the above equation, we must first find a particular solution.

To find a particular solution, we must consider the form of the function $f(t)$. Suppose that a particular solution to (C.2) is given by $y_p(t)$. Since $y_p(t)$ is a solution to (C.2), when we substitute $y_p(t)$ into (C.2), the left-hand side of (C.2) must equal $f(t)$. Therefore, $y_p(t)$ and its derivatives must be comprised of terms that resemble the terms of $f(t)$. Thus, by examining $f(t)$, we can deduce a general expression for $y_p(t)$ containing one or more unknown coefficients. We, then, solve for these unknown coefficients. This solution technique is sometimes referred to as the **method of undetermined coefficients**.

Table C.1 shows the general form of $y_p(t)$ that should be nominally used in the case of several frequently encountered forms for $f(t)$. There is, however, one caveat. The general expression chosen for $y_p(t)$ must not include any terms that are linearly dependent on terms in the solution to the corresponding complementary equation. If linearly dependent terms exist in our preliminary choice of $y_p(t)$, we must replace each term $v(t)$ with $t^m v(t)$ where m is the smallest positive integer eliminating this linear dependence. To illustrate how this solution process works in more detail, we will now consider a few example problems.

Example C.4. Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = 5t^2 + 3t + 8.$$

Solution. We begin by considering the form of the function on the right-hand side of the given differential equation. Since terms in t^2, t^1, and t^0 yield terms in t^2, t^1, and t^0 when substituted into the left-hand side of the given equation, we deduce that a particular solution is of the form

$$y_p(t) = a_2 t^2 + a_1 t + a_0. \tag{C.3}$$

Differentiating $y_p(t)$, we obtain

$$y_p'(t) = 2a_2 t + a_1 \quad \text{and}$$
$$y_p''(t) = 2a_2.$$

$f(t)$	$y_p(t)$
$c_0 + c_1 t + \ldots + c_n t^n$	$p_0 + p_1 t + \ldots + p_n t^n$
$c e^{at}$	$p e^{at}$
$c \sin bt$ or $c \cos bt$	$p \sin bt + q \cos bt$
$(c_0 + c_1 t + \ldots + c_n t^n) e^{at}$	$(p_0 + p_1 t + \ldots + p_n t^n) e^{at}$
$(c_0 + c_1 t + \ldots + c_n t^n) \cos bt$ or $(c_0 + c_1 t + \ldots + c_n t^n) \sin bt$	$(p_0 + p_1 t + \ldots + p_n t^n) \cos bt + (q_0 + q_1 t + \ldots + q_n t^n) \sin bt$
$(c_0 + c_1 t + \ldots + c_n t^n) e^{at} \cos bt$ or $(c_0 + c_1 t + \ldots + c_n t^n) e^{at} \sin bt$	$(p_0 + p_1 t + \ldots + p_n t^n) e^{at} \cos bt + (q_0 + q_1 t + \ldots + q_n t^n) e^{at} \sin bt$

Table C.1: Forms for particular solution.

Substituting $y_p(t)$ and its derivatives into the given differential equation yields

$$2a_2 + 4\left[2a_2 t + a_1\right] + 5\left[a_2 t^2 + a_1 t + a_0\right] = 5t^2 + 3t + 8$$
$$\Rightarrow \quad 2a_2 + 8a_2 t + 4a_1 + 5a_2 t^2 + 5a_1 t + 5a_0 = 5t^2 + 3t + 8$$
$$\Rightarrow \quad [5a_2]t^2 + [8a_2 + 5a_1]t + [2a_2 + 4a_1 + 5a_0] = 5t^2 + 3t + 8.$$

Comparing the left- and right-hand sides of the above equation, we see that

$$5a_2 = 5,$$
$$8a_2 + 5a_1 = 3, \quad \text{and}$$
$$2a_2 + 4a_1 + 5a_0 = 8.$$

Solving this system of equations yields $a_0 = 2$, $a_1 = -1$, and $a_2 = 1$. Therefore, from (C.3), the given differential equation has the particular solution

$$y_p(t) = t^2 - t + 2.$$

$\square$

Example C.5. Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = e^{-2t}.$$

Solution. We begin by considering the form of the function on the right-hand side of the given differential equation. Since terms in e^{-2t} yield terms in e^{-2t} when substituted into the left-hand side of the equation, we deduce that the particular solution $y_p(t)$ is of the form

$$y_p(t) = a e^{-2t}.$$

Differentiating $y_p(t)$, we obtain

$$y_p'(t) = -2a e^{-2t} \quad \text{and}$$
$$y_p''(t) = 4a e^{-2t}.$$

Substituting $y_p(t)$ and its derivatives into the given differential equation yields

$$4a e^{-2t} + 4\left[-2a e^{-2t}\right] + 5\left[a e^{-2t}\right] = e^{-2t}$$
$$\Rightarrow \quad 4a e^{-2t} - 8a e^{-2t} + 5a e^{-2t} = e^{-2t}$$
$$\Rightarrow \quad a e^{-2t} = e^{-2t}.$$

Comparing the left- and right-hand sides of the above equation, we have that $a = 1$. Therefore, the given differential equation has the particular solution

$$y_p(t) = e^{-2t}.$$

$\square$

Example C.6. Find a particular solution of the differential equation

$$y''(t) + 4y'(t) + 5y(t) = \sin t.$$

Solution. To begin, we examine the form of the function of the right-hand side of the given differential equation. Since terms in $\sin t$ yield terms in $\sin t$ and $\cos t$ when substituted into the left-hand side of the given equation, we deduce that the particular solution $y_p(t)$ is of the form

$$y_p(t) = a_1 \cos t + a_2 \sin t.$$

Differentiating $y_p(t)$, we obtain

$$y_p'(t) = -a_1 \sin t + a_2 \cos t \quad \text{and}$$
$$y_p''(t) = -a_1 \cos t - a_2 \sin t.$$

Substituting $y_p(t)$ and its derivatives into the given differential equation yields

$$[-a_1 \cos t - a_2 \sin t] + 4[-a_1 \sin t + a_2 \cos t] + 5[a_1 \cos t + a_2 \sin t] = \sin t$$
$$\Rightarrow \quad [-a_1 + 4a_2 + 5a_1] \cos t + [-a_2 - 4a_1 + 5a_2] \sin t = \sin t$$
$$\Rightarrow \quad [4a_1 + 4a_2] \cos t + [4a_2 - 4a_1] \sin t = \sin t.$$

Comparing the left- and right-hand sides of the above equation, we have that

$$4a_1 + 4a_2 = 0 \quad \text{and}$$
$$4a_2 - 4a_1 = 1.$$

Solving this system of equations yields $a_1 = -\frac{1}{8}$ and $a_2 = \frac{1}{8}$. Therefore, the given differential equation has the particular solution

$$y_p(t) = -\frac{1}{8} \cos t + \frac{1}{8} \sin t.$$

$\square$

C.5   General Solution of Nonhomogeneous Equations

With every nonhomogeneous constant-coefficient linear differential equation

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t),$$

we can associate a homogeneous equation

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = 0$$

called the complementary equation. The complementary equation is formed by simply setting the function $x(t)$ (and its derivatives) to zero in the original equation.

As it turns out, in order to find the solution of a nonhomogeneous equation, we must find a particular solution to the given equation and also a general solution to its complementary equation. This process is more precisely specified by the theorem below.

Theorem C.2. *A general solution of the linear differential equation*

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t)$$

has the form

$$y(t) = y_c(t) + y_p(t),$$

where $y_c(t)$ is a general solution of the associated complementary equation and $y_p(t)$ is any particular solution of the given equation.

Example C.7. Suppose that we have the differential equation

$$y''(t) + 2y'(t) + 2y(t) = -2t + 4.$$

(a) Find the general solution of this equation.
(b) Find the solution if $y(0) = 1$ and $y'(0) = 0$.

Solution. (a) First, we need to find the general solution $y_c(t)$ of the corresponding complementary equation

$$y''(t) + 2y'(t) + 2y(t) = 0.$$

This equation has the characteristic equation

$$s^2 + 2s + 2 = 0.$$

Solving for the roots of this equation, we have

$$s = \frac{-2 \pm \sqrt{2^2 - 4(2)}}{2}$$
$$= -1 \pm j.$$

Therefore, the characteristic equation has a pair of complex conjugate roots $-1 \pm j$, each root being of multiplicity 1. From this, we know that the complementary equation has a general solution of the form

$$y_c(t) = e^{-t}[a_1 \cos t + b_1 \sin t]. \tag{C.4}$$

Now, we must find a particular solution $y_p(t)$ of the given differential equation. We consider the form of the function on the right-hand side of the given equation. Since terms in t^1 and t^0 yield terms in t^1 and t^0 when substituted into the left-hand side of the equation, we deduce that $y_p(t)$ is of the form

$$y_p(t) = c_1 t + c_0.$$

Differentiating $y_p(t)$, we obtain

$$y_p'(t) = c_1 \quad \text{and}$$
$$y_p''(t) = 0.$$

Substituting $y_p(t)$ and its derivatives into the given differential equation yields

$$2c_1 + 2[c_1 t + c_0] = -2t + 4$$
$$\Rightarrow \quad [2c_1]t + [2c_1 + 2c_0] = -2t + 4.$$

Comparing the left- and right-hand sides of the above equation, we have

$$2c_1 = -2 \quad \text{and}$$
$$2c_1 + 2c_0 = 4.$$

Solving this system of equations yields $c_0 = 3$ and $c_1 = -1$. Therefore, the particular solution is given by

$$y_p(t) = -t + 3. \tag{C.5}$$

Combining the results of (C.4) and (C.5), we conclude that the given equation has the general solution

$$y(t) = y_c(t) + y_p(t)$$
$$= e^{-t}[a_1 \cos t + b_1 \sin t] - t + 3.$$

(b) We compute the derivative of $y(t)$ as

$$y'(t) = -e^{-t}[a_1 \cos t + b_1 \sin t] + e^{-t}[-a_1 \sin t + b_1 \cos t] - 1.$$

From the given initial conditions, we have

$$0 = -a_1 + b_1 - 1 \quad \text{and}$$
$$1 = a_1 + 3.$$

Solving for a_1 and b_1 yields $a_1 = -2$ and $b_1 = -1$. Therefore, we have

$$y(t) = e^{-t}[-2 \cos t - \sin t] - t + 3$$
$$= -e^{-t}[2 \cos t + \sin t] - t + 3.$$

$\square$

Example C.8. Suppose that we have the differential equation

$$y''(t) + 3y'(t) + 2y(t) = e^{-t}.$$

(a) Find the general solution of this equation.
(b) Determine the solution if $y(0) = -1$ and $y'(0) = 1$.

Solution. First, we need to find the general solution $y_c(t)$ of the corresponding complementary equation

$$y''(t) + 3y'(t) + 2y(t) = 0.$$

This equation has the characteristic equation

$$s^2 + 3s + 2 = 0$$

which can be factored as

$$(s+2)(s+1) = 0.$$

Thus, the characteristic equation has roots at -2 and -1, each of multiplicity 1. From this, we can deduce that

$$y_c(t) = a_1 e^{-t} + a_2 e^{-2t}. \tag{C.6}$$

Now, we need to find a particular solution $y_p(t)$ of the given differential equation. Since $y_c(t)$ contains a term with e^{-t}, we deduce that $y_p(t)$ is of the form

$$y_p(t) = cte^{-t}.$$

Differentiating $y_p(t)$, we obtain

$$y'_p(t) = ce^{-t} - cte^{-t} \quad \text{and}$$
$$y''_p(t) = -ce^{-t} - c[e^{-t} - te^{-t}] = -2ce^{-t} + cte^{-t}.$$

Substituting $y_p(t)$ and its derivatives into the given differential equation yields

$$[-2ce^{-t} + cte^{-t}] + 3[ce^{-t} - cte^{-t}] + 2cte^{-t} = e^{-t}$$
$$\Rightarrow \quad [-2c + 3c]e^{-t} + [c - 3c + 2c]te^{-t} = e^{-t}$$
$$\Rightarrow \quad ce^{-t} = e^{-t}.$$

Comparing the left- and right-hand sides of this equation, we conclude $c = 1$. Therefore, we have

$$y_p(t) = te^{-t}. \tag{C.7}$$

Combining the results from (C.6) and (C.7), we have

$$y(t) = y_c(t) + y_p(t)$$
$$= a_1 e^{-t} + a_2 e^{-2t} + te^{-t}.$$

(b) We compute the derivative of $y(t)$ as

$$y'(t) = -a_1 e^{-t} - 2a_2 e^{-2t} + e^{-t} - te^{-t}.$$

From the given initial conditions, we have

$$1 = -a_1 - 2a_2 + 1 \quad \text{and}$$
$$-1 = a_1 + a_2.$$

Solving for a_1 and a_2 yields

$$a_1 = -2$$
$$a_2 = 1.$$

Therefore, we have that

$$y(t) = -2e^{-t} + e^{-2t} + te^{-t}.$$

$\square$

C.6 Problems

C.1 Find the general solution to each of the following differential equations:
(a) $8y''(t) + 6y'(t) + y(t) = 0$;
(b) $y'''(t) + 5y''(t) + 17y'(t) + 13y(t) = 0$; **[Hint: One root of the characteristic equation is -1.]**; and
(c) $y''(t) + 9y'(t) + 20y(t) = 0$;
(d) $y''(t) + 2y'(t) + y(t) = 0$.

C.2 Find a particular solution to each of the differential equations below. In each case, the solution $y_c(t)$ of the corresponding complementary equation is given.
(a) $y''(t) + 3y'(t) + 2y(t) = t^2$ with $y_c(t) = c_1 e^{-2t} + c_2 e^{-3t}$;
(b) $y''(t) + 3y'(t) + 2y(t) = e^{-3t} + t$ with $y_c(t) = c_1 e^{-2t} + c_2 e^{-3t}$;
(c) $y''(t) + 4y'(t) + 3y(t) = e^{-t}$ with $y_c(t) = c_1 e^{-t} + c_2 e^{-3t}$; and
(d) $y''(t) + 2y'(t) + y(t) = \sin t$ with $y_c(t) = c_0 e^{-t} + c_1 t e^{-t}$.

C.3 Suppose that we have the differential equation

$$\frac{d^2}{dt^2} y(t) + 3 \frac{d}{dt} y(t) + 2y(t) = t + 1.$$

(a) Find the general solution to this equation.
(b) Determine the solution in the case that $y(0) = -\frac{1}{4}$ and $y'(0) = -\frac{1}{2}$.

C.4 Suppose that we have the differential equation

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6y(t) = 2e^{-3t}.$$

(a) Find the general solution of this equation.
(b) Determine the solution in the case that $y(0) = 0$ and $y'(0) = 1$.

C.5 Find the general solution to each of the following differential equations:
(a) $y''(t) + 7y'(t) + 12y(t) = 6t^2 - 5t + 18$;
(b) $y''(t) + 7y'(t) + 12y(t) = e^{-3t}$;
(c) $y''(t) + 4y'(t) + 8y(t) = e^{-t}$; and
(d) $y''(t) + 2y'(y) + 5y(t) = 1 + e^{-t}$.

C.7 MATLAB Problems

C.101 Use the `dsolve` function in MATLAB to solve each of the differential equations in Problem C.1.

C.102 Use the `dsolve` function in MATLAB to solve Problem C.4.

Appendix D

Miscellaneous Information

D.1 Integrals

$$\int x^n dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{for } n \neq -1 \\ \ln|x| + C & \text{for } n = -1 \end{cases}$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int x e^x \, dx = e^x(x-1) + C$$

$$\int x e^{ax} \, dx = \frac{1}{a^2} e^{ax}(ax-1) + C$$

$$\int x^2 e^{ax} \, dx = \frac{1}{a^3} e^{ax}(a^2 x^2 - 2ax + 2) + C$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}[a\cos bx + b\sin bx]}{a^2 + b^2} + C$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}[a\sin bx - b\cos bx]}{a^2 + b^2} + C$$

$$\int x \cos x \, dx = \cos x + x \sin x + C$$

$$\int x \sin x \, dx = \sin x - x \cos x + C$$

$$\int u \, dv = uv - \int v \, du$$

D.2 Derivatives

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sin x = \cos x$$

D.3   Arithmetic and Geometric Series

The sum of an arithmetic series is given by

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n(2a+d(n-1))}{2}. \tag{D.1}$$

The sum of a geometric series is given by

$$\sum_{k=0}^{n-1}ar^k = a\frac{r^n-1}{r-1} \quad \text{for } r \neq 1. \tag{D.2}$$

The sum of an infinite geometric series is given by

$$\sum_{k=0}^{\infty}ar^k = \frac{a}{1-r} \quad \text{for } |r| < 1. \tag{D.3}$$

Appendix E

MATLAB

E.1 Introduction

MATLAB is a software tool that is useful for solving a wide variety of problems arising in engineering applications. The MATLAB software is a product of a company called The MathWorks. Extensive information on this software (including detailed guides and manuals) is available, free of charge, from the company's web site (http://www.mathworks.com). A number of helpful books on MATLAB are also available [1, 2]. In this appendix, a reasonably detailed introduction to MATLAB is also provided.

E.2 Octave

Although MATLAB is very powerful, it is a commercial software product. Therefore, MATLAB is not free. Fortunately, an open-source MATLAB-like software package is available called Octave. Octave is available for download from its official web site http://www.octave.org. This software is included in several major Linux distributions (e.g., Fedora Core and Red Hat). As well, Octave is also available for the Cygwin environment under Microsoft Windows. (For more details about Cygwin, see http://www.cygwin.org.)

E.3 Invoking MATLAB

On most UNIX systems, the MATLAB software is started by invoking the matlab command.

E.3.1 UNIX

The MATLAB software is invoked using a command line of the form:

    matlab *[options]*

The matlab command supports a number of options including the following:

-help or -h
> Display information on MATLAB options.

-nodisplay
> Disable all graphical output. The MATLAB desktop will not be started.

-nojvm
> Disable all Java support by not starting the Java virtual machine. In particular, the MATLAB desktop will not be started.

Table E.1: Keys for Command Line Editing

Key	Action
Up arrow	Recall previous line
Down arrow	Recall next line
Left arrow	Move left one character
Right arrow	Move right one character
Home	Move to beginning of line
End	Move to end of line
Ctrl-C	Cancel current line

`-nodesktop`
> Disable the MATLAB desktop. Use the current terminal for commands.

`-display` *displayname*
> Specify the X display to use for graphics output.

Like most UNIX programs, MATLAB uses the X Windows System for rendering graphics output. The `DISPLAY` environment variable provides the default display setting for graphics output. If necessary, one can override the default display setting by explicitly specifying the display to use via the `-display` option.

When running MATLAB remotely, it may be necessary to disable the desktop (with the `-nodesktop` option). When the desktop is enabled, MATLAB tends to require an obscene amount of network bandwidth (much more than what a standard DSL link can provide). With the desktop disabled, one should be able to use graphics functionality when running MATLAB remotely without too much trouble with a standard high-speed Internet connection. (Forget about using graphics functionality over a dialup link, however. The X Windows protocol requires far too much bandwidth for that.)

E.3.2 Microsoft Windows

Unfortunately, the author has not used MATLAB under Microsoft Windows. So, he cannot comment on the specifics of running MATLAB under this operating system.

E.4 Command Line Editor

In MATLAB, several keys are quite useful for editing purposes, as listed in Table E.1. For example, the arrow keys can be used to perform editing in the usual way.

E.5 MATLAB Basics

Arguably, one of the most helpful commands in MATLAB is the `help` command. This command can be used to obtain information on many of the operators, functions, and commands available in MATLAB. For example, to find information on the `help` command, one can type:

```
help help
```

In a similar vein, the `doc` command can be used to obtain detailed documentation on many of the functions and commands in MATLAB. For example, to display documentation on the `doc` command, one can type:

```
doc doc
```

Table E.2: Predefined Variables

Variable	Description
pi	π
i	$\sqrt{-1}$
j	$\sqrt{-1}$
nan	not-a-number (NaN)
inf	infinity
ans	last expression evaluated that was not assigned to variable
date	date
clock	wall clock
realmin	smallest usable positive real number
realmax	largest usable positive real number
bitmax	largest usable positive integer

Table E.3: Operators

Symbol	Description
+	addition
-	subtraction
*	multiplication
/	right division
\	left division
^	exponentiation
'	conjugate transpose
.*	element-wise multiplication
./	element-wise division
.^	element-wise exponentiation
.'	transpose

E.5.1 Identifiers

Identifiers (i.e., variable/function names) are case sensitive and may consist of uppercase and lowercase letters, underscores, and digits, but the first character cannot be a digit. Although an identifier can be arbitrarily long, only the first n characters are significant, where n depends on the particular version of MATLAB being used. Any characters after the first n are simply ignored. (The namelengthmax function can be used to find the precise value of n.) Several variables are automatically predefined by MATLAB as listed in Table E.2. You can assign a new value to a predefined variable, but its original value will be lost.

E.5.2 Basic Functionality

In MATLAB, comments begin with a percent sign character and continue to the end of line. Some of the operators supported by MATLAB are listed in Table E.3. Some math functions provided by MATLAB are listed in Tables E.4, E.5, E.6, E.7, and E.8.

Beware of the sinc function in MATLAB. In MATLAB, the sinc function is defined as $\mathrm{sinc}\, x = \frac{sin(\pi x)}{\pi x}$. This is different from the definition of the sinc function employed in this document.

Example E.1. Below are some examples of some very basic calculations done using MATLAB.

```
a = [1 2 3; 4 5 6; 7 8 9] % 3 x 3 array
b = [1 2 3
     4 5 6
```

Table E.4: Elementary Math Functions

Name	Description
abs	magnitude of complex number
angle	principal argument of complex number
imag	imaginary part of complex number
real	real part of complex number
conj	conjugate of complex number
round	round to nearest integer
fix	round towards zero
floor	round towards $-\infty$
ceil	round towards ∞
sign	signum function
rem	remainder (with same sign as dividend)
mod	remainder (with same sign as divisor)

Table E.5: Other Math-Related Functions

Name	Description
min	minimum value
max	maximum value
mean	mean value
std	standard deviation
median	median value
sum	sum of elements
prod	product of elements
cumsum	cumulative sum of elements
cumprod	cumulative product of elements
polyval	evaluate polynomial
cart2pol	Cartesian-to-polar coordinate conversion
pol2cart	polar-to-Cartesian coordinate conversion

Table E.6: Exponential and Logarithmic Functions

Name	Description
exp	exponential function
log	natural logarithmic function
log10	base-10 logarithmic function
sqrt	square root function

Table E.7: Trigonometric Functions

Name	Description
sin	sine function
cos	cosine function
tan	tangent function
asin	arcsine function
acos	arccosine function
atan	arctangent function
atan2	two-argument form of arctangent function

Table E.8: Radix Conversion Functions

Name	Description
dec2bin	convert decimal to binary
bin2dec	convert binary to decimal
dec2hex	convert decimal to hexadecimal
hex2dec	convert hexadecimal to decimal
dec2base	convert decimal to arbitrary radix
base2dec	convert arbitrary radix to decimal

```
      7 8 9] % 3 x 3 array
a - b
x = [1; 3; -1] % 3-dimensional column vector
y = x .* x + 3 * x + 2
y = a * x

t = 5;
s = t ^ 2 + 3 * t - 7;

z = 3 + 4 * j; % complex number in Cartesian form
z = 20 * exp(j * 10); % complex number in polar form
```

□

The `disp` function prints a single string. For example, the following code fragment prints "Hello, world" (followed by a newline character):

```
disp('Hello, world');
```

The `sprintf` function provides very sophisticated string formatting capabilities, and is often useful in conjunction with the `disp` function. The use of the `sprintf` function is illustrated by the following code fragment:

```
name = 'Jane Doe';
id = '06020997';
mark = 91.345678912;
disp(sprintf('The student %s (ID %s) received a grade of %4.2f%%.', ...
  name, id, mark));
```

The `sprintf` function is very similar in spirit to the function of the same name used in the C programming language.

E.6 Arrays

Frequently, it is necessary to determine the dimensions of an array (i.e., matrix or vector). For this purpose, MATLAB provides two very useful functions as listed in Table E.9. The function `size` can be used to determine the number of rows and/or columns in an array:

- `size(a)` returns a row vector with the number of rows and columns in `a` as elements (in that order);

- `size(a, 1)` returns the number of rows in `a`; and

- `size(a, 2)` returns the number of columns in `a`.

The function `length` is used to find the maximum of the two array dimensions. That is, `length(a)` is equivalent to `max(size(a))`. Usually, the `length` function is used in conjunction with arrays that are known to be row/column vectors.

Example E.2. Suppose that `a = [1 2 3 4; 5 6 7 8]`. Then, `size(a)` returns `[2 4]`, `size(a, 1)` returns 2, `size(a, 2)` returns 4, and `length(a)` returns 4. □

Table E.9: Array Size Functions

Name	Description
size	query array dimensions
length	query vector/array dimension

E.6.1   Arrays with Equally-Spaced Elements

Often, it is necessary to specify a vector with equally-spaced elements. As a result, MATLAB provides a compact means for specifying such a vector. In particular, an expression of the following form is employed:

  start : *step* : *end*

The above expression is equivalent to a row vector with its first element equal to *start* and each of the subsequent elements increasing in value by *step* until the value would exceed *end*. Note that *step* may be negative.

Example E.3. Here, we give some examples of abbreviated forms of vectors.

Abbreviated Form	Long Form
1 : 4	[1 2 3 4]
0 : 0.2 : 1	[0 0.2 0.4 0.6 0.8 1]
1 : -1 : -2	[1 0 -1 -2]
0 : 10 : 25	[0 10 20]
-1.5 : -1 : -4	[-1.5 -2.5 -3.5]

□

E.6.2   Array Subscripting

Suppose that we have an array a. We can access elements of the array by specifying the rows and columns in which the elements are contained. In particular, a(*rowspec,* *colspec*) is the array consisting of the elements of a that are in the rows specified by *rowspec* and columns specified by *colspec*. Here, *rowspec* is either a vector containing row indices or the special value ":" which means "all rows". Similarly, *colspec* is either a vector containing column indices or the special value ":" which means "all columns". We can also access elements of the array a by specifying a 1-dimensional element index, where elements in the array are numbered in column-major order. That is, a (*indspec*) is the vector of elements of a that have the indices specified by *indspec*. Here, *indspec* is either a vector containing element indices or the special value ":" which means "all elements".

Example E.4. Suppose that a is a 10×10 matrix and x is 10×1 vector. Then, we have the results listed in the table below.

Expression	Meaning
a(1, :)	first row of a
a(:, 1)	first column of a
a(1 : 50)	first 50 elements of a arranged in a row vector
a(1 : 10)	first 10 elements of a arranged in a row vector (i.e., the first column of a)
a(1 : 2 : 10, :)	odd-indexed rows of a
a(:, 2 : 2 : 10)	even-indexed columns of a
a(1 : 5, :)	rows 1 to 5 of a
a(:, 6 : 10)	columns 6 to 10 of a
a(1 : 2, 9 : 10)	submatrix consisting of elements that are in rows 1,2 and also in columns 9,10
x(1 : 3)	first three elements of x (arranged as a row or column vector to match x)

□

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Table E.10: Special Matrix/Vector Functions

Name	Description
eye	identity matrix
ones	matrix of ones
zeros	matrix of zeros
diag	diagonal matrix
rand	random matrix
linspace	vector with linearly spaced elements
logspace	vector with logarithmically spaced elements

Table E.11: Basic Array Manipulation Functions

Name	Description
rot90	rotate array by 90 degrees
fliplr	flip array horizontally
flipud	flip array vertically
reshape	change array dimensions

E.6.3 Other Array Functions

Certain types of matrices tend to be used frequently in code. For this reason, MATLAB provides functions for generating some common forms of matrices. These functions are listed in Table E.10.

MATLAB provides functions for performing some common operations to matrices. These are listed in Table E.11.

E.7 Scripts

Instead of interactively entering MATLAB code for immediate execution, code can be placed in a file and then executed. Normally, MATLAB code is placed in what are called M-files. The term "M file" originates from the fact that these files use a name ending in the suffix ".m". To create an M-file script, one simply creates a file with a name ending in the suffix ".m". Then, the code in the M-file can be invoked by using a command with the same name as the M-file but without the ".m" extension. For reasons that will become apparent shortly, the base name of the M-file (i.e., the name without the ".m" extension) must be a valid MATLAB identifier. For example, 2foobar.m is not a valid M-file name since "2foobar" is not a valid MATLAB identifier. (Recall that MATLAB identifiers cannot start with a digit such as "2".) Also, in order for MATLAB to find an M-file, the file must be in one of the directories listed in the MATLAB search path. We will explain how to query and change the MATLAB search path later in this section. Before doing so, however, we provide a few examples of M-file scripts below.

Example E.5. Suppose that we have an M-file script called hello.m with the following contents:

```
% Print a greeting.
disp('Hello, World.');
```

Then, the code in this file can be executed by simply typing the following in MATLAB:

```
hello
```

That is, we invoke the code in the M-file by using the base name of the file. (It is tacitly assumed that the file hello.m has been placed in one of the directories listed in the MATLAB search path.) □

Example E.6. In order to save some typing, we can create a file called main.m containing the following:

```
a = [
   0.9501 0.8913 0.8214 0.9218;
   0.2311 0.7621 0.4447 0.7382;
```

```
    0.6068 0.4565 0.6154 0.1763;
    0.4860 0.0185 0.7919 0.4057;
];
y0 = a * [1 2 3 4].'
y1 = a * [-1 2.5 3 4].'
y3 = a * [41 -22 3 4].'
```

Then, to invoke the code in the above file, we simply type the following in our MATLAB session:

```
main
```

(Again, it is tacitly assumed that the file main.m has been placed in one of the directories listed in the MATLAB search path.) □

Generally, one should avoid giving a script file a name that is associated with a previously defined variable or function, as this leads to the potential for naming conflicts. For example, it would be a bad idea to name a script file as sin.m, since the sin function is already builtin to MATLAB.

Clearly, MATLAB needs a means to locate M-file scripts, since there are usually many directories in which a user might choose to place a script. For this purpose, the MATLAB search path is used. The MATLAB search path is a list of directories in which MATLAB looks for M-files. In order for the code in an M-file to be successfully located by MATLAB and executed, the M-file must be stored in a directory listed in the MATLAB search path. The MATLAB search path can be queried with the path command:

```
path
```

This command will output all of the directories in the MATLAB search path (i.e., all of the directories in which MATLAB will look for M-file scripts).

A new directory can be added to the MATLAB search path with the addpath command:

```
addpath dirname
```

This adds the directory *dirname* to the MATLAB search path.

A few other commands are also sometimes useful in the context of M-file scripts. These commands are described below.

The working directory for MATLAB can be changed using the cd command:

```
cd dirname
```

The working directory is changed to the directory named *dirname*.

The current working directory can be queried with the pwd command:

```
pwd
```

This command will display the current working directory.

E.8 Relational and Logical Operators

The relational and logical operators provided by MATLAB are listed in Tables E.12 and E.13, respectively. Some functions that are also quite useful in relational and logical expressions are listed in Table E.14. As far as boolean expressions are concerned, MATLAB considers any nonzero number to be true and zero to be false.

Example E.7. Suppose that a = [1 2 3 4 5] and b = [5 4 3 2 1]. Then, we have that

Expression	Value
a > b	[0 0 0 1 1]
a == b	[0 0 1 0 0]
a < b	[1 1 0 0 0]
a >= 2 & a <= 3	[0 1 1 0 0]
a < 2 | a > 4	[1 0 0 0 1]
~a	[0 0 0 0 0]

Table E.12: Relational Operators

Symbol	Description
<	less than
<=	less than or equal
>	greater than
>=	greater than or equal
==	equal
˜=	not equal

Table E.13: Logical Operators

Symbol	Description
&	and
|	or
˜	not

Table E.14: Relational and Logical Functions

Name	Description
any	any element nonzero
all	all elements nonzero
find	find nonzero elements
exist	check if variables exist
isnan	detect NaNs
finite	detect infinities
isempty	detect empty matrices
isstr	detect strings
strcmp	compare strings

□

Example E.8. The following code fragment illustrates how one might use some of the relational/logical functions such as all, any, and find.

```
a = [1 2 3; 4 5 6; 7 8 9];
if all(a > 0)
    disp('All matrix elements are positive.');
end
if all(a < 0)
    disp('All matrix elements are negative.');
end
if ˜any(a == 0)
    disp('All matrix elements are nonzero.');
end
if all(real(a) == a)
    disp('All matrix elements are real.');
end
i = find(a >= 8);
disp('The following matrix elements are greater than or equal to 8:');
disp(a(i));
```

□

Table E.15: Operator Precedence

Operators	Precedence Level
()	highest
.' ' .^ ^	
+ - ˜	
.* * ./ .\ / \	
+ -	
:	
< <= > >= == ˜=	
&	
|	lowest

Unary + and - have higher precedence than the binary forms.
In older versions of MATLAB, & and | have the same precedence.

E.9   Operator Precedence

The operator precedence is shown in Table E.15.

E.10   Control Flow

In the sections that follow, we introduce the conditional execution and looping constructs available in MATLAB.

E.10.1   If-Elseif-Else

The **if**-**elseif**-**else** construct allows groups of statements to be conditionally executed, and has a number of variants. The simplest variant (i.e., the **if** variant) has the form:

> **if** *expression*
> >   *statements*
>
> **end**

If the expression *expression* is true, then the statements *statements* are executed. The next simplest variant (i.e., the **if**-**else** variant) has the form:

> **if** *expression*$_1$
> >   *statements*$_1$
>
> **else**
> >   *statements*$_2$
>
> **end**

If the expression *expression*$_1$ is true, then the statements *statements*$_1$ are executed. Otherwise, the statements *statements*$_2$ are executed. Finally, the most general variant has the form:

> **if** *expression*$_1$
> >   *statements*$_1$
>
> **elseif** *expression*$_2$
> >   *statements*$_2$
> >
> > $\vdots$
>
> **elseif** *expression*$_{n-1}$
> >   *statements*$_{n-1}$
>
> **else**
> >   *statements*$_n$
>
> **end**

Note that the **elseif** and **else** clauses are optional.

Example E.9. The code fragment below provides a simple example of the use of an **if**-**elseif**-**else** construct. The code tests the sign of the variable x and prints an appropriate message.

```
if x > 0
    disp('x is positive');
elseif x < 0
    disp('x is negative');
else
    disp('x is neither positive nor negative');
end
```

□

E.10.2 Switch

The **switch** construct provides another means to conditionally execute groups of statements. The general form of this construct is as follows:

```
switch expression
case test_expression₁
    statements₁
case test_expression₂
    statements₂
         ⋮
case test_expressionₙ₋₁
    statementsₙ₋₁
otherwise
    statementsₙ
end
```

The switch expression *expression* is compared to each of the test expressions $test\_expression_1$, $test\_expression_2$, ..., $test\_expression_{n-1}$ in order. The first test expression, say $test\_expression_k$, matching the expression *expression* has its corresponding statements $statements_k$ executed. If none of the test expressions match the switch expression and an **otherwise** clause is present, the statements $statements_n$ in this clause are executed. The switch expression must be either a scalar or string.

Example E.10. The code fragment below examines the real variable n and prints some information concerning its value.

```
n = 5;
switch mod(n, 2)
case 0
    disp('number is even integer');
case 1
    disp('number is odd integer');
case {0.5, 1.5}
    disp('number is half integer');
otherwise
    disp('number is not an integer');
end
```

□

Example E.11. The code fragment below converts a mass specified in a variety of units to kilograms.

```
x = 100; % input mass
units = 'lb'; % units for input mass
switch units
case {'g'}
    y = 0.001 * x;
case {'kg'}
    y = x;
case {'lb'}
    y = 0.4536 * x;
otherwise
    error('unknown units');
end
disp(sprintf('%f %s converts to %f kg', x, units, y));
```

□

E.10.3   For

The **for** construct allows a group of statements to be repeated a fixed number of times. This construct has the general form:

> **for** *variable = array*
>   *statements*
> **end**

The statements *statements* are executed once for each value in the array *array*, where the variable *variable* is set to the corresponding array value each time.

Example E.12 (Degree to radian conversion). The following code fragment outputs a table for converting angles from units of degrees to radians.

```
disp('Degrees    Radians');
for theta_degrees = -5 : 0.5 : 5
    theta_radians = theta_degrees * pi / 180;
    disp(sprintf('%7.1f    %7.4f', theta_degrees, theta_radians));
end
```

□

Example E.13. The code fragment below applies a linear transformation (represented by the matrix a) to each column of the matrix [0 2 4 6; 1 3 5 7].

```
a = [1 0; 1 -1];
for v = [0 2 4 6; 1 3 5 7]
    disp(a * v);
end
```

□

E.10.4   While

The **while** construct allows a group of statements to be executed an indefinite number of times. This construct has the form:

> **while** *expression*
>   *statements*
> **end**

The statements *statements* are executed repeatedly as long as the condition *expression* is true.

Example E.14. In the code fragment below, we compute the smallest machine-representable positive real number that, when added to one, is still greater than one. (Due to finite-precision effects, there is a lower bound on this quantity.) This quantity is essentially the builtin `eps` variable in MATLAB.

```
epsilon = 1;
while (1 + epsilon / 2) > 1
    epsilon = epsilon / 2;
end
disp(epsilon);
```

$\square$

E.10.5 Break and Continue

Sometimes, it may necessary to prematurely break out of a loop or prematurely continue with its next iteration. This is accomplished via **break** and **continue**.

Example E.15. The two code fragments given below are equivalent. The first one employs a **break** statement, while the other does not.

```
% Code fragment 1
done = 0;
while 1
    if done
        break % Terminate (i.e., break out of) loop.
    end
    % Do something here.
    % If we are finished, set done to one.
end

% Code fragment 2
done = 0;
while ˜done
    % Do something here.
    % If we are finished, set done to one.
end
```

$\square$

Example E.16. The code fragment below gives an example of the use of the **continue** statement.

```
a = [1 0 3 2 0];
for i = a
    if i == 0
        % Skip over the processing of a zero element in the array.
        continue
    end
    % Process the nonzero array element.
    disp(i);
end
```

The above code will print only the nonzero elements of the array `a`. $\square$

E.11 Functions

MATLAB supports user-defined functions. To create a user-defined function, the code for the function is placed in an M-file. In this sense, user-defined functions are very similar to script files. For this reason, most of the material on script files in Section E.7 is also applicable here. There is, however, one key difference between a script and function file. A function file must include a **function** directive (whereas a script file must not). This directive is primarily used to indicate the number of input and output arguments for a function.

The first (non-comment) line in function file must contain the **function** directive. This directive has the form:

function [$argout_1, argout_2, \ldots, argout_n$] = *funcname*($argin_1, argin_2, \ldots, argin_m$)

This indicates that the function has the name *funcname*, the m input arguments $argin_1, argin_2, \ldots, argin_m$, and n output arguments $argout_1, argout_2, \ldots, argout_n$. The function name *funcname* should always be the same as the base name of the file in which the function is stored (i.e., the file name without the ".m" suffix). Immediately following the line containing the **function** directive, one should provide comments to be printed in response to a help inquiry for the function. The code in a function file executes until either the end of file is reached or a **return** statement is encountered.

In MATLAB all input arguments to a function are passed by value. For this reason, changes to the input arguments made inside of a function will not propagate to the caller. Also, any variables accessed/manipulated inside of a function are local in scope to that function.

Example E.17 (Sinc function). Unfortunately, the sinc function provided by MATLAB is not quite the same as the sinc function as defined in these notes. Here, we define a function mysinc that is compatible with the definition herein.

```
function y = mysinc(x)
% mysinc - Compute the sinc function (as defined in these notes)
% mysinc(x) returns a matrix whose elements are the sinc of the
% elements of x

% Initialize the output array to all ones.
y = ones(size(x));
% Determine the indices of all nonzero elements in the input array.
i = find(x);
% Compute the sinc function for all nonzero elements.
% The zero elements are already covered, since the output
% array was initialized to all ones above.
y(i) = sin(x(i)) ./ (x(i));
return
```

The command help mysinc will result in MATLAB printing the first block of comments from the above function file. □

Example E.18 (Factorial function). The contents of the file named myfactorial.m is as follows:

```
function y = myfactorial(x)
% myfactorial - compute factorial

y = 1;
for n = 2 : x
        y = y * n;
end
return
```

The above code is invoked by calling the myfactorial function. For example, myfactorial(4) returns the value 24. □

Table E.16: Special Predefined Function Variables

Name	Description
nargin	number of input arguments
nargout	number of output arguments
varargin	variable-length input argument
varargout	variable-length output argument

In MATLAB, functions may take a variable number of input arguments and may return a variable number of output arguments. In order for a function to determine the number of input and output arguments and access these arguments, several variables are automatically defined upon entry to a function. These variables are listed in Table E.16. In what follows, we give some examples of functions that take a variable number of input arguments.

Example E.19 (Function with variable number of input arguments). The contents of the file named mysum.m is as follows:

```
function y = mysum(a, b, c)
% mysum - calculate the sum (of one to three quantities)

if nargin == 1
    % function called with one argument
    y = a;
elseif nargin == 2
    % function called with two arguments
    y = a + b;
elseif nargin == 3
    % function called with three arguments
    y = a + b + c;
else
    error('invalid number of arguments');
end
return
```

The above code is invoked by calling the mysum function. For example, mysum(1) returns the value 1, mysum(1, 2) returns the value 3, and mysum(1, 2, 3) returns the value 6. □

Perhaps, we would like to write a function similar to the one in the previous example, except we would like to be able to handle an arbitrary number of input arguments (possibly many more than three). This can be accomplished by using the special predefined varargin variable.

Example E.20 (Variable number of input arguments). Suppose that we would like to write a function that returns the sum of its input arguments, but allows the number of input arguments to be arbitrary. We can achieve this functionality with the following code:

```
function y = mysum2(varargin)
% mysum2 - Compute the sum of the input arguments

if nargin == 0
    y = [];
    return
end
y = varargin{1};
for i = 2 : nargin
    y = y + varargin{i};
end
```

Table E.17: Basic Plotting Functions

Name	Description
plot	linear x-y plot
loglog	log log x-y plot
semilogx	semi-log x-y plot (x-axis logarithmic)
semilogy	semi-log x-y plot (y-axis logarithmic)
polar	polar plot
bar	bar chart
stem	stem plot
pcolor	pseudocolor (checkerboard) plot

Table E.18: Other Graphing Functions/Commands

Name	Description
axis	control axis scaling and appearance
hold	hold current plot
subplot	multiple axes in single figure
figure	create figure

Table E.21: Marker Styles

Symbol	Marker Style
.	point
o	circle
x	cross
+	plus sign
*	asterisk
s	square
d	diamond
v	triangle (down)
^	triangle (up)
<	triangle (left)
>	triangle (right)
p	pentagram
h	hexagram

Table E.20: Line Colors

Symbol	Line Color
b	blue
g	green
r	red
c	cyan
m	magenta
y	yellow
k	black
w	white

Table E.19: Line Styles

Symbol	Line Style
–	solid
:	dotted
–.	dash dot
––	dashed

```
return
```

The above code is invoked by calling the mysum2 function. For example, mysum2(1) returns the value 1, mysum2(1, 2, 3) returns the value 6, and mysum2(1, 1, 1, 1, 1, 1, 1, 1) returns the value 8. □

E.12 Graphing

MATLAB has a very rich set of graphing capabilities. Herein, we will try to illustrate some of these capabilities by way of examples.

Some of the basic plotting functions are listed in Table E.17 and several other graphing-related functions/commands are given in Table E.18.

When generating plots, it is sometimes desirable to be able to specify line styles, line colors, and marker styles. The supported line styles, line colors, and marker styles are listed in Tables E.19, E.20, and E.21, respectively.

Example E.21 (Simple plot). Suppose that we want to plot the function $\sin t$ for t in the interval $[-4\pi, 4\pi]$. This can be accomplished with the following code:
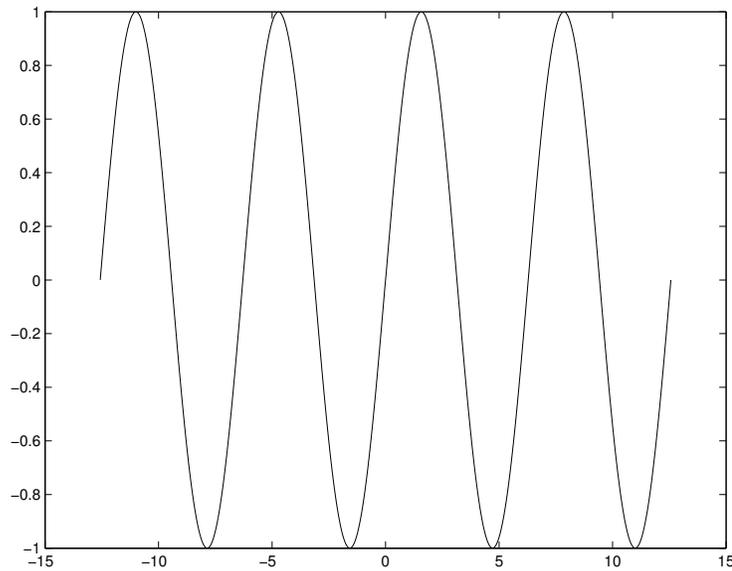
Figure E.1: Plot from example.

Table E.22: Graph Annotation Functions

Name	Description
title	graph title
xlabel	x-axis label
ylabel	y-axis label
grid	grid lines
text	arbitrarily-positioned text
gtext	mouse-positioned text

```
t = linspace(-4 * pi, 4 * pi, 500);
y = sin(t);
plot(t, y);
```

The resulting plot is shown in Figure E.1. □

Often, we need to add annotations to plots (e.g., a title, axis labels, etc.). This is accomplished using the functions listed in Table E.22. Sometimes, we wish to use special symbols (such as Greek letters) in annotations. Fortunately, numerous symbols are available in MATLAB as listed in Table E.23.

Example E.22 (Annotated plot). Here, we plot the function $\alpha(\omega) = |\omega|^2 \sin \omega$. Special symbols are used in the axis labels (in order to display Greek letters).

```
w = linspace(-10 * pi, 10 * pi, 500);
a = abs(w) .^ 2 .* sin(w);
plot(w, a);
xlabel('\omega');
ylabel('\alpha');
title('Plot of \alpha versus \omega');
```

The resulting plot is shown in Figure E.2. □

Example E.23 (Multiple plots on single axes). Suppose that we want to plot both $\sin t$ and $\cos t$ on the same axes for t in the interval $[-2\pi, 2\pi]$. This can be accomplished with the following code:

Table E.23: Special Symbols for Annotation Text

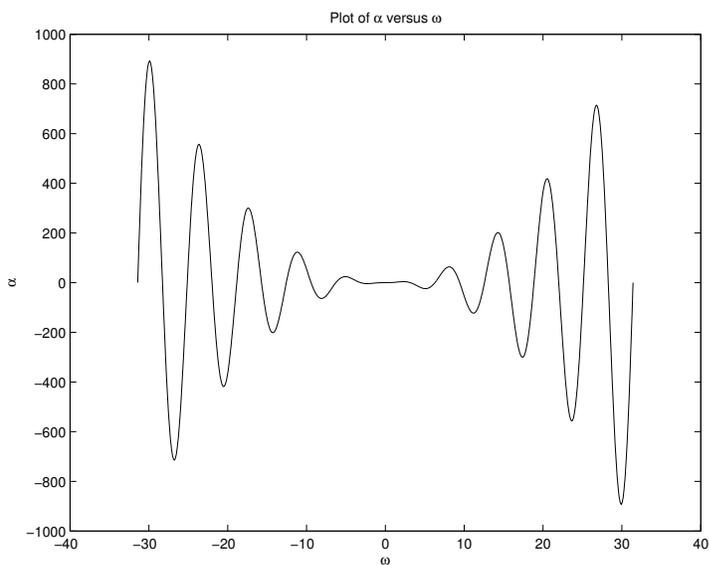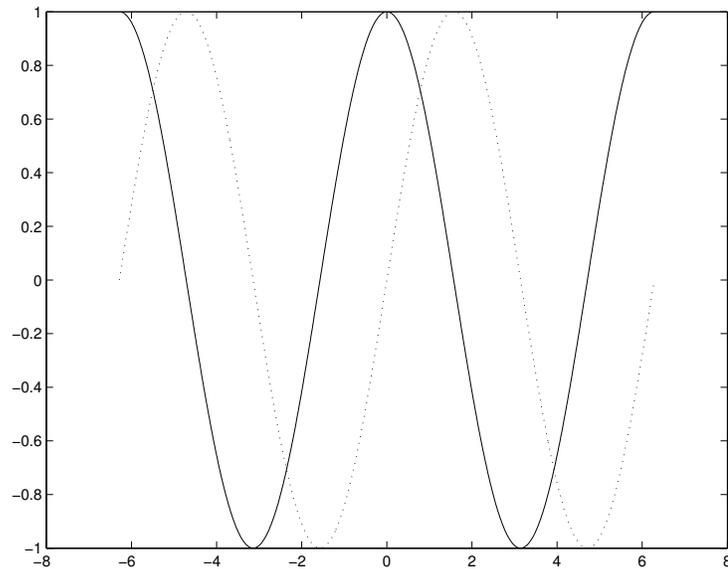| String	Symbol
\alpha	α
\beta	β
\delta	δ
\gamma	γ
\omega	ω
\theta	θ
\Delta	Δ
\Omega	Ω

Figure E.2: Plot from example.

Figure E.3: Plot from example.

```
t = linspace(-2 * pi, 2 * pi, 500);
plot(t, sin(t), ':'); % plot using dotted line
hold on % cause the second plot not to erase the first
plot(t, cos(t), '-'); % plot using solid line
hold off % allow the next plot to erase previous plots
```

The resulting plot is shown in Figure E.3.   □

Example E.24 (Multiple axes on same figure). We would like to plot the functions $\cos t$, $\sin t$, $\arccos t$, and $\arcsin t$ using four separate plots on the same figure. This can be done as follows:

```
t = linspace(-pi, pi, 500);
subplot(2, 2, 1); % select first subplot in 2-by-2 layout
plot(t, cos(t));
title('cos(t)')
subplot(2, 2, 2); % select second subplot in 2-by-2 layout
plot(t, sin(t));
title('sin(t)');
t = linspace(-1, 1, 500);
subplot(2, 2, 3); % select third subplot in 2-by-2 layout
plot(t, acos(t));
title('acos(t)')
subplot(2, 2, 4); % select fourth subplot in 2-by-2 layout
plot(t, asin(t));
title('asin(t)')
```

The resulting graphs are shown in Figure E.4.   □

E.13   Printing

To print copies of figures, the `print` command is employed. This command has various forms, but one of the simplest is as follows:
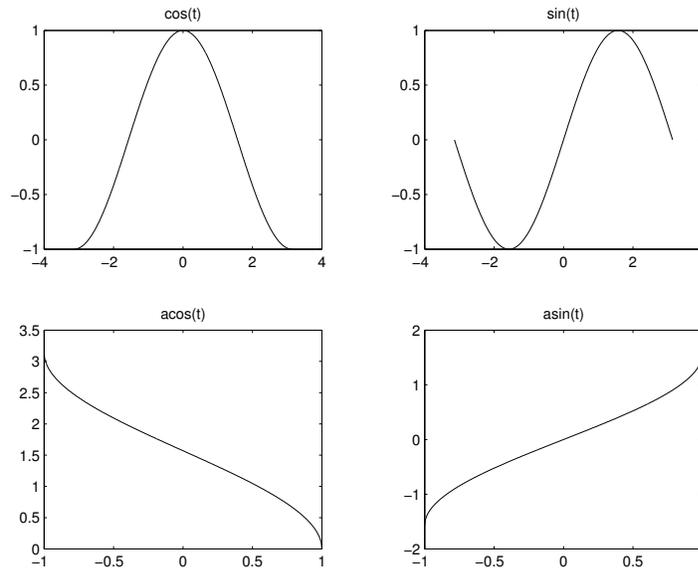
Figure E.4: Plot from example.

```
print -ddevice filename
```

Here, *device* specifies the output format and *filename* is the name of the file in which to save the output. For example, for a PostScript printer, the `ps` format should be used. For more details on additional options, type `help print`.

E.14 Symbolic Math Toolbox

Symbolic computation is sometimes quite helpful in solving engineering problems. For example, a very complicated formula or equation involving several variables might need to be simplified without assuming specific values for the variables in the formula/equation. The Symbolic Math Toolbox provides MATLAB with such symbolic computation capabilities.

E.14.1 Symbolic Objects

The Symbolic Math Toolbox defines a new data type called a **symbolic object**. The toolbox uses symbolic objects to represent symbolic variables and expressions. In essence, a symbolic object can have as its value any valid mathematical expression.

Symbolic objects can be used in many of the same ways as non-symbolic objects. One must, however, keep in mind that performing a computation symbolically is quite different than performing it non-symbolically. Generally speaking, symbolic computation is much slower than non-symbolic computation.

E.14.2 Creating Symbolic Objects

A symbolic object is created using either the `sym` function or `syms` directive. For example, a symbolic object `f` with the value $ax^2 + bx + c$ is created as follows:

```
f = sym('a * x ^ 2 + b * x + c');
```

Sometimes, it is necessary to create one or more symbolic objects, each with its value equal to its name. Obviously, this can be accomplished with the `sym` function. For example, we can write:

```
x = sym('x');
y = sym('y');
z = sym('z');
```

This common construct, however, has an abbreviated form. Instead of writing the above lines of code, we can simply say:

```
syms x y z;
```

E.14.3   Manipulating Symbolic Objects

Symbolic objects can often be used in the same way as non-symbolic objects. For example, we can do things like:

```
f = sym('t + 1');
g0 = f ^ 3 - 2 * f - 21;
g1 = cos(f) * sin(f / 3);
```

Symbolic objects are quite distinctive from other types of objects in MATLAB. For example, the following two lines of code have very different effects:

```
x = pi;
x = sym('pi');
```

It is important to understand the difference in what these two lines of code do.

To substitute some expression/variable for another variable, use the subs function. For example, to substitute $t + 1$ for t in the expression t^2, we can use the following code:

```
f = sym('t ^ 2');
g = subs(f, 't', 't + 1')
```

To factor a symbolic expression, use the factor function. For example, suppose that we want to factor the polynomial $t^2 + 3t + 2$. This could be accomplished with the following code:

```
f = sym('t ^ 2 + 3 * t + 2');
g = factor(f)
```

After executing the preceding code, g is associated with the expression $(t - 1)(t - 2)$. Note that the factor function will only produce factors with real roots.

To simplify a symbolic expression, use the simplify function. For example, suppose that we want to substitute $2t + 1$ for t in the expression $t^2 - 1$ and simplify the result. This can be accomplished with the following code:

```
f = sym('t ^ 2 - 1');
g = simplify(subs(f, 't', '2 * t + 1'))
```

To expand an expression, use the expand function. For example, to compute $(t + 1)^{10}$, we can use the following code:

```
f = sym('(t+1)^10');
g = expand(f)
```

To display an expression in a human-friendly format, use the pretty function. For example, to compute $\frac{1}{2}t^2 + \frac{1}{3}(t + 1)^{-4}$ in an expanded and beautified format, type:

```
f = sym('((1/2) * t^2 + (1/3) * (t+1))^(-4)');
pretty(expand(f))
```

Table E.24: Functions related to signal processing

Name	Description
besself	Bessel analog filter design
bode	bode plot
butter	Butterworth analog/digital filter design
freqs	frequency response of analog filter
impulse	compute impulse response
lsim	simulate system
step	compute step response
tf	generate system model from transfer function coefficients

Example E.25 (Sum of arithmetic series). Consider a sum of the form

$$S(a,d,n) \triangleq \sum_{k=0}^{n-1}(a+kd)$$

(i.e., the sum of an arithmetic series). Suppose that we did not happen to know that this sum can be calculated as

$$S(a,d,n) = \tfrac{1}{2}n\left(2a + d(n-1)\right).$$

We could determine a general formula for the sum using the following code:

```
f = sym('a + k * d');
simple(symsum(f, 'k', '0', 'n - 1'))
```

The code yields the result `1/2*n*(2*a+d*n-d)`. Clearly, this result is equivalent to the known expression for $S(a,d,n)$ given above. □

E.14.4 Plotting Symbolic Expressions

To plot a symbolic expression in one variable, use the `ezplot` function. For example, to plot the function $f(t) = 3t^2 - 4t + 2$, we can use the following code:

```
f = sym('3 * t ^ 2 - 4 * t + 2');
ezplot(f);
```

The range of the independent variable may optionally be specified. For example, to plot the expression with t in the interval $[-1,1]$, type:

```
ezplot(f, [-1 1]);
```

E.15 Signal Processing

MATLAB provides many functions that are helpful in the analysis of signals and systems. Some of the more useful functions are listed in Table E.24. In what follows, we provide several examples of how some of these functions can be used.

E.15.1 Frequency Responses

In signal processing, we often need to evaluate the frequency responses of LTI systems. Usually such systems have a frequency response that is rational. That is, the frequency response $H_F(\omega)$ is of the form

$$H_F(\omega) = \frac{\sum_{k=0}^{M-1} \alpha_k \omega^k}{\sum_{k=0}^{N-1} \beta_k \omega^k},$$

where α_k and β_k are complex constants. Fortunately, a function of this form can be evaluated quite easily in MATLAB using the `polyval` function.

Often, it is more convenient to characterize a LTI system by its transfer function rather than its frequency response. Most systems of interest have a transfer function that is rational. That is, the transfer function $H_{TF}(s)$ is of the form

$$H_{TF}(s) = \frac{\sum_{k=0}^{M-1} a_k s^k}{\sum_{k=0}^{N-1} b_k s^k},$$

where a_k and b_k are complex constants. Although we could compute the frequency response by using `polyval` to evaluate the transfer function at points on the imaginary axis, there is an easier way to accomplish this task. In particular, we can simply use the `freqs` function provided by MATLAB.

Example E.26 (Computing and plotting frequency responses with freqs). Consider the LTI system with transfer function $H(s)$, where

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

(This system is a second-order Butterworth lowpass filter with a cutoff frequency of 1 rad/s.) Suppose that we would like to evaluate the frequency response at various values of ω. This is equivalent to evaluating the transfer function $H(s)$ at points on the imaginary axis. To this end, we can employ the `freqs` function in MATLAB. More specifically, we can calculate and plot the magnitude and phase responses of the above system with the code given below.

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tfnum = [0 0 1];
tfdenom = [1 sqrt(2) 1];

% Plot the magnitude and phase responses.
freqs(tfnum, tfdenom, linspace(0, 2, 500));
```

□

Example E.27 (Plotting frequency responses). Suppose that we would like to have a function that behaves in a similar way to the MATLAB `freqs` function, but with a few differences in how plotting is performed. In particular, we would like the plots generated with the magnitude response as a unitless quantity and the phase response in unwrapped form. This can be accomplished with the code given below.

```
function [freqresp, omega] = myfreqs(tfnum, tfdenom, omega)
% The myfreqs function has essentially the same interface as the MATLAB freqs
% function, but performs plotting slightly differently.
% The magnitude response is plotted as a unitless quantity (not in decibels).
% The phase response is plotted with the phase unwrapped.

% If the frequencies have been specified as an input argument, then simply
% pass them through to the real freqs function.
if nargin >= 3
        [freqresp, omega] = freqs(tfnum, tfdenom, omega);
else
        [freqresp, omega] = freqs(tfnum, tfdenom);
end

% If no output arguments were specified, plot the frequency response.
if nargout == 0
```

```
    % Compute the magnitude response as a unitless quantity.
    magresp = abs(freqresp);

    % Compute the phase response with the phase unwrapped.
    phaseresp = unwrap(angle(freqresp));

    % On the first of two graphs, plot the magnitude response.
    subplot(2, 1, 1);
    plot(omega, magresp);
    title('Magnitude Response');
    xlabel('Frequency (rad/s)');
    ylabel('Magnitude (unitless)');

    % On the second of two graphs, plot the phase response.
    subplot(2, 1, 2);
    plot(omega, phaseresp);
    title('Phase Response');
    xlabel('Frequency (rad/s)');
    ylabel('Angle (rad)');

end
```

$\square$

E.15.2   Impulse and Step Responses

Sometimes, we need to determine the response of a LTI system to a specific input. Two inputs of particular interest are the Dirac delta function $\delta(t)$ and unit-step function $u(t)$. Fortunately, it is quite easy to compute impulse and step responses, as illustrated by the example below.

Example E.28 (Computing impulse and step responses). Consider the LTI system with the transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Suppose that we wish to calculate and plot the impulse and step responses of this system. This can be accomplished with the code given below.

```
% Initialize the numerator and denominator coefficients of the transfer
% function.
tfnum = [0 0 1];
tfdenom = [1 sqrt(2) 1];

% Determine the system model associated with the given transfer function.
sys = tf(tfnum, tfdenom);

% Plot the impulse response.
subplot(2, 1, 1);
impulse(sys);

% Plot the step response.
subplot(2, 1, 2);
step(sys);
```

Executing the above code produces the plots shown in Figure E.5. $\square$
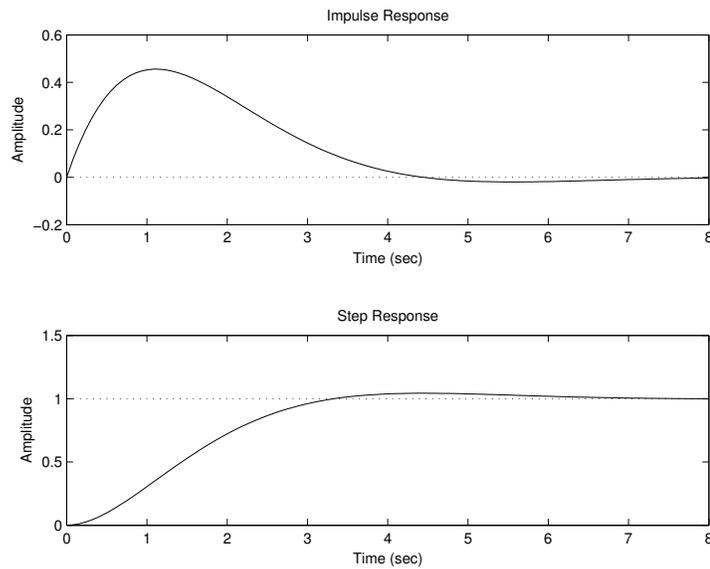
Figure E.5: Plot from example.

E.15.3 Filter Design

Example E.29 (Butterworth filter design). Suppose that we want to design a tenth-order Butterworth lowpass filter with a cutoff frequency of 100 rad/s. We can accomplish this using the code below.

```
% Calculate the transfer function coefficients for a tenth-order Butterworth
% lowpass filter with a cutoff frequency of 100 rad/s.
[tfnum, tfdenom] = butter(10, 100, 's');

% Plot the frequency response of the filter.
freqs(tfnum, tfdenom);
```

This generates a filter with the frequency response shown in Figure E.6. □

Example E.30 (Bessel filter design). Suppose that we want to design a tenth-order Bessel lowpass filter with a cutoff frequency of 100 rad/s. We can achieve this with the code given below.

```
% Calculate the transfer function coefficients for a tenth-order Bessel
% lowpass filter with a cutoff frequency of 100 rad/s.
[tfnum, tfdenom] = besself(10, 100);

% Plot the frequency response of the filter.
freqs(tfnum, tfdenom);
```

This generates a filter with the frequency response shown in Figure E.7. □

E.16 Miscellany

Some other functions that might be useful are listed in Table E.25.
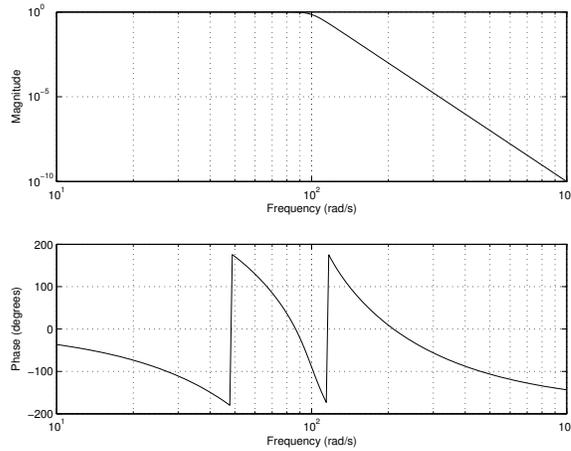
Figure E.6: Frequency response of the Butterworth lowpass filter.
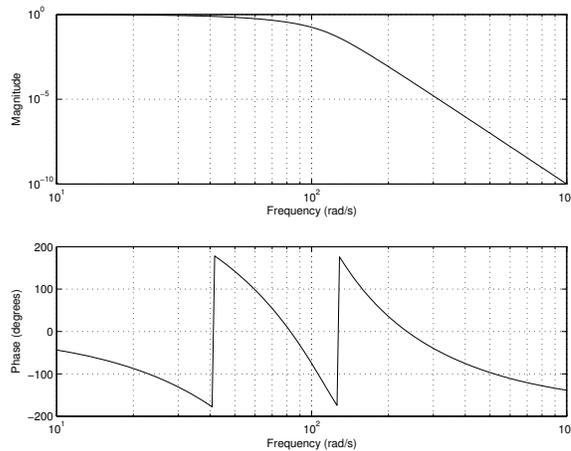
Figure E.7: Frequency response of the Bessel lowpass filter.

Table E.25: Miscellaneous Functions/Commands

Name	Description
roots	find roots of polynomial
clear	clear a variable
diary	log MATLAB session
echo	echo commands on execution (for debugging)
quit	quit MATLAB
format	output format for numbers

E.17 Problems

E.101 Indicate whether each of the following is a valid MATLAB identifier (i.e., variable/function name):
(a) `4ever`
(b) `$rich$`
(c) `foobar`
(d) `foo_bar`
(e) `_foobar`

E.102 Let T_C, T_F, and T_K denote the temperature measured in units of Celsius, Fahrenheit, and Kelvin, respectively. Then, these quantities are related by

$$T_F = \tfrac{9}{5}T_C + 32 \quad \text{and}$$
$$T_K = T_C + 273.15.$$

Write a program that generates a temperature conversion table. The first column of the table should contain the temperature in Celsius. The second and third columns should contain the corresponding temperatures in units of Fahrenheit and Kelvin, respectively. The table should have entries for temperatures in Celsius from -50 to 50 in steps of 10.

E.103 Let $F(\omega)$ denote the complex-valued function of the real variable ω given by

$$F(\omega) = \frac{1}{j\omega + 1}.$$

Write a program to plot $|F(\omega)|$ and $\arg F(\omega)$ for ω in the interval $[-10, 10]$. Use `subplot` to place both plots on the same figure.

E.104 In what follows, we consider a simple algorithm for generating a set of 2^d points in the complex plane (where d is a positive integer). For $n = 0, 1, \ldots, 2^d - 1$, this algorithm computes the nth point p_n as

$$p_n = \sum_{k=0}^{d-1} a_k \beta^{k+1},$$

where $\beta = \frac{1}{\sqrt{2}} e^{-j\pi/4}$, $a_k \in \{0, 1\}$, and $n = \sum_{k=0}^{d-1} a_k 2^k$. Note that the binary sequence $a_{d-1}, \ldots, a_1, a_0$ is simply the d-bit binary representation of the integer n, where a_0 corresponds to the least-significant bit. For example, if $d = 3$, then the relationship between n and a_2, a_1, a_0 is as shown in the following table:

n	a_2	a_1	a_0
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

(a) Write a function called `twindragon` that calculates and plots the point set obtained by using the above algorithm for a specified value of the parameter d. The value of d should be passed as an input argument to the function. [Hint: It is possible to implement this function in about 15 to 20 lines of code. The `polyval` and `dec2bin` functions should be quite helpful.]
(b) Using the function developed in part (a), plot the set of points obtained with the parameter d equal to 12. In the limit as d approaches infinity, the resulting set of points converges to the well-known twin-dragon fractal set.

Although choosing $d = 12$ should be sufficient to see the approximate shape of the point set in the limiting case, larger values of d will yield better approximations, albeit at the expense of significantly more computation. You should obtain a plot resembling that shown below.
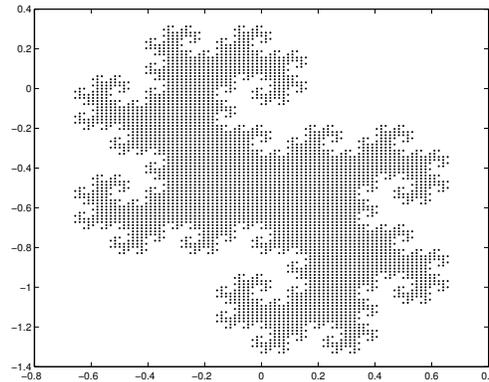
E.105 In what follows, let $\min(a, b)$ denote the minimum of a and b. For a complex number z, we define an iterative process that generates the complex-number sequence $(v_0, v_1, v_2, \ldots)$, where

$$v_n = \begin{cases} v_{n-1}^2 + z & \text{for } n \geq 1 \\ 0 & \text{for } n = 0. \end{cases}$$

Let $g(z)$ denote the smallest value of n for which $|v_n| > 10$, where the sequence $(v_0, v_1, v_2, \ldots)$ is defined in terms of z as given above. In the case that $|v_n|$ never exceeds 10, $g(z)$ is simply defined to be ∞. Let $f(z) = \min(g(z), 10)$. For example, we have the following:

$$\begin{aligned}
\text{For } z = 0: \quad & (v_0, v_1, v_2, \ldots) = (0, 0, 0, \ldots), \quad g(z) = \infty, \quad \text{and} \quad f(z) = \min(\infty, 10) = 10. \\
\text{For } z = 2: \quad & (v_0, v_1, v_2, v_3) = (0, 2, 6, 38), \quad g(z) = 3, \quad \text{and} \quad f(z) = 3. \\
\text{For } z = 2j: \quad & (v_0, v_1, v_2, v_3) = (0, 2j, -4 + 2j, 12 - 14j), \quad g(z) = 3, \quad \text{and} \quad f(z) = 3. \\
\text{For } z = j: \quad & (v_0, v_1, v_2, v_3, v_4, v_5, \ldots) = (0, j, -1 + j, -j, -1 + j, -j, \ldots), \\
& g(z) = \infty, \quad \text{and} \quad f(z) = \min(\infty, 10) = 10.
\end{aligned}$$

Write a function called `mandelbrotfunc` to compute the function $f(z)$ for any (complex) value of z. Using the function `mandelbrotfunc`, evaluate $f(z)$ on an evenly-spaced rectangular grid (in the complex plane) having width 128, height 128, and with the bottom-left and top-right corners of the grid corresponding to the points $-2.25 - 1.5j$ and $1 + 1.5j$, respectively. Store these computed values into a 128×128 matrix. Then, using the `pcolor` function in MATLAB, plot the contents of this matrix. After the call to `pcolor`, use the command "`shading interp`" to eliminate the grid lines from the plot (which causes the plot to look somewhat less attractive).

A complete solution to this problem requires less than 25 lines of code (excluding comments). A correct solution should yield a plot resembling the one shown below. Incidentally, the innermost region in the plot is an approximation to the famous Mandelbrot (fractal) set.
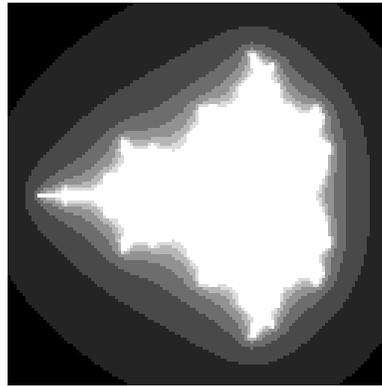
E.106 Suppose that the vector `v` is defined by the following line of code:

```
v = [0 1 2 3 4 5]
```

Write an expression in terms of `v` that yields a new vector of the same dimensions as `v`, where each element t of the original vector `v` has been replaced by the given quantity below. In each case, the expression should be as short as possible.
(a) $2t - 3$;
(b) $1/(t+1)$;
(c) $t^5 - 3$; and
(d) $|t| + t^4$.

E.107 (a) Write a function called `unitstep` that takes a single real argument t and returns $u(t)$, where

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Modify the function from part (a) so that it takes a single vector argument $t = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}^T$ (where $n \geq 1$ and $t_1, t_2, \ldots, t_n$ are real) and returns the vector $\begin{bmatrix} u(t_1) & u(t_2) & \cdots & u(t_n) \end{bmatrix}^T$. Your solution must employ a looping construct (e.g., a for loop).
(c) With some ingenuity, part (b) of this problem can be solved using only two lines of code, without the need for any looping construct. Find such a solution. [Hint: In MATLAB, to what value does an expression like "`[-2 -1 0 1 2] >= 0`" evaluate?]

E.108 In this problem, we consider an algorithm for generating an ordered list of n points in the plane $\{p_0, p_1, \ldots, p_{n-1}\}$. The first point p_0 is chosen as the origin (i.e., $p_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$), with the remaining points being given by the formula

$$p_i = p_{i-1} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{i-1} \begin{bmatrix} i \\ 0 \end{bmatrix}.$$

(a) Using MATLAB, write a function called `drawpattern` that takes n and θ as input arguments (in that order) with θ being specified in degrees, and then computes and plots the points $\{p_0, p_1, \ldots, p_{n-1}\}$ connected by straight lines (i.e., draw a line from p_0 to p_1, p_1 to p_2, p_2 to p_3, and so on). When performing the plotting, be sure to use `axis('equal')` in order to maintain the correct aspect ratio for the plot. For illustrative purposes, the plots produced for two sets of θ and n values are shown in Figure E.8.
(b) Generate the plots obtained by invoking `drawpattern` with $n = 100$ and θ set to each of the following values: $89°$, $144°$, and $154°$. [Note: In MATLAB, the `sin` and `cos` functions take values in radians, not degrees.]
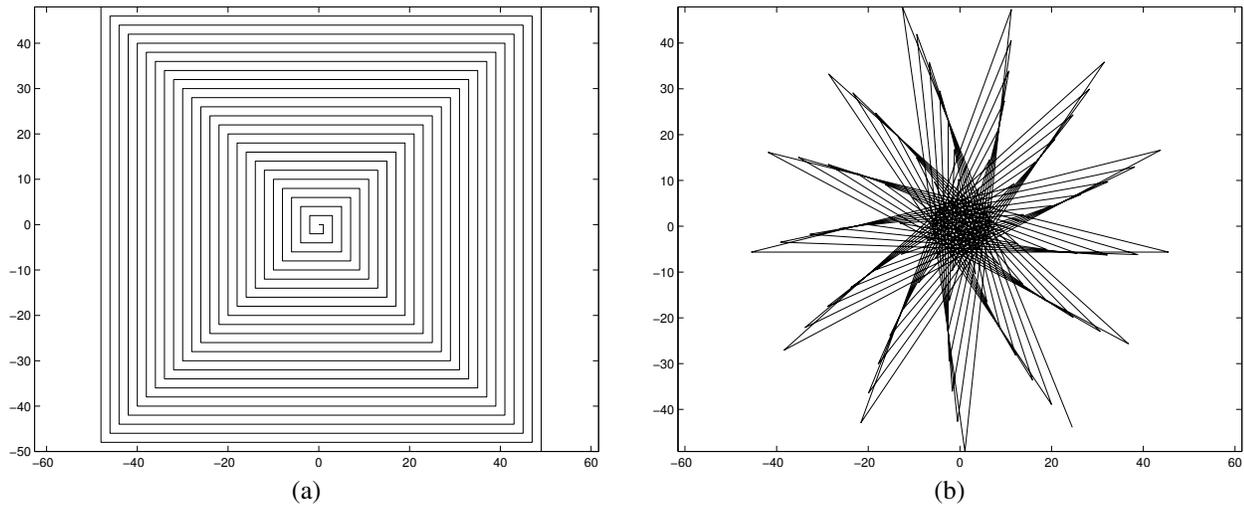
Figure E.8: Sample plots obtained from drawpattern. (a) $\theta = 90°$ and $n = 100$; (b) $\theta = 166°$ and $n = 100$.

Appendix F

Review Problems

F.1 A communication channel heavily distorts high frequencies but does not significantly affect very low frequencies. Determine which of the following signals would be least distorted by the communication channel:
(a) $x_1(t) = \delta(t)$
(b) $x_2(t) = 5$
(c) $x_3(t) = 10e^{j1000t}$
(d) $x_4(t) = 1/t$

Answer:

$x_2(t)$

F.2 Consider a system consisting of a communication channel with input $x(t)$ and output $y(t)$. Since the channel is not ideal, $y(t)$ is typically a distorted version of $x(t)$. Suppose that the channel can be modelled as a causal LTI system with impulse response $h(t) = e^{-t}u(t) + \delta(t)$. Determine whether we can devise a physically-realizable stable system that recovers $x(t)$ from $y(t)$. If such a system exists, find its impulse response $g(t)$.

Answer:

$g(t) = \delta(t) - e^{-2t}u(t)$

F.3 A signal $x(t)$ has magnitude spectrum $M(\omega) = 1$ and phase spectrum $P(\omega) = \omega$. Find $x(t)$.

Answer:

$x(t) = \delta(t+1)$

F.4 Let $x(t) = \text{sinc}(5t - 3)$. By direct application of the Nyquist sampling theorem, determine the lowest sampling rate ω_s at which $x(t)$ can be sampled such that it can be exactly reconstructed from its samples.

Answer:

10 rad/s

F.5 A causal LTI system has impulse response $h(t)$, system function $H(s)$, and the following characteristics: 1) $H(s)$ is rational with one pole at -2 and no zeros; and 2) $h(0^+) = 4$. Find $h(t)$.

Answer:

$h(t) = 4e^{-2t}u(t)$

F.6 A causal LTI system with input $x(t)$ and output $y(t)$ is characterized by the differential equation $\frac{d}{dt}y(t) + 3y(t) = 2x(t)$. Find the impulse response $h(t)$ of the system.

Answer:

$h(t) = 2e^{-3t}u(t)$

F.7 A causal LTI system has input $x(t)$ and output $y(t)$. If the input $x_1(t) = e^{-2t}u(t)$ produces the output $y_1(t) = e^{-3t}u(t)$, find the unit-step response $s(t)$ of the system.

Answer:

$s(t) = \frac{2}{3}u(t) + \frac{1}{3}e^{-3t}u(t)$

F.8 A causal stable LTI system $\mathcal{H}$ has impulse response $h(t)$ and system function $H(s)$. The function $H(s)$ is rational, contains a pole at -2 and does not have a zero at the origin. The other poles and zeros of $H(s)$ are unknown. Determine whether each of the statements below is true, false, or uncertain (i.e., insufficient information to determine).
(a) $\mathcal{F}\{e^{3t}h(t)\}$ converges;
(b) $\int_{-\infty}^{\infty} h(t)dt = 0$;
(c) $th(t)$ is the impulse response of a causal and stable system;
(d) $\frac{d}{dt}h(t)$ has at least one pole in its Laplace transform expression.

Answer:

(a) false; (b) false; (c) true; (d) true

F.9 Find the inverse Laplace transform $x(t)$ of $X(s) = e^{-7s}\frac{6s+13}{(s+2)(s+3)}$ for $\text{Re}\{s\} > -2$.

Answer:

$x(t) = e^{-2(t-7)}u(t-7) + 5e^{-3(t-7)}u(t-7)$

F.10 Let $X(s)$ denote the Laplace transform of $x(t)$. Suppose that $x(t)$ is absolutely integrable and $X(s)$ is rational with a pole at 1 and the other poles unknown. Determine if $x(t)$ can be each of the following:
(a) left sided;
(b) right sided;
(c) two sided;
(d) finite duration.

Answer:

(a) yes; (b) no; (c) yes; (d) no

F.11 For each of the cases given below, determine whether $x(t)$ is periodic, and if it is, find its fundamental period T.
(a) $x(t) = 3\cos(\sqrt{2}t) + 7\cos 2t$
(b) $x(t) = (3\cos 2t)^3$
(c) $x(t) = 7\cos(35t + 3) + 5\sin(15t - 2)$

Answer:

(a) not periodic; (b) π-periodic; (c) $(2\pi/5)$-periodic

F.12 A communication channel can be well approximated by a LTI system with impulse response $h(t) = \frac{1000}{\pi}\text{sinc}(1000t)$. Determine which of the following signals would be least distorted by this channel:
(a) $x_1(t) = \delta(t)$
(b) $x_2(t) = u(t)$
(c) $x_3(t) = \cos 100t$

Answer:

$x_3(t)$

F.13 Given that $e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{\omega^2+a^2}$, find the Fourier transform $X(\omega)$ of $x(t) = \frac{2a}{t^2+a^2}$.

Answer:

$X(\omega) = 2\pi e^{-a|\omega|}$

F.14 Simplify each of the following expressions:

(a) $\frac{(\omega^2+1)\delta(\omega-1)}{\omega^2+9}$

(b) $\frac{(\sin k\omega)\delta(\omega)}{\omega}$

Answer:

(a) $\frac{1}{5}\delta(\omega-1)$; (b) $k\delta(\omega)$

F.15 Let $E\{x(t)\}$ denote the energy of the signal $x(t)$. Show that:

(a) $E\{ax(t)\} = a^2 E\{x(t)\}$ where a is a real constant;

(b) $E\{x(t-t_0)\} = E\{x(t)\}$ where t_0 is a real constant;

(c) $E\{x(at)\} = \frac{1}{|a|} E\{x(t)\}$ where a is a nonzero real constant

F.16 Evaluate each of the following:

(a) $\int_{-\infty}^{\infty} e^{t-1} \cos[\frac{\pi}{2}(t-5)]\delta(t-3)dt$

(b) $\int_{-\infty}^{\infty} \delta(2t-3)\sin \pi t\, dt$

Answer:

(a) $-e^2$; (b) $-\frac{1}{2}$

F.17 Let $x_e(t)$ and $x_o(t)$ denote the even and odd parts of $x(t)$, respectively. Show that:

(a) $\int_{-\infty}^{\infty} x_e(t)x_o(t)dt = 0$;

(b) $\int_{-\infty}^{\infty} x(t)dt = \int_{-\infty}^{\infty} x_e(t)dt$

F.18 Consider a system with input $x(t)$ and output $y(t)$. For each of the equations given below, determine whether the corresponding system is linear.

(a) $3y(t) + 2 = x(t)$

(b) $y(t) = \frac{d}{dt}x(t)$

(c) $y(t) = tu(t)x(t)$

Answer:

(a) nonlinear; (b) linear; (c) linear

F.19 Consider a system with input $x(t)$ and output $y(t)$. For each of the equations given below, determine whether the corresponding system is time invariant.

(a) $y(t) = \int_{-4}^{4} x(\tau)d\tau$

(b) $y(t) = [\frac{d}{dt}x(t)]^2$

(c) $y(t) = tu(t)x(t)$

Answer:

(a) not time invariant; (b) time invariant; (c) not time invariant

F.20 Consider a system with input $x(t)$ and output $y(t)$. For each of the equations given below, determine whether the corresponding system is additive and/or homogeneous.

(a) $y(t) = \frac{x^2(t)}{\frac{d}{dt}x(t)}$

Answer:

(a) homogeneous, not additive

F.21 Consider a system with input $x(t)$ and output $y(t)$. For each of the equations given below, determine whether the corresponding system is causal.

(a) $y(t) = x(at)$ where a is a nonzero real constant;

(b) $y(t) = tu(t)x(t)$;

(c) $y(t) = x(t-a)$ where a is a strictly negative real constant

Answer:

(a) causal if and only if $a = 1$; (b) causal; (c) noncausal

F.22 Consider a system with input $x(t)$ and output $y(t)$. For each of the equations given below, determine whether the corresponding system is invertible.
(a) $y(t) = \frac{d}{dt}x(t)$
(b) $y(t) = \cos x(t)$
(c) $y(t) = e^{x(t)}$, where $x(t)$ is real

Answer:

(a) not invertible; (b) not invertible; (c) invertible $(x(t) = \ln y(t))$

F.23 Consider a system with input $x(t)$ and output $y(t)$. For each of the equations given below, determine whether the corresponding system is BIBO stable.
(a) $y(t) = u(t)x(t)$

Answer:

(a) stable

F.24 Show that $x(t) * \delta(t - t_0) = x(t - t_0)$, where t_0 is an arbitrary real constant.

F.25 Consider a LTI system with impulse response $h(t)$. For each of the cases below, determine whether the system is BIBO stable.
(a) $h(t) = u(t - 1) - u(t - 2)$

Answer:

(a) stable

F.26 Consider a LTI system with impulse response $h(t)$. For each of the cases below, determine whether the system is causal.
(a) $h(t) = u(t + 1) - u(t - 1)$
(b) $h(t) = e^{-5t}u(t - 1)$

Answer:

(a) not causal (b) causal

F.27 Use the Laplace transform to compute the convolution $y(t) = e^{-at}u(t) * e^{-bt}u(t)$, where $a, b > 0$. [Note that there are two cases to consider, namely $a = b$ and $a \neq b$.]

Answer:

if $a \neq b$, $y(t) = \frac{1}{b-a}(e^{-at} - e^{-bt})u(t)$; if $a = b$, $y(t) = te^{-at}u(t)$

F.28 Find the inverse Laplace transform $x(t)$ of $X(s) = \frac{-3}{(s+2)(s-1)}$ if the ROC is
(a) $-2 < \text{Re}\{s\} < 1$;
(b) $\text{Re}\{s\} > 1$;
(c) $\text{Re}\{s\} < -2$

Answer:

(a) $x(t) = e^{-2t}u(t) + e^t u(-t)$; (b) $x(t) = [e^{-2t} - e^t]u(t)$; (c) $x(t) = [-e^{-2t} + e^t]u(-t)$

F.29 A rational function $F(s)$ has a first-order pole at -1, a second-order pole at -2, and a first-order zero at 0. The function is known not to have any other poles or zeros. If $F(1) = 1$, find $F(s)$.

Answer:

$F(s) = \frac{18s}{(s+1)(s+2)^2}$

F.30 Let $X(\omega)$ denote the Fourier transform of $x(t)$. Show that (a) if $x(t)$ is even, then $X(\omega) = 2\int_0^\infty x(t)\cos \omega t \, dt$; and (b) if $x(t)$ is odd, then $X(\omega) = -2j\int_0^\infty x(t)\sin \omega t \, dt$.

F.31 Let $x(t)$ be a real function with even part $x_e(t)$ and odd part $x_o(t)$. Let $X(\omega)$, $X_e(\omega)$, and $X_o(\omega)$ denote the Fourier transforms of $x(t)$, $x_e(t)$, and $x_o(t)$, respectively. Show that (a) $X_e(\omega) = \text{Re}\{X(\omega)\}$ and (b) $X_o(\omega) = \text{Im}\{X(\omega)\}$.

F.32 For each of the cases below, find the Fourier transform $X(\omega)$ of the signal $x(t)$.
(a) $x(t) = \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right]$;
(b) $x(t) = \delta(t+a) + \delta(t-a)$;

Answer:

(a) $X(\omega) = u(\omega)$; (b) $X(\omega) = 2\cos a\omega$;

F.33 Show that if $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$ then $x(t+a) + x(t-a) \overset{\mathscr{F}}{\longleftrightarrow} 2X(\omega)\cos a\omega$.

F.34 Show that, if a signal $x(t)$ has bandwidth B, then $x^n(t)$ has bandwidth nB.

F.35 Using Parseval's relation, evaluate the integral $\int_{-\infty}^{\infty} \text{sinc}^2(kx)dx$, where k is a real constant.

Answer:

π/k

F.36 In wireless communication channels, the transmitted signal is propagated simultaneously along multiple paths of varying lengths. Consequently, the signal received from the channel is the sum of numerous delayed and amplified/attenuated versions of the original transmitted signal. In this way, the channel distorts the transmitted signal. This is commonly referred to as the multipath problem. In what follows, we examine a simple instance of this problem.

Consider a LTI communication channel with input $x(t)$ and output $y(t)$. Suppose that the transmitted signal $x(t)$ propagates along two paths. Along the intended direct path, the channel has a delay of T and gain of one. Along a second (unintended indirect) path, the signal experiences a delay of $T + \tau$ and gain of a. Thus, the received signal $y(t)$ is given by $y(t) = x(t-T) + ax(t-T-\tau)$. Find the transfer function $H(s)$ of a system that can be connected in series with the output of the communication channel in order to recover the (delayed) signal $x(t-T)$ without any distortion.

Answer:

$H(s) = \frac{1}{1+ae^{-s\tau}}$

F.37 A signal $x(t)$ is bandlimited to 22 kHz (i.e., only has spectral content for frequencies f in the range $[-22000, 22000]$). Due to excessive noise, the portion of the spectrum that corresponds to frequencies f satisfying $|f| > 20000$ has been badly corrupted and rendered useless. (a) Determine the minimum sampling rate for $x(t)$ that would allow the uncorrupted part of the spectrum to be recovered. (b) Suppose now that the corrupted part of the spectrum were eliminated by filtering prior to sampling. In this case, determine the minimum sampling rate for $x(t)$.

Answer:

(a) 42 kHz; (b) 40 kHz

F.38 Suppose that we want to transmit a binary sequence $y[n]$ (where $y[n]$ is either zero or one) over a continuous-time communication channel. To do this, we choose to represent each binary sample of $y[n]$ with a continuous-time pulse. Using the sampling theorem, show that it is possible to transmit the bits of $y[n]$ at a rate of $2B$ bits per second over an ideal (i.e., noise-free) channel of bandwidth B. As it turns out, this is the theoretical upper bound on the data transmission rate, assuming that each pulse is used to represent only a single bit of data. (Hint: According to the sampling theorem, a continuous-time signal of bandwidth B can be constructed from $2B$ pieces of information.)

F.39 An audio signal $x(t)$ consists of two perfect sinusoidal tones at 440 Hz and 880 Hz. The signal $x(t)$ is sampled at a rate of f_s Hz and then played back on a loudspeaker. Determine how many tones will be heard on the

loudspeaker and their frequencies, if f_s equals (a) 500 Hz and (b) 2000 Hz. (You can safely assume that a human can hear frequencies from 20 Hz to 20 kHz.)

Answer:

(a) Two tones are heard with frequencies 60 Hz and 120 Hz; (b) Two tones are heard at 440 Hz and 880 Hz

F.40 Prove that a signal cannot be both timelimited and bandlimited. [Hint: Let $X(\omega)$ denote the Fourier transform of $x(t)$. Suppose that $x(t)$ is timelimited and also bandlimited such that $X(\omega) = 0$ for $|\omega| \geq B$. Due to the bandlimited nature of $x(t)$, we have that $X(\omega) = X(\omega) \text{rect}\left(\frac{\omega}{2B'}\right)$ for $B' > B$. Then, show that the inverse Fourier transform of the preceding equation leads to a contradiction. To do this, you will need to show that the convolution of a timelimited signal with a non-timelimited signal must be a non-timelimited signal.]

F.41 The bandwidth of a LTI system is most simply defined as the bandwidth of the system's frequency response $H(\omega)$. Explain why a (LTI) communication channel with (finite) bandwidth B cannot be used to (reliably) transmit a signal with bandwidth greater than B.

F.42 For each of the cases below, using the graphical method, compute the convolution $y(t) = x(t) * h(t)$.
(a) $x(t) = \text{rect}\left(\frac{t}{2a}\right)$, $h(t) = \text{rect}\left(\frac{t}{2a}\right)$; and
(b) $x(t) = \text{rect}\left(\frac{t}{a}\right)$, $h(t) = \text{rect}\left(\frac{t}{a}\right)$.

Answer:

(a) $y(t) = 2a\,\text{tri}\left(\frac{t}{4a}\right)$; (b) $y(t) = a\,\text{tri}\left(\frac{t}{2a}\right)$

F.43 A common problem in real-world instrumentation systems is electromagnetic interference caused by 60 Hz power lines. In particular, 60 Hz power lines can often introduce a significant amount of interference (i.e., noise) at 60 Hz and its higher harmonics (i.e., 120 Hz, 180 Hz, 240 Hz, and so on). Consider a causal system with an impulse response of the form $h(t) = a[u(t) - u(t - b)]$, where a and b are nonzero real constants. With an appropriate choice of a and b, such a system can be made to reject interference at 60 Hz and all of its higher harmonics. Find a and b.

Answer:

$a \neq 0$ and $b = \frac{1}{60}$

F.44 Find the impulse response $h(t)$ of the causal LTI system with input $x(t)$ and output $y(t)$ characterized by the given equation, for each of the cases below.
(a) $4y''(t) = 2x(t) - x'(t)$.

Answer:

(a) $\left[\frac{1}{2}t - \frac{1}{4}\right]u(t)$

F.45 For each of the cases below, find the zeros and poles of $F(s)$ and their orders.
(a) $F(s) = \frac{s-3}{s^5 + 7s}$.

Answer:

(a) first-order zero at 3; first-order poles at 0, $\sqrt[4]{7}e^{j\pi/4}$, $\sqrt[4]{7}e^{j3\pi/4}$, $\sqrt[4]{7}e^{-j3\pi/4}$, and $\sqrt[4]{7}e^{-j\pi/4}$.

F.46 Given that $x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega)$ and $y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(\omega)$, use a Fourier transform table and properties of the Fourier transform to express $Y(\omega)$ in terms of $X(\omega)$ for each of the cases below.
(a) $y(t) = t(\cos 3t)x(t)$;
(b) $y(t) = (t-1)^{100}x^*(t-1)$.

Answer:

(a) $Y(\omega) = \frac{j}{2}X'(\omega - 3) + \frac{j}{2}X'(\omega + 3)$ (where the prime symbol denotes the first derivative); (b) $Y(\omega) = e^{-j\omega}\left(\frac{d}{d\omega}\right)^{100}[X^*(-\omega)]$.

F.47 Let $x_1(t) = \mathrm{sinc}(at - b_1)$ and $x_2(t) = \mathrm{sinc}(at - b_2)$, where a, b_1, and b_2 are real constants with $a \neq 0$. Using properties of the Fourier transform and a Fourier transform table, compute the convolution $y(t) = x_1(t) * x_2(t)$.

Answer:

$y(t) = \frac{\pi}{|a|} \mathrm{sinc}(at - b_1 - b_2)$

F.48 Let $y_1(t)$ and $y_2(t)$ be signals bandlimited to frequencies in the range $[-\omega_b, \omega_b]$. Suppose that these signals are sampled at a frequency ω_s satisfying the Nyquist condition to produce the sequences

$$x_1[n] = y_1(nT) \quad \text{and} \quad x_2[n] = y_2(nT),$$

where $T = \frac{2\pi}{\omega_s}$. Now, consider the signal $y(t) = y_1(t) * y_2(t)$. Suppose that $y(t)$ is also sampled with period T to produce the sequence

$$x[n] = y(nT).$$

(a) Show that $y(t)$ is bandlimited to frequencies in the range $[-\omega_b, \omega_b]$, meaning that it must be possible to recover $y(t)$ exactly from its samples.
(b) Show that the samples of $y(t)$ can be computed as follows:

$$x[n] = \frac{2\pi}{\omega_s} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k].$$

(c) Explain how we might use the above results to compute the (continuous-time) convolution of bandlimited signals using a (discrete-time) computer.

F.49 Consider a bandlimited signal $x(t)$ that has been sampled at a frequency ω_s (in radians) satisfying the Nyquist condition to produce the sequence $y[n]$. Find an expression for the Fourier transform $X(\omega)$ of the signal $x(t)$ in terms of its samples $y[n]$ and ω_s. [Hint: Recall that, from the sampling theorem, $x(t) = \sum_{n=-\infty}^{\infty} y[n] \mathrm{sinc}(\frac{\omega_s}{2}t - \pi n)$.]

Answer:

$X(\omega) = \frac{2\pi}{\omega_s} \sum_{n=-\infty}^{\infty} y[n] e^{-j2\pi n\omega/\omega_s} \mathrm{rect}(\omega/\omega_s)$

F.50 Find the system function $H(s)$ of the system with input $X(s)$ and output $Y(s)$ as shown in the figure. Determine whether the system is BIBO stable. (Note that all subsystems are causal.)
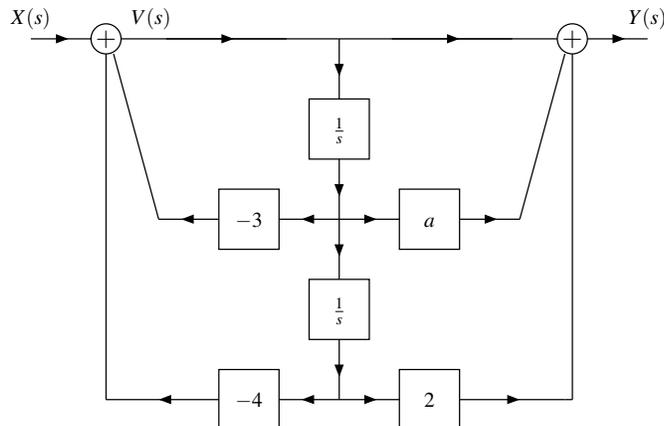
Answer:

$H(s) = \frac{s^2 + as + 2}{s^2 + 3s + 4}$; system is BIBO stable

F.51 Find the system function $H(s)$ of the system with input $X(s)$ and output $Y(s)$ as shown in the figure. Determine for what values of the parameter a the system is BIBO stable.
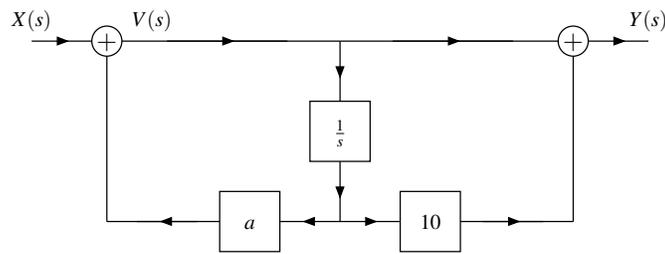
Answer:

$H(s) = \frac{s+10}{s-a}$; system is BIBO stable if $a < 0$

F.52 Find the system function $H(s)$ of the system with input $X(s)$ and output $Y(s)$ as shown in the figure. Determine whether the system is BIBO stable. All subsystems in the figure are causal.
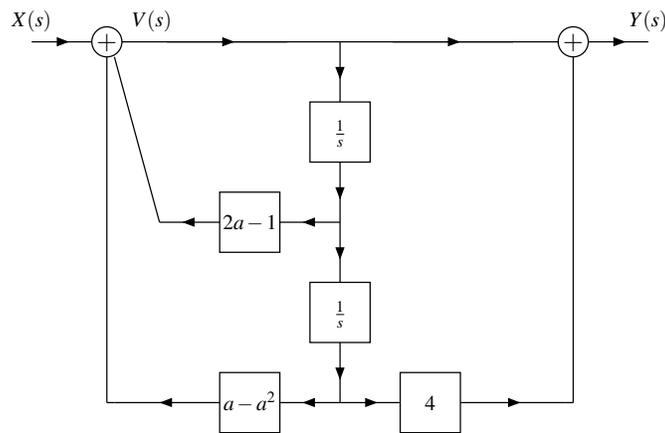
Answer:

$H(s) = \frac{s^2+4}{s^2+(1-2a)s+a^2-a}$; BIBO stable if $a < 0$

F.53 Consider the system with input $X(s)$ and output $Y(s)$ as shown in the figure, where $P(s) = 1/s$, $C(s) = as+3$, $Q(s) = 1$, and a is a real constant. All subsystems in the figure are causal. (a) Find the system function $H(s)$. (b) Determine for what values of the parameter a the system is BIBO stable.
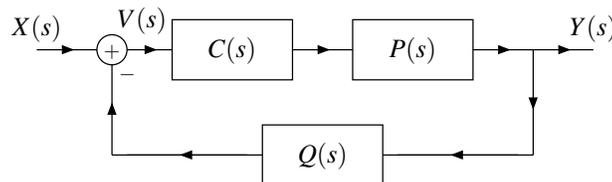
Answer:

(a) $H(s) = \frac{as+3}{s+as+3}$; (b) $a > -1$

F.54 Let $X(\omega)$ denote the Fourier transform of $x(t)$. For each case below, indicate whether $x(t)$ is: real, even/odd, periodic, finite duration, finite energy (if this can be determined).

(a)   (b)   (c)

(d)   (e)
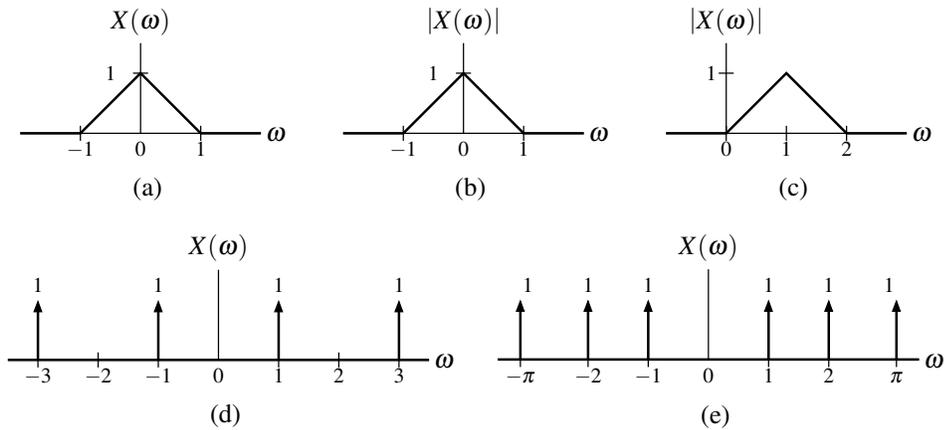
Answer:

(a) real, even, aperiodic, not finite duration, finite energy; (b) aperiodic, finite energy, not finite duration; (c) not real, aperiodic, not finite duration, finite energy, not even/odd; (d) real, even, periodic (fundamental period $\frac{2\pi}{1} = 2\pi$), not finite duration, not finite energy; (e) real, even, aperiodic, not finite duration, not finite energy

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