Lecture Slides for Signals and Systems
Edition 2020-05-ECE260

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2. M. D. Adams, *Multiresolution Signal and Geometry Processing: Filter Banks, Wavelets, and Subdivision (Version 2013-09-26)*, University of Victoria, Victoria, BC, Canada, Sept. 2013, xxxviii + 538 pages, ISBN 978-1-55058-507-0 (print), ISBN 978-1-55058-508-7 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/waveletbook.

3 M. D. Adams, *Lecture Slides for Multiresolution Signal and Geometry Processing (Version 2015-02-03)*, University of Victoria, Victoria, BC, Canada, Feb. 2015, xi + 587 slides, ISBN 978-1-55058-535-3 (print), ISBN 978-1-55058-536-0 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/waveletbook.

4 M. D. Adams, *Continuous-Time Signals and Systems*, Edition 2.0, University of Victoria, Victoria, BC, Canada, Feb. 2020, xxx + 366 pages, ISBN 978-1-55058-657-2 (print), ISBN 978-1-55058-658-9 (PDF). Available from Google Books, Google Play Books, University of Victoria Bookstore, and author's web site http://www.ece.uvic.ca/~mdadams/sigsysbook.

Part 0

Preface

About These Lecture Slides

- This document constitutes a detailed set of lecture slides on signals and systems, covering both the continuous-time and discrete-time cases.
- These slides are organized in such a way as to facilitate the teaching of a course that covers: only the continuous-time case, or only the discrete-time case, or both the continuous-time and discrete-time cases.
- To teach a course on only the continuous-time case, these slides can be used in conjunction with the following textbook:
  - M. D. Adams, *Continuous-Time Signals and Systems*, Edition 2.0, University of Victoria, Victoria, BC, Canada, Feb. 2020. xxx + 366 pages. ISBN 978-1-55058-657-2 (print), ISBN 978-1-55058-658-9 (PDF). Available online from http://www.ece.uvic.ca/~mdadams/sigsysbook.
- The author is currently in the process of developing a new textbook that covers both the continuous-time and discrete-time cases. These lecture slides are also intended for use with this new textbook, when it becomes available.

Typesetting Conventions

- In a definition, the term being defined is often typeset in a font **like this**.
- To emphasize particular words, the words are typeset in a font *like this*.

Part 1
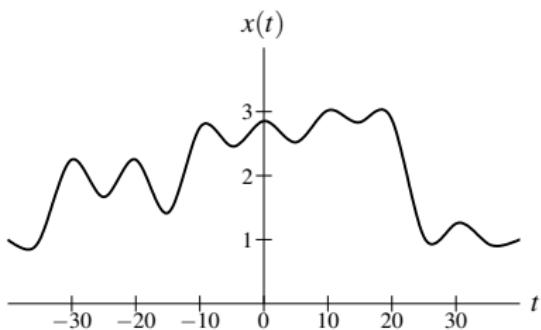
Introduction

Signals

- A **signal** is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- For a function f, in the expression $f(t_1, t_2, \ldots, t_n)$, each of the $\{t_k\}$ is called an **independent variable**, while the function value itself is referred to as a **dependent variable**.
- Some examples of signals include:
    - a voltage or current in an electronic circuit
    - the position, velocity, or acceleration of an object
    - a force or torque in a mechanical system
    - a flow rate of a liquid or gas in a chemical process
    - a digital image, digital video, or digital audio
    - a stock market index
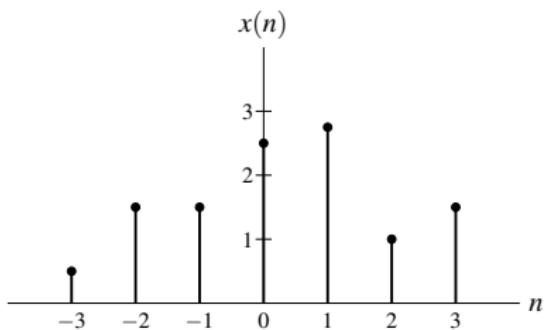
Classification of Signals

- Number of independent variables (i.e., dimensionality):
    - A signal with *one* independent variable is said to be **one dimensional** (e.g., audio).
    - A signal with *more than one* independent variable is said to be **multi-dimensional** (e.g., image).
- Continuous or discrete independent variables:
    - A signal with *continuous* independent variables is said to be **continuous time (CT)** (e.g., voltage waveform).
    - A signal with *discrete* independent variables is said to be **discrete time (DT)** (e.g., stock market index).
- Continuous or discrete dependent variable:
    - A signal with a *continuous* dependent variable is said to be **continuous valued** (e.g., voltage waveform).
    - A signal with a *discrete* dependent variable is said to be **discrete valued** (e.g., digital image).
- A *continuous-valued CT* signal is said to be **analog** (e.g., voltage waveform).
- A *discrete-valued DT* signal is said to be **digital** (e.g., digital audio).

Graphical Representation of Signals
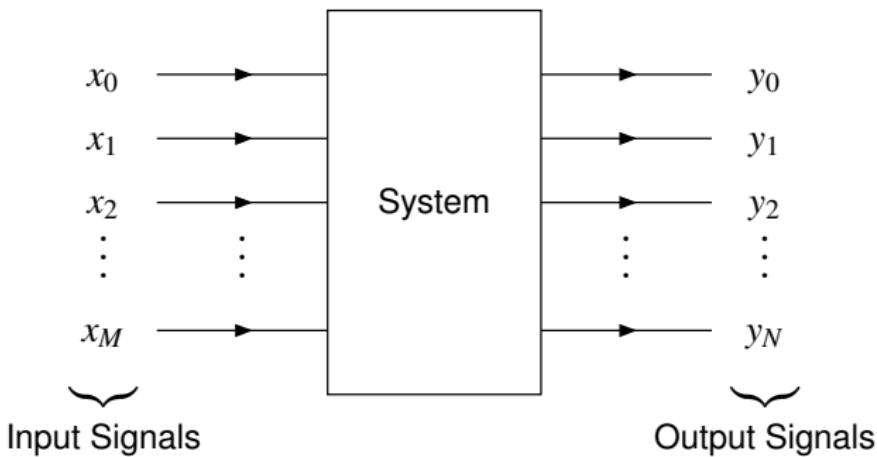
Continuous-Time (CT) Signal

Discrete-Time (DT) Signal
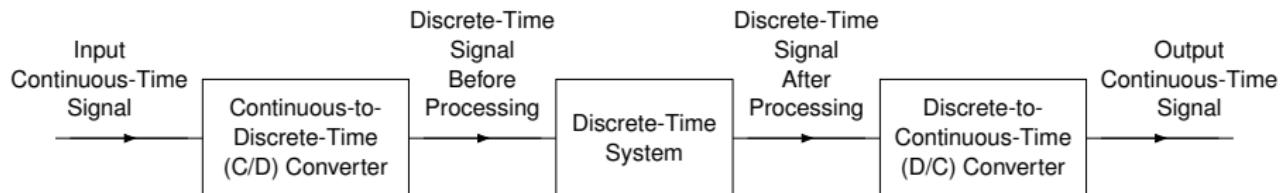
- A **system** is an entity that processes one or more input signals in order to produce one or more output signals.

Input Signals — System — Output Signals
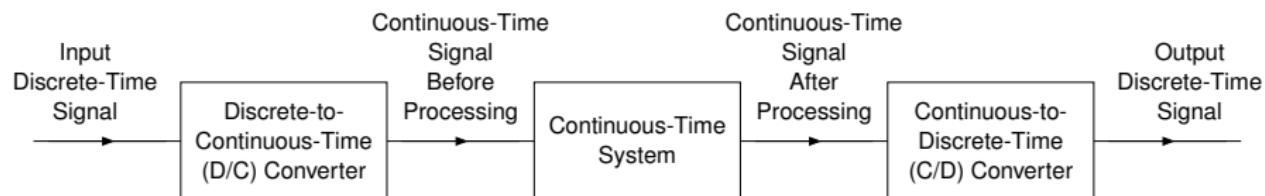
Classification of Systems

- Number of inputs:
  - A system with *one* input is said to be **single input (SI)**.
  - A system with *more than one* input is said to be **multiple input (MI)**.
- Number of outputs:
  - A system with *one* output is said to be **single output (SO)**.
  - A system with *more than one* output is said to be **multiple output (MO)**.
- Types of signals processed:
  - A system can be classified in terms of the *types of signals* that it processes.
  - Consequently, terms such as the following (which describe signals) can also be used to describe systems:
    - one-dimensional and multi-dimensional,
    - continuous-time (CT) and discrete-time (DT), and
    - analog and digital.
  - For example, a continuous-time (CT) system processes CT signals and a discrete-time (DT) system processes DT signals.

Signal Processing Systems
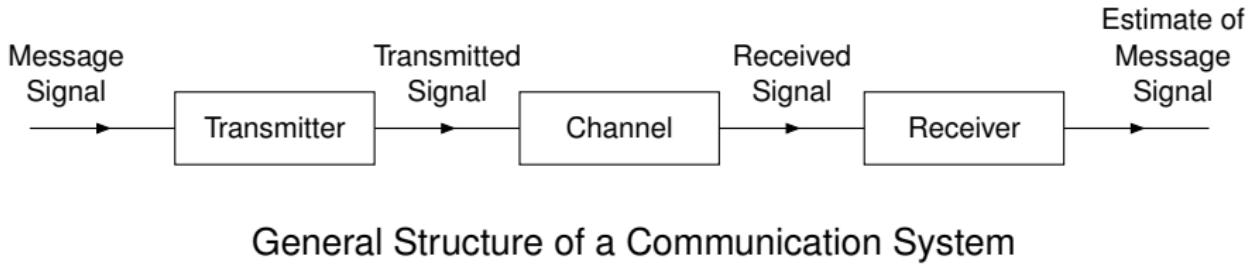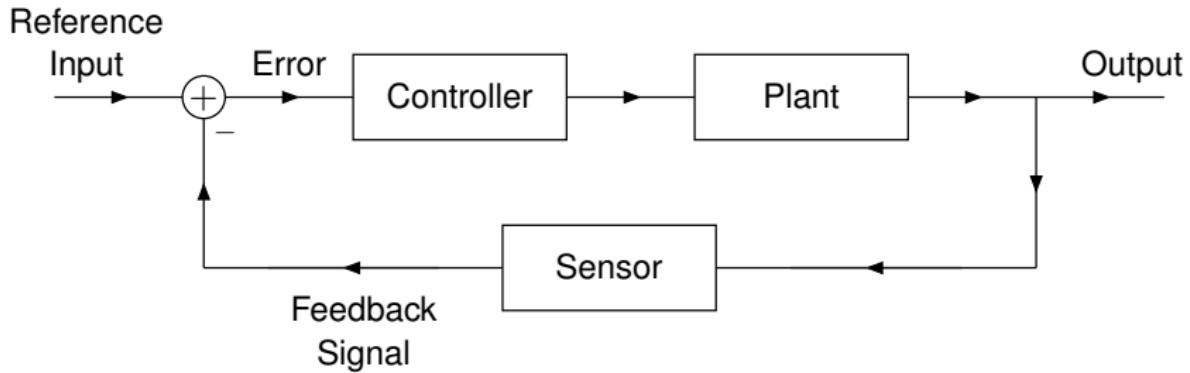
Processing a Continuous-Time Signal With a Discrete-Time System

Processing a Discrete-Time Signal With a Continuous-Time System

Message Signal → Transmitter → Transmitted Signal → Channel → Received Signal → Receiver → Estimate of Message Signal

General Structure of a Communication System

General Structure of a Feedback Control System

Why Study Signals and Systems?

- Engineers build systems that process/manipulate signals.
- We need a formal mathematical framework for the study of such systems.
- Such a framework is necessary in order to ensure that a system will meet the required specifications (e.g., performance and safety).
- If a system fails to meet the required specifications or fails to work altogether, negative consequences usually ensue.
- When a system fails to operate as expected, the consequences can sometimes be catastrophic.

System Failure Example: Tacoma Narrows Bridge

- The (original) Tacoma Narrows Bridge was a suspension bridge linking Tacoma and Gig Harbor (WA, USA).
- This mile-long bridge, with a 2,800-foot main span, was the third largest suspension bridge at the time of opening.
- Construction began in Nov. 1938 and took about 19 months to build at a cost of $6,400,000.
- On July 1, 1940, the bridge opened to traffic.
- On Nov. 7, 1940 at approximately 11:00, the bridge collapsed during a moderate (42 miles/hour) wind storm.
- The bridge was supposed to withstand winds of up to 120 miles/hour.
- The collapse was due to wind-induced vibrations and an *unstable mechanical system*.
- Repair of the bridge was not possible.
- Fortunately, a dog trapped in an abandoned car was the only fatality.

Image of bridge collapse omitted for copyright reasons.

A video of the bridge collapse can be found at
https://youtu.be/j-zczJXSxnw.

Part 2

Preliminaries

Section 2.1

Functions, Sequences, System Operators, and Transforms

Sets

- A **rational number** is a number of the form x/y, where x and y are integers and $y \neq 0$ (i.e., a ratio of integers).
- For example, $-\frac{5}{3}$, $\frac{17}{11}$, and $0 = \frac{0}{1}$ are rational numbers, whereas π and e are irrational numbers (i.e., not rational).
- The symbols employed to denote several commonly-used sets are as follows:

Symbol	Set
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{Q}$	rational numbers

- For two integers a and b, we define the following notation for sets of consecutive integers:

$$[a\mathinner{\ldotp\ldotp}b] = \{x \in \mathbb{Z} : a \le x \le b\},$$
$$[a\mathinner{\ldotp\ldotp}b) = \{x \in \mathbb{Z} : a \le x < b\},$$
$$(a\mathinner{\ldotp\ldotp}b] = \{x \in \mathbb{Z} : a < x \le b\}, \quad \text{and}$$
$$(a\mathinner{\ldotp\ldotp}b) = \{x \in \mathbb{Z} : a < x < b\}.$$

- In this notation, a and b indicate the endpoints of the range for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicate whether each endpoint is included in the set.

- For example:
  - $[0\mathinner{\ldotp\ldotp}4]$ denotes the set of integers $\{0, 1, 2, 3, 4\}$;
  - $[0\mathinner{\ldotp\ldotp}4)$ denotes the set of integers $\{0, 1, 2, 3\}$; and
  - $[0\mathinner{\ldotp\ldotp}N-1]$ and $[0\mathinner{\ldotp\ldotp}N)$ both denote the set of integers $\{0, 1, 2, \ldots, N-1\}$.

Notation for Intervals on the Real Line

- For two real numbers a and b, we define the following notation for intervals on the real line:
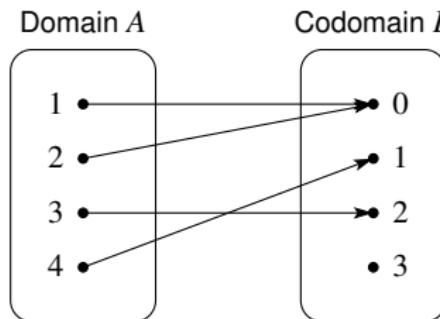
$$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$
$$(a,b) = \{x \in \mathbb{R} : a < x < b\},$$
$$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}, \quad \text{and}$$
$$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

- In this notation, a and b indicate the endpoints of the interval for the set, and the type of brackets used (i.e., parenthesis versus square bracket) indicate whether each endpoint is included in the set.

- For example:
  - $[0, 100]$ denotes the set of all real numbers from 0 to 100, including both 0 and 100;
  - $(-\pi, \pi]$ denotes the set of all real numbers from $-\pi$ to π, excluding $-\pi$ but including π; and
  - $[-\pi, \pi)$ denotes the set of all real numbers from $-\pi$ to π, including $-\pi$ but excluding π.

Mappings

- A **mapping** is a relationship involving two sets that associates each element in one set, called the **domain**, with an element from the other set, called the **codomain**.

- The notation $f : A \to B$ denotes a mapping f whose domain is the set A and whose codomain is the set B.

- Example:

$$f : A \to B$$
$$A = \{1, 2, 3, 4\}$$
$$B = \{0, 1, 2, 3\}$$
$$f(x) = \begin{cases} 0 & x \in \{1, 2\} \\ 1 & x = 4 \\ 2 & x = 3. \end{cases}$$

- Although many types of mappings exist, the types of most relevance to our study of signals and systems are: functions, sequences, system operators, and transforms.

Functions

- A **function** is a mapping where the domain is a set that is *continuous* in nature, such as the real numbers or complex numbers.
- In practice, the codomain is typically either the real numbers or complex numbers.
- Functions are also commonly referred to as **continuous-time (CT) signals**.
- Example:
  - Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = t^2$ (i.e., f is the squaring function).
  - The function f maps each real number t to the real number $f(t) = t^2$.
  - The domain and codomain are the real numbers.
  - Note that f is a *function*, whereas $f(t)$ is a *number* (namely, the value of the function f evaluated at t).
- Herein, we will focus almost exclusively on functions of a single independent variable (i.e., one-dimensional functions).

Sequences

- A **sequence** is a mapping where the domain is a set that is *discrete* in nature, such as the integers, or a subset thereof.
- In practice, the codomain is typically either the real numbers or complex numbers.
- Sequences are also commonly referred to as **discrete-time (DT) signals**.
- Example:
  - Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(n) = n^2$, where $\mathbb{Z}^+$ denotes the set of (strictly) positive integers (i.e., f is the sequence of perfect squares).
  - The sequence f maps each (strictly) positive integer n to the (strictly) positive integer $f(n) = n^2$.
  - The domain and codomain are $\mathbb{Z}^+$ (i.e., the positive integers).
  - Note that f is a *sequence*, whereas $f(n)$ is a *number* (namely, the value of the sequence f evaluated at n).
- As a matter of notation, the nth element of a sequence x is denoted as either $x(n)$ or x_n.
- Herein, we will focus almost exclusively on sequences with a single independent variable (i.e., one-dimensional sequences).

SKIP SLIDE

- For a real-valued function f of a real variable and an arbitrary real number t, the expression f denotes the function f itself and the expression $f(t)$ denotes the value of the function f evaluated at t.

- That is, f is a *function* and $f(t)$ is a *number*.

- Unfortunately, the practice of using $f(t)$ to denote the function f is quite common, although strictly speaking this is an abuse of notation.

- In contexts where imprecise notation may lead to problems, one should be careful to clearly distinguish between a function and its value.

- For the real-valued functions f and g of a real variable and an arbitrary real number t:
    □ The expression $f + g$ denotes a *function*, namely, the function formed by adding the functions f and g.
    □ The expression $f(t) + g(t)$ denotes a *number*, namely, the sum of: 1) the value of the function f evaluated at t; and 2) the value of the function g evaluated at t.

- Similar comments as the ones made above for functions also hold in the case of sequences.

- To express that two functions f and g are equal, we can write either:
  1. $f = g$; or
  2. $f(t) = g(t)$ for all t.

- Of the preceding two expressions, the first (i.e., $f = g$) is usually preferable, as it is less verbose.

- For the functions f and g and an operation $\circ$ that is defined pointwise for functions (such as addition, subtraction, multiplication, and division), the following relationship holds:

$$(f \circ g)(t) = f(t) \circ g(t).$$

- Some operations $\circ$ involving functions (such as convolution, to be discussed later) cannot be defined in a pointwise manner, in which case $(f \circ g)(t)$ is a valid mathematical expression, while $f(t) \circ g(t)$ is not.

- Again, similar comments as the ones made above for functions also hold in the case of sequences.

System Operators

- A **system operator** is a mapping used to represent a system.

- We will focus exclusively on the case of single-input single-output systems.

- A (single-input single-output) **system operator** maps a function or sequence representing the input of a system to a function or sequence representing the output of the system.

- The domain and codomain of a system operator are sets of *functions or sequences*, not sets of numbers.

- Example:
  - Let $\mathcal{H} : F \to F$ such that $\mathcal{H}x(t) = 2x(t)$ (for all $t \in \mathbb{R}$) and F is the set of functions mapping $\mathbb{R}$ to $\mathbb{R}$.
  - The system $\mathcal{H}$ maps a function to a function.
  - In particular, the domain and codomain are each F, which is a set of functions.
  - The system $\mathcal{H}$ multiplies its input function x by a factor of 2 in order to produce its output function $\mathcal{H}x$.
  - Note that $\mathcal{H}x$ is a function, not a number.

Remarks on Operator Notation for CT Systems

- For a system operator $\mathcal{H}$ and a function x, $\mathcal{H}x$ is the function produced as the output of the system $\mathcal{H}$ when the input is the function x.

- Brackets around the operand of an operator are ***often omitted when not required*** for grouping.

- For example, for an operator $\mathcal{H}$, a function x, and a real number t, we would normally prefer to write:
    1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
    2. $\mathcal{H}x(t)$ instead of the equivalent expression $\mathcal{H}(x)(t)$.

- Also, note that $\mathcal{H}x$ is a ***function*** and $\mathcal{H}x(t)$ is a ***number*** (namely, the value of the function $\mathcal{H}x$ evaluated at t).

- In the expression $\mathcal{H}(x_1 + x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1 + x_2) \not\equiv \mathcal{H}x_1 + x_2$ (where "$\not\equiv$" means "not equivalent").

- When multiple operators are applied, they group from ***right to left***.

- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the function x, the expression $\mathcal{H}_2\mathcal{H}_1x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

- For a system operator $\mathcal{H}$ and a sequence x, $\mathcal{H}x$ is the sequence produced as the output of the system $\mathcal{H}$ when the input is the sequence x.

- Brackets around the operand of an operator are *often omitted when not required* for grouping.

- For example, for an operator $\mathcal{H}$, a sequence x, and an integer n, we would normally prefer to write:
    1. $\mathcal{H}x$ instead of the equivalent expression $\mathcal{H}(x)$; and
    2. $\mathcal{H}x(n)$ instead of the equivalent expression $\mathcal{H}(x)(n)$.

- Also, note that $\mathcal{H}x$ is a *sequence* and $\mathcal{H}x(n)$ is a *number* (namely, the value of the sequence $\mathcal{H}x$ evaluated at n).

- In the expression $\mathcal{H}(x_1 + x_2)$, the brackets are needed for grouping, since $\mathcal{H}(x_1 + x_2) \not\equiv \mathcal{H}x_1 + x_2$ (where "$\not\equiv$" means "not equivalent").

- When multiple operators are applied, they group from *right to left*.

- For example, for the operators $\mathcal{H}_1$ and $\mathcal{H}_2$, and the sequence x, the expression $\mathcal{H}_2\mathcal{H}_1x$ means $\mathcal{H}_2[\mathcal{H}_1(x)]$.

- Later, we will be introduced to several types of mappings known as transforms.
- Transforms have a mathematical structure similar to system operators.
- That is, transforms map functions/sequences to functions/sequences.
- Due to this similar structure, many of the earlier comments about system operators also apply to the case of transforms.
- For example, the Fourier transform (introduced later) is denoted as $\mathcal{F}$ and the result of applying the Fourier transform operator to the function/sequence x is denoted as $\mathcal{F}x$.
- Some examples of transforms of interest in the study of signals and systems are listed on the next slide.

Examples of Transforms

Name	Domain	Codomain
CT Fourier Series	T-periodic functions (with domain $\mathbb{R}$)	sequences (with domain $\mathbb{Z}$)
CT Fourier Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{R}$)
Laplace Transform	functions (with domain $\mathbb{R}$)	functions (with domain $\mathbb{C}$)
DT Fourier Series	N-periodic sequences (with domain $\mathbb{Z}$)	N-periodic sequences (with domain $\mathbb{Z}$)
DT Fourier Transform	sequences (with domain $\mathbb{Z}$)	2π-periodic functions (with domain $\mathbb{R}$)
Z Transform	sequences (with domain $\mathbb{Z}$)	functions (with domain $\mathbb{C}$)

Section 2.2

Properties of Signals

Even Symmetry
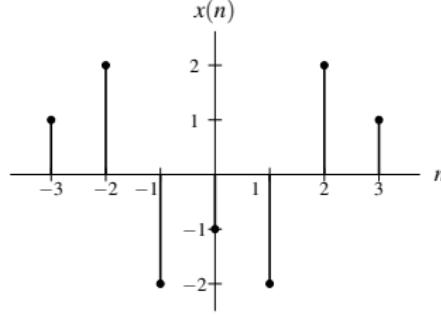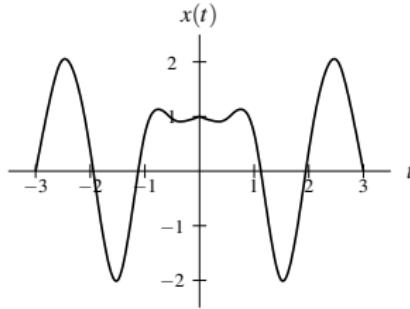
- A function x is said to be **even** if it satisfies

$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- Geometrically, the graph of an even signal is *symmetric* with respect to the vertical axis.

- Some examples of even signals are shown below.

Odd Symmetry
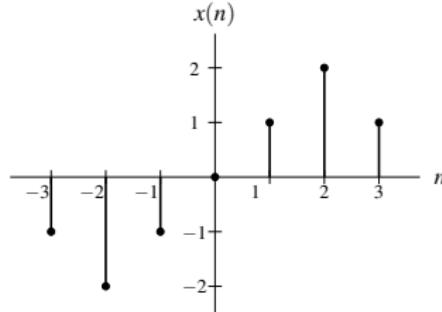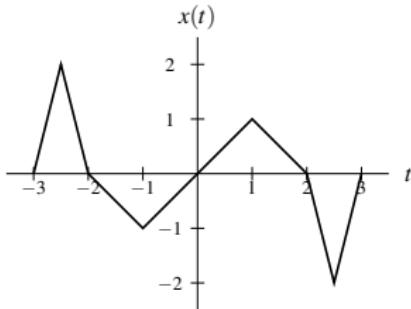
- A function x is said to be **odd** if it satisfies

$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- An odd signal x must be such that $x(0) = 0$.

- Geometrically, the graph of an odd signal is *symmetric* with respect to the origin.

- Some examples of odd signals are shown below.

Conjugate Symmetry

- A function x is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- The real part of a conjugate symmetric function or sequence is even.
- The imaginary part of a conjugate symmetric function or sequence is odd.
- An example of a conjugate symmetric function is a complex sinusoid $x(t) = \cos \omega t + j \sin \omega t$, where ω is a real constant.
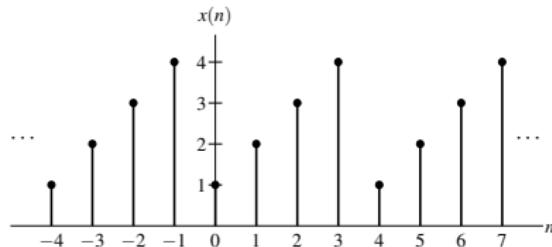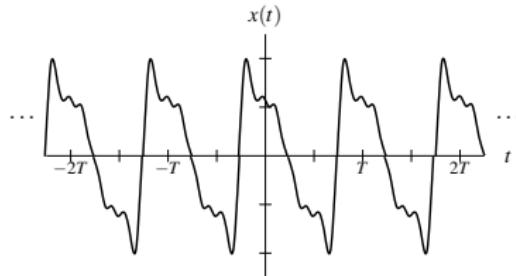
- A function x is said to be **periodic** with **period T** (or T**-periodic**) if, for some strictly-positive real constant T, the following condition holds:

$$x(t) = x(t + T) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **periodic** with **period N** (or N**-periodic**) if, for some strictly-positive integer constant N, the following condition holds:

$$x(n) = x(n + N) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$
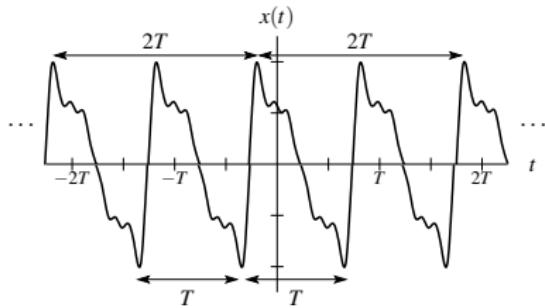
- Some examples of periodic signals are shown below.

- A function/sequence that is not periodic is said to be **aperiodic**.
- A T-periodic function x is said to have **frequency** $\frac{1}{T}$ and **angular frequency** $\frac{2\pi}{T}$.
- An N-periodic sequence x is said to have **frequency** $\frac{1}{N}$ and **angular frequency** $\frac{2\pi}{N}$.

- The period of a periodic signal is *not unique*. That is, a signal that is periodic with period T is also periodic with period kT, for every (strictly) positive integer k.

- The smallest period with which a signal is periodic is called the **fundamental period** and its corresponding frequency is called the **fundamental frequency**.

Part 3

Continuous-Time (CT) Signals and Systems

Section 3.1

Independent- and Dependent-Variable Transformations

Time Shifting (Translation)
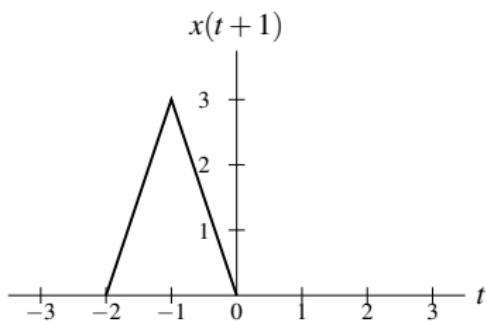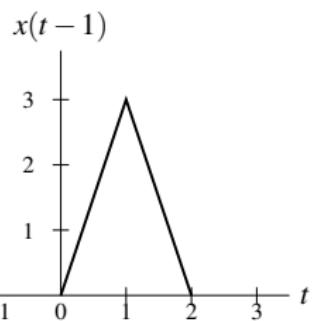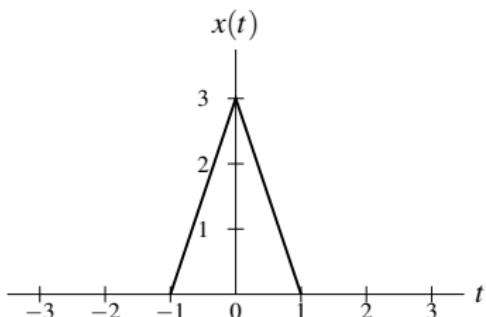
- **Time shifting** (also called **translation**) maps the input function x to the output function y as given by

$$y(t) = x(t - b),$$

where b is a real number.

- Such a transformation shifts the function (to the left or right) along the time axis.

- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).

- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).
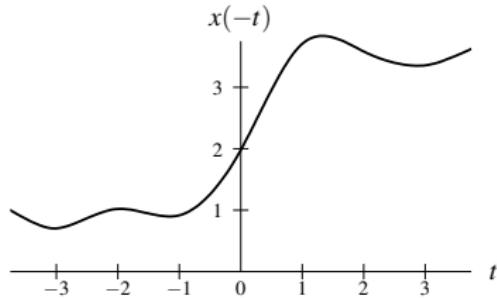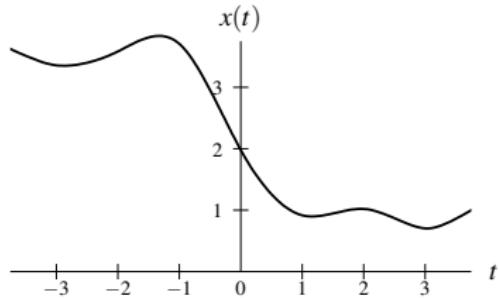
Time Reversal (Reflection)

■ **Time reversal** (also known as **reflection**) maps the input function x to the output function y as given by

$$y(t) = x(-t).$$

■ Geometrically, the output function y is a reflection of the input function x about the (vertical) line $t = 0$.

Time Compression/Expansion (Dilation)
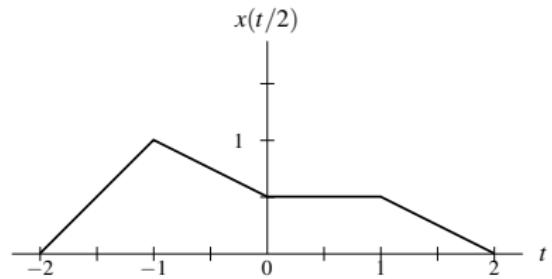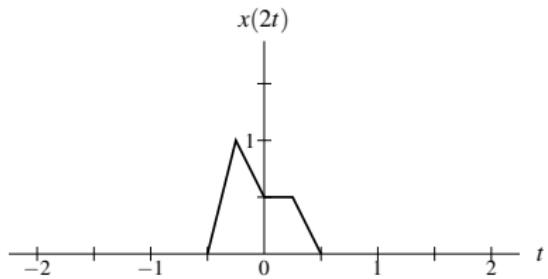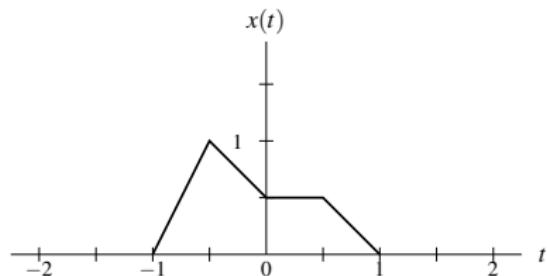
- **Time compression/expansion** (also called **dilation**) maps the input function x to the output function y as given by

$$y(t) = x(at),$$

  where a is a *strictly positive* real number.

- Such a transformation is associated with a compression/expansion along the time axis.

- If $a > 1$, y is *compressed* along the horizontal axis by a factor of a, relative to x.

- If $a < 1$, y is *expanded* (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

- **Time scaling** maps the input function x to the output function y as given by

$$y(t) = x(at),$$

where a is a *nonzero* real number.

- Such a transformation is associated with a dilation (i.e., compression/expansion along the time axis) and/or time reversal.

- If $|a| > 1$, the function is *compressed* along the time axis by a factor of $|a|$.

- If $|a| < 1$, the function is *expanded* (i.e., stretched) along the time axis by a factor of $\left|\frac{1}{a}\right|$.

- If $|a| = 1$, the function is neither expanded nor compressed.

- If $a < 0$, the function is also time reversed.

- Dilation (i.e., expansion/compression) and time reversal *commute*.

- Time reversal is a special case of time scaling with $a = -1$; and time compression/expansion is a special case of time scaling with $a > 0$.

Combined Time Scaling and Time Shifting

- Consider a transformation that maps the input function x to the output function y as given by

$$y(t) = x(at - b),$$
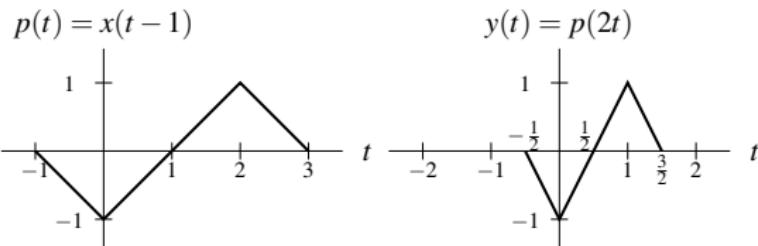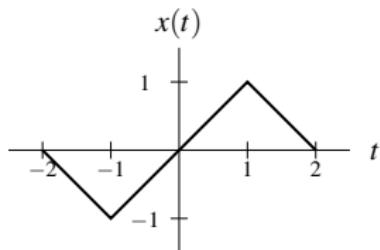
where a and b are real numbers and $a \neq 0$.

- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.

- Since time scaling and time shifting *do not commute*, we must be particularly careful about the order in which these transformations are applied.

- The above transformation has two distinct but equivalent interpretations:
    1. first, time shifting x by b, and then time scaling the result by a;
    2. first, time scaling x by a, and then time shifting the result by b/a.

- Note that the time shift is not by the same amount in both cases.

- In particular, note that when time scaling is applied first followed by time shifting, the time shift is by b/a, not b.

Exercise 3.3

time shift by 1 and then time scale by 2

$p(t) = x(t-1)$

$y(t) = p(2t)$

Given x as shown below, find $y(t) = x(2t-1)$.

$x(t)$

time scale by 2 and then time shift by $\frac{1}{2}$

$q(t) = x(2t)$

$y(t) = q(t - 1/2)$

Two Perspectives on Independent-Variable Transformations

- A transformation of the independent variable can be viewed in terms of
    1. the effect that the transformation has on the *function*; or
    2. the effect that the transformation has on the *horizontal axis*.

- This distinction is important because such a transformation has *opposite* effects on the function and horizontal axis.

- For example, the (time-shifting) transformation that replaces t by $t - b$ (where b is a real number) in $x(t)$ can be viewed as a transformation that
    1. shifts the function x *right* by b units; or
    2. shifts the horizontal axis *left* by b units.

- In our treatment of independent-variable transformations, we are only interested in the effect that a transformation has on the *function*.

- If one is not careful to consider that we are interested in the function perspective (as opposed to the axis perspective), many aspects of independent-variable transformations will not make sense.
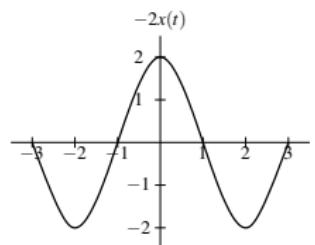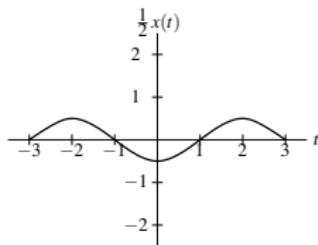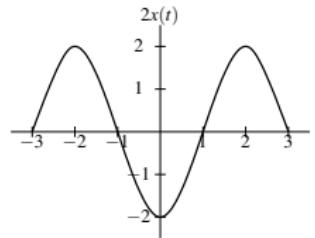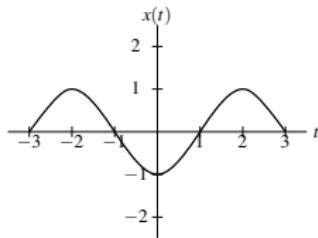
- **Amplitude scaling** maps the input function x to the output function y as given by

$$y(t) = ax(t),$$

  where a is a real number.

- Geometrically, the output function y is *expanded/compressed* in amplitude and/or *reflected* about the horizontal axis.
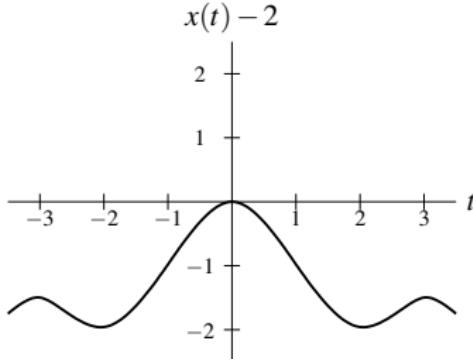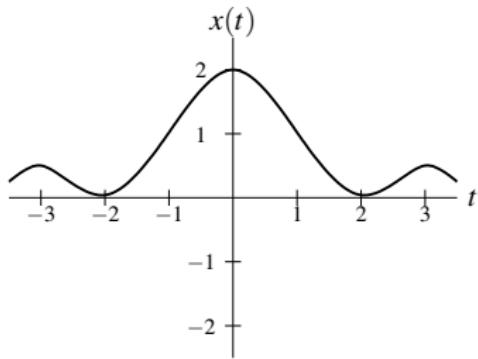
- **Amplitude shifting** maps the input function x to the output function y as given by

$$y(t) = x(t) + b,$$

where b is a real number.

- Geometrically, amplitude shifting adds a *vertical displacement* to x.

Combined Amplitude Scaling and Amplitude Shifting

- We can also combine amplitude scaling and amplitude shifting transformations.
- Consider a transformation that maps the input function x to the output function y, as given by

$$y(t) = ax(t) + b,$$

where a and b are real numbers.

- Equivalently, the above transformation can be expressed as

$$y(t) = a\left[x(t) + \tfrac{b}{a}\right].$$

- The above transformation is equivalent to:
  1. first amplitude scaling x by a, and then amplitude shifting the resulting function by b; or
  2. first amplitude shifting x by b/a, and then amplitude scaling the resulting function by a.

Section 3.2

Properties of Functions

Symmetry and Addition/Multiplication

- Sums involving even and odd functions have the following properties:
  - □ The sum of two even functions is even.
  - □ The sum of two odd functions is odd.
  - □ The sum of an even function and odd function is neither even nor odd, provided that neither of the functions is identically zero.

- That is, the *sum* of functions with the *same type of symmetry* also has the *same type of symmetry*.

- Products involving even and odd functions have the following properties:
  - □ The product of two even functions is even.
  - □ The product of two odd functions is even.
  - □ The product of an even function and an odd function is odd.

- That is, the *product* of functions with the *same type of symmetry* is *even*, while the *product* of functions with *opposite types of symmetry* is *odd*.

- Every function x has a ***unique*** representation of the form

$$x(t) = x_{\mathsf{e}}(t) + x_{\mathsf{o}}(t),$$

  where the functions x_{e} and x_{o} are ***even*** and ***odd***, respectively.

- In particular, the functions x_{e} and x_{o} are given by

$$x_{\mathsf{e}}(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_{\mathsf{o}}(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right].$$

- The functions x_{e} and x_{o} are called the **even part** and **odd part** of x, respectively.

- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

- **Sum of periodic functions.** For two periodic functions x_1 and x_2 with fundamental periods T_1 and T_2, respectively, and the sum $y = x_1 + x_2$:
    1. The sum y is periodic if and only if the ratio T_1/T_2 is a *rational number* (i.e., the quotient of two integers).
    2. If y is periodic, its fundamental period is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$), where $T_1/T_2 = q/r$ and q and r are integers and *coprime* (i.e., have no common factors). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)

- Although the above theorem only directly addresses the case of the sum of two functions, the case of N functions (where $N > 2$) can be handled by applying the theorem repeatedly $N - 1$ times.
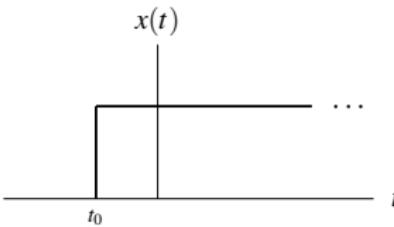
Right-Sided Functions

- A function x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

(i.e., x is *only potentially nonzero to the right of* t_0).

- An example of a right-sided function is shown below.

- A function x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

- A causal function is a *special case* of a right-sided function.
- A causal function is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.
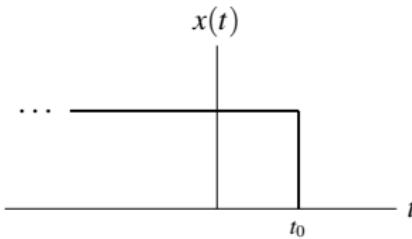
Left-Sided Functions

- A function x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e., x is *only potentially nonzero to the left of t_0*).

- An example of a left-sided function is shown below.
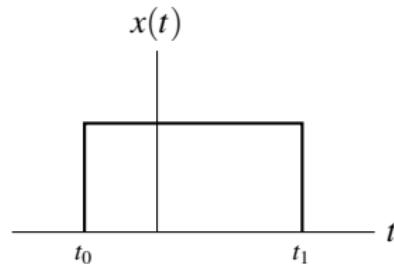
- Similarly, a function x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$
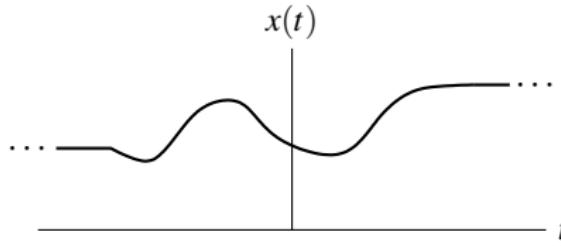
- An anticausal function is a *special case* of a left-sided function.
- An anticausal function is not to be confused with an anticausal system. In these two contexts, the word "anticausal" has very different meanings.

Finite-Duration and Two-Sided Functions

- A function that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration function is shown below.

- A function that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided function is shown below.

Bounded Functions

- A function x is said to be **bounded** if there exists some (*finite*) positive real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e., $x(t)$ is *finite* for all t).

- For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \text{ for all } t \quad \text{and} \quad |\cos t| \leq 1 \text{ for all } t.$$

- In contrast, the tangent function and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t \to \pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t| \to \infty} |p(t)| = \infty.$$

- The **energy** E contained in the function x is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

- A signal with finite energy is said to be an **energy signal**.
- The **average power** P contained in the function x is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

- A signal with (nonzero) finite average power is said to be a **power signal**.

Section 3.3

Elementary Functions
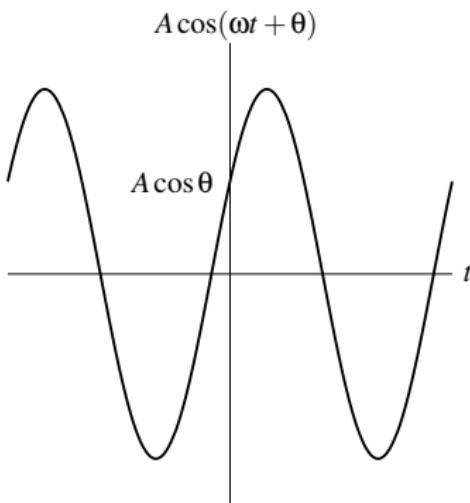
Real Sinusoidal Functions

- A **real sinusoidal function** is a function of the form

$$x(t) = A\cos(\omega t + \theta),$$

where A, ω, and θ are *real* constants.
- Such a function is periodic with *fundamental period* $T = \frac{2\pi}{|\omega|}$ and *fundamental frequency* $|\omega|$.
- A real sinusoid has a plot resembling that shown below.

- A **complex exponential function** is a function of the form
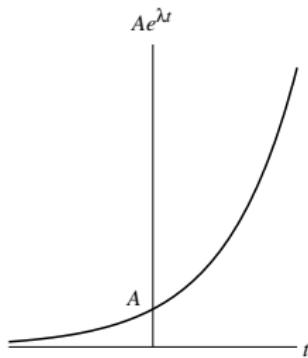
$$x(t) = Ae^{\lambda t},$$

  where A and λ are *complex* constants.
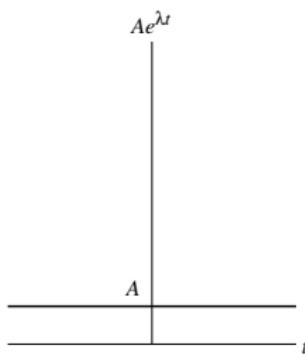
- A complex exponential can exhibit one of a number of *distinct modes of behavior*, depending on the values of its parameters A and λ.

- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

Real Exponential Functions
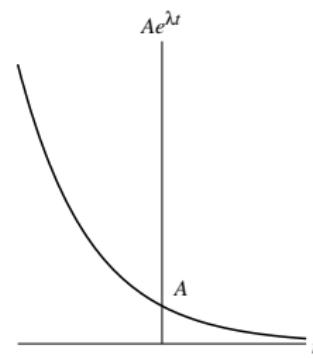
- A **real exponential function** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be *real* numbers.
- A real exponential can exhibit one of *three distinct modes* of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ *increases* exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ *decreases* exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the *constant A*.

$$\lambda > 0 \qquad\qquad \lambda = 0 \qquad\qquad \lambda < 0$$

Complex Sinusoidal Functions

- A complex sinusoidal function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is **complex** and λ is **purely imaginary** (i.e., $\text{Re}\{\lambda\} = 0$).

- That is, a **complex sinusoidal function** is a function of the form

$$x(t) = Ae^{j\omega t},$$

  where A is **complex** and ω is **real**.
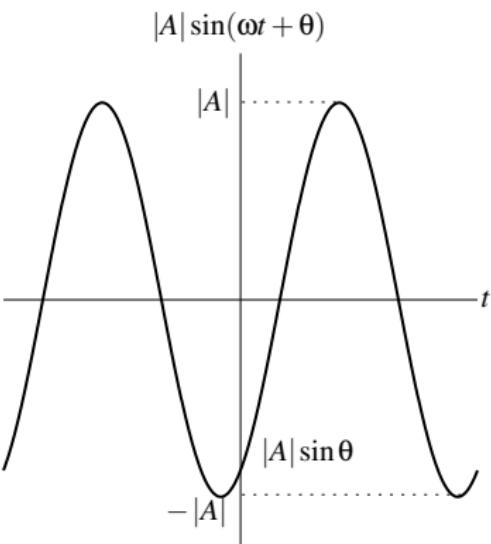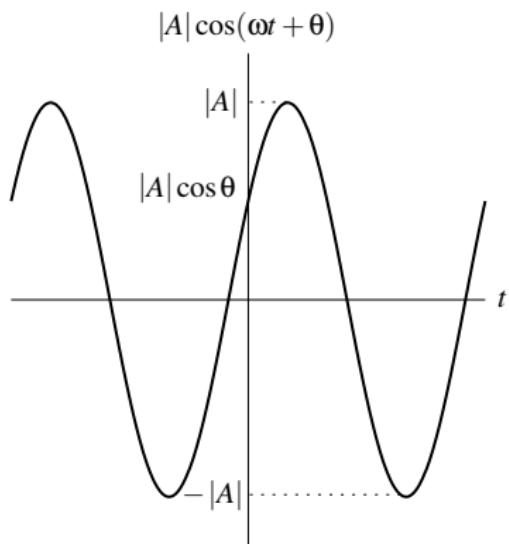
- By expressing A in polar form as $A = |A|\, e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A| \cos(\omega t + \theta)}_{\text{Re}\{x(t)\}} + j \underbrace{|A| \sin(\omega t + \theta)}_{\text{Im}\{x(t)\}}.$$
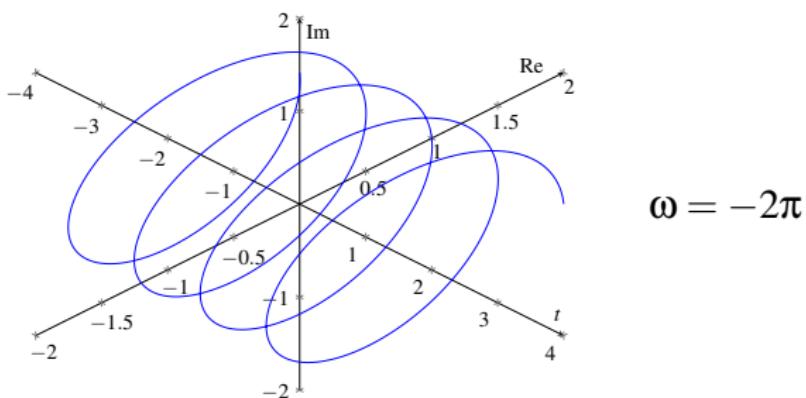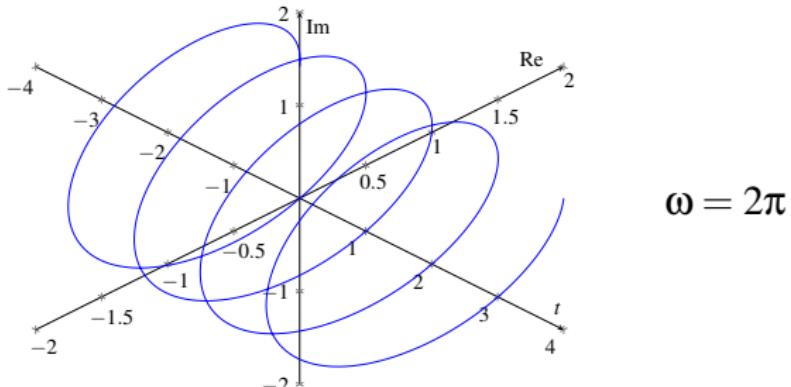
- Thus, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ are the same except for a time shift.

- Also, x is periodic with **fundamental period $T = \frac{2\pi}{|\omega|}$** and **fundamental frequency** $|\omega|$.

■ The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.

$\omega = 2\pi$

$\omega = -2\pi$

General Complex Exponential Functions
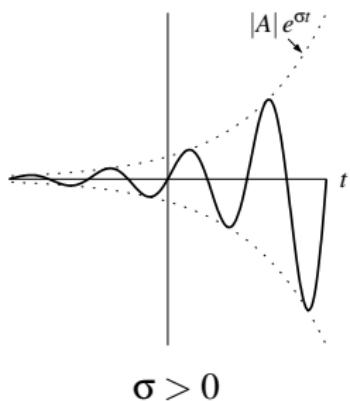
- In the most general case of a complex exponential function $x(t) = Ae^{\lambda t}$, A and λ are both *complex*.

- Letting $A = |A|\, e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A|\, e^{\sigma t} \cos(\omega t + \theta)}_{\text{Re}\{x(t)\}} + j \underbrace{|A|\, e^{\sigma t} \sin(\omega t + \theta)}_{\text{Im}\{x(t)\}}.$$
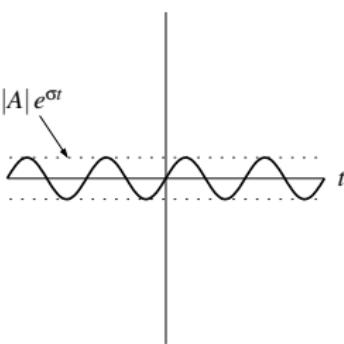
- Thus, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ are each the product of a real exponential and real sinusoid.

- One of *three distinct modes* of behavior is exhibited by $x(t)$, depending on the value of σ.

- If $\sigma = 0$, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ are *real sinusoids*.

- If $\sigma > 0$, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ are each the *product of a real sinusoid and a growing real exponential*.

- If $\sigma < 0$, $\text{Re}\{x\}$ and $\text{Im}\{x\}$ are each the *product of a real sinusoid and a decaying real exponential*.

- The *three modes of behavior* for $\text{Re}\{x\}$ and $\text{Im}\{x\}$ are illustrated below.

$\sigma > 0$ $\sigma = 0$ $\sigma < 0$

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t).$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

$$A\cos(\omega t + \theta) = \frac{A}{2}\left[e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}\right] \quad \text{and}$$

$$A\sin(\omega t + \theta) = \frac{A}{2j}\left[e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}\right].$$
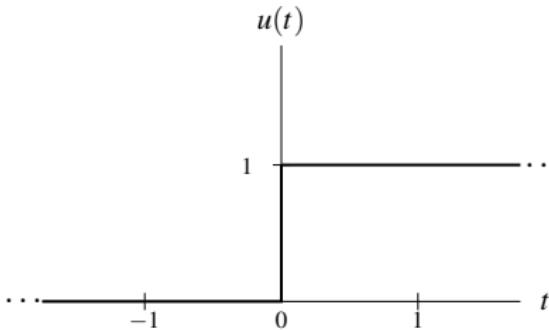
- Note that, above, we are simply **_restating results_** from the (appendix) material on complex analysis.

Unit-Step Function

- The **unit-step function** (also known as the **Heaviside function**), denoted u, is defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which u is used in practice, the actual ***value of*** $u(0)$ is unimportant. Sometimes values of 0 and $\frac{1}{2}$ are also used for $u(0)$.

- A plot of this function is shown below.
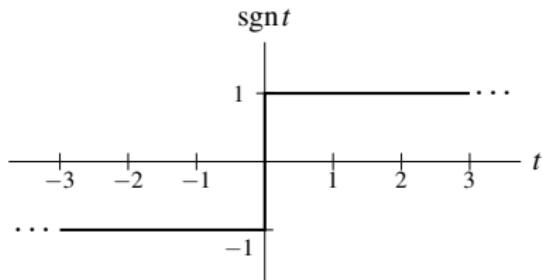
Signum Function

- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

- From its definition, one can see that the signum function simply computes the *sign* of a number.
- A plot of this function is shown below.
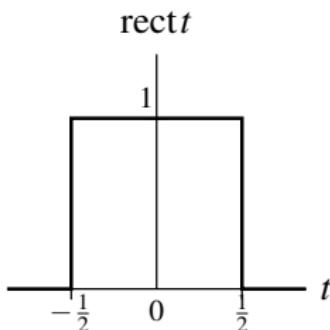
Rectangular Function

- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\text{rect}\, t = \begin{cases} 1 & -\frac{1}{2} \le t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual **value of** $\text{rect}\, t$ **at** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.

- A plot of this function is shown below.

$\text{rect}\, t$
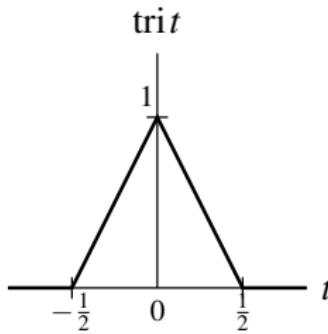
- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\operatorname{tri} t = \begin{cases} 1 - 2\left|t\right| & \left|t\right| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this function is shown below.
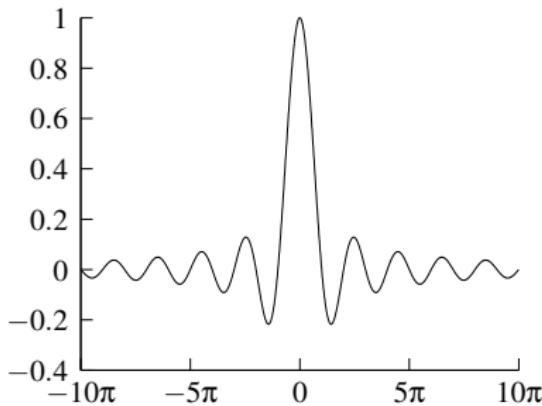
Cardinal Sine Function

■ The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc} t = \frac{\sin t}{t}.$$

■ By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.

■ A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc} t$ do not die out for finite t.]

Floor and Ceiling Functions

- The **floor function**, denoted $\lfloor \cdot \rfloor$, is a function that maps a real number x to the largest integer not more than x.

- In other words, the floor function rounds a real number to the nearest integer in the direction of negative infinity.

- For example,

$$\left\lfloor -\tfrac{1}{2} \right\rfloor = -1, \quad \left\lfloor \tfrac{1}{2} \right\rfloor = 0, \quad \text{and} \quad \lfloor 1 \rfloor = 1.$$

- The **ceiling function**, denoted $\lceil \cdot \rceil$, is a function that maps a real number x to the smallest integer not less than x.

- In other words, the ceiling function rounds a real number to the nearest integer in the direction of positive infinity.

- For example,

$$\left\lceil -\tfrac{1}{2} \right\rceil = 0, \quad \left\lceil \tfrac{1}{2} \right\rceil = 1, \quad \text{and} \quad \lceil 1 \rceil = 1.$$

■ Several useful properties of the floor and ceiling functions include:

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z};$$

$$\lceil x+n \rceil = \lceil x \rceil + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z};$$

$$\lceil x \rceil = -\lfloor -x \rfloor \quad \text{for } x \in \mathbb{R};$$

$$\lfloor x \rfloor = -\lceil -x \rceil \quad \text{for } x \in \mathbb{R};$$

$$\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0; \quad \text{and}$$

$$\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil = \left\lceil \frac{m+1}{n} \right\rceil - 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0.$$
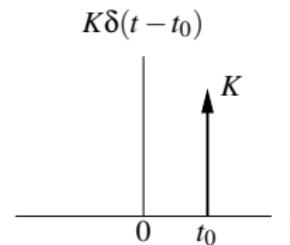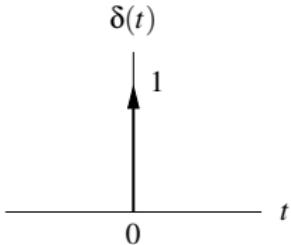
Unit-Impulse Function

- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a *generalized function*. Consequently, the δ function sometimes behaves in unusual ways.

- Graphically, the delta function is represented as shown below.
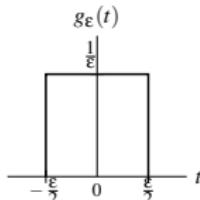
Unit-Impulse Function as a Limit

- Define

$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

- The function g_ε has a plot of the form shown below.

- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)dt = 1$.
- The function δ can be obtained as the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

- That is, δ can be viewed as a *limiting case of a rectangular pulse* where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.
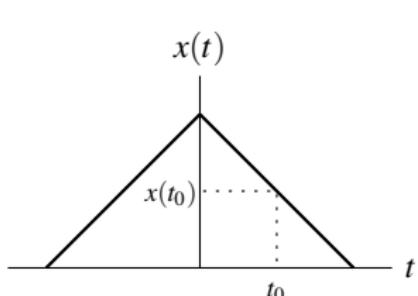
- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$
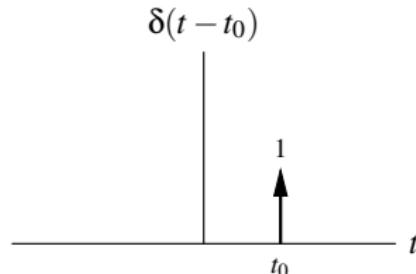
- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0).$$
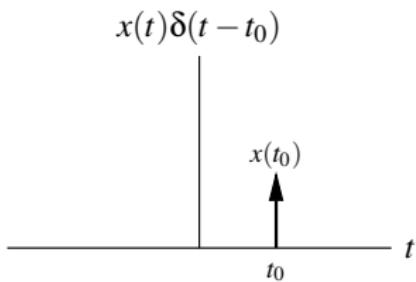
Function x

Time-Shifted Unit-Impulse Function

Product

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x is a *rectangular pulse* of height one, with a *rising edge at a* and *falling edge at b*).

- The function x can be equivalently written as

$$x(t) = u(t - a) - u(t - b)$$

(i.e., the difference of two time-shifted unit-step functions).

- Unlike the original expression for x, this latter expression for x *does not involve multiple cases*.

- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

- The idea from the previous slide can be extended to handle any function that is defined in a *piecewise manner* (i.e., via an expression involving multiple cases).
- That is, by using unit-step functions, we can always collapse a formula involving multiple cases into a single expression.
- Often, simplifying a formula in this way can be quite beneficial.

Section 3.4

Continuous-Time (CT) Systems

CT Systems

- A system with input x and output y can be described by the equation

$$y = \mathcal{H}x,$$

where $\mathcal{H}$ denotes an operator (i.e., transformation).

- Note that the operator $\mathcal{H}$ *maps a function to a function* (not a number to a number).

- Alternatively, we can express the above relationship using the notation

$$x \xrightarrow{\mathcal{H}} y.$$
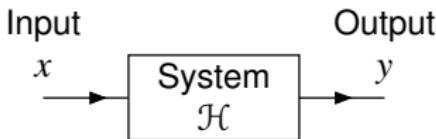
- If clear from the context, the operator $\mathcal{H}$ is often omitted, yielding the abbreviated notation

$$x \rightarrow y.$$

- Note that the symbols "$\rightarrow$" and "=" have *very different* meanings.
- The symbol "$\rightarrow$" should be read as "*produces*" (not as "equals").
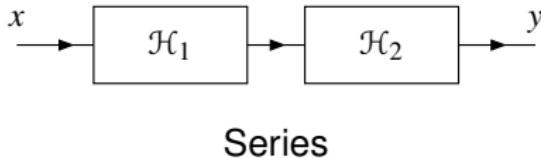
Block Diagram Representations

■ Often, a system defined by the operator $\mathcal{H}$ and having the input x and output y is represented in the form of a *block diagram* as shown below.

Input   Output

x → | System $\mathcal{H}$ | → y
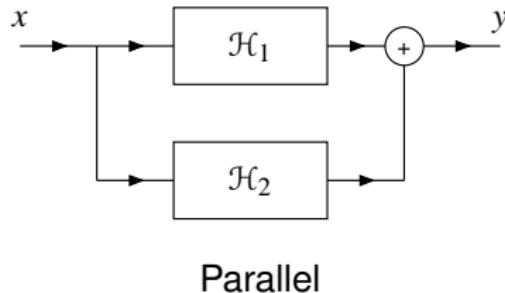
- *Two basic ways* in which systems can be interconnected are shown below.

Series

Parallel

- A **series** (or **cascade**) connection ties the output of one system to the input of the other.
- The overall series-connected system is described by the equation

$$y = \mathcal{H}_2 \mathcal{H}_1 x.$$

- A **parallel** connection ties the inputs of both systems together and sums their outputs.
- The overall parallel-connected system is described by the equation

$$y = \mathcal{H}_1 x + \mathcal{H}_2 x.$$
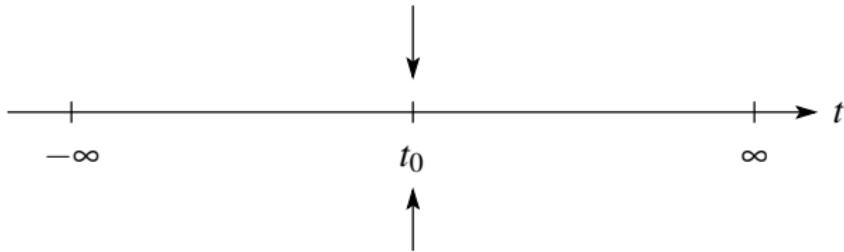
Section 3.5

Properties of (CT) Systems

- A system $\mathcal{H}$ is said to be **memoryless** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t \neq t_0$.
- In other words, a memoryless system is such that the value of its output at any given point in time can depend on the value of its input at only the *same* point in time.
- A system that is not memoryless is said to have **memory**.
- Although simple, a memoryless system is *not very flexible*, since its current output value cannot rely on past or future values of the input.

If the system $\mathcal{H}$ is memoryless,
the output $\mathcal{H}x$ at t_0
can depend on the input x
only at t_0.

$-\infty$ t_0 ∞ t
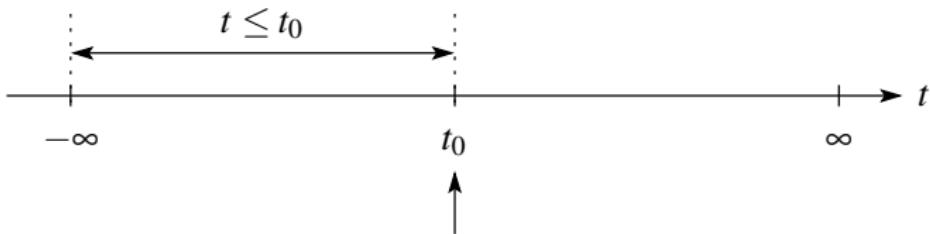
Consider the calculation of the
output $\mathcal{H}x$ at t_0.

- A system $\mathcal{H}$ is said to be **causal** if, for every real constant t_0, $\mathcal{H}x(t_0)$ does not depend on $x(t)$ for some $t > t_0$.
- In other words, a causal system is such that the value of its output at any given point in time can depend on the value of its input at only the *same or earlier points* in time (i.e., *not later points in time*).
- If the independent variable t represents time, a system must be causal in order to be *physically realizable*.
- Noncausal systems can sometimes be useful in practice, however, since the independent variable *need not always represent time* (e.g., the independent variable might represent position).
- A memoryless system is always causal, although the converse is not necessarily true.

If the system $\mathcal{H}$ is causal,
the output $\mathcal{H}x$ at t_0
can depend on the input x
only at points $t \leq t_0$.

Consider the calculation of the
output $\mathcal{H}x$ at t_0.

Invertibility
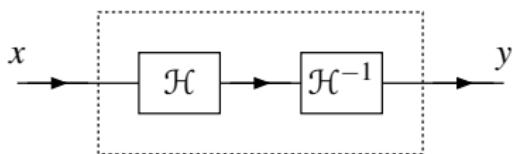
- The **inverse** of a system $\mathcal{H}$ (if it exists) is another system $\mathcal{H}^{-1}$ such that, for every function x,

$$\mathcal{H}^{-1}\mathcal{H}x = x$$
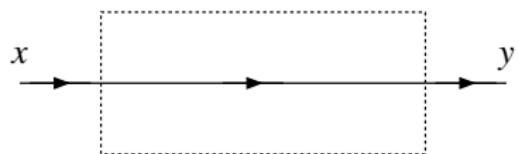
(i.e., the system formed by the cascade interconnection of $\mathcal{H}$ followed by $\mathcal{H}^{-1}$ is a system whose input and output are equal).

- A system is said to be **invertible** if it has a corresponding inverse system (i.e., its inverse exists).
- Equivalently, a system is invertible if its input can always be *uniquely* determined from its output.
- An invertible system will always produce *distinct outputs* from any two *distinct inputs* (i.e., $x_1 \neq x_2 \Rightarrow \mathcal{H}x_1 \neq \mathcal{H}x_2$).
- To show that a system is *invertible*, we simply find the *inverse system*.
- To show that a system is *not invertible*, we find *two distinct inputs* that result in *identical outputs* (i.e., $x_1 \neq x_2$ and $\mathcal{H}x_1 = \mathcal{H}x_2$).
- In practical terms, invertible systems are "nice" in the sense that their *effects can be undone*.

- A system $\mathcal{H}^{-1}$ being the inverse of $\mathcal{H}$ means that the following two systems are equivalent (i.e., $\mathcal{H}^{-1}\mathcal{H}$ *is an identity*):

System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$

System 2: $y = x$

Bounded-Input Bounded-Output (BIBO) Stability

- A system $\mathcal{H}$ is said to be **bounded-input bounded-output (BIBO) stable** if, for every bounded function x, $\mathcal{H}x$ is bounded (i.e., $|x(t)| < \infty$ for all t implies that $|\mathcal{H}x(t)| < \infty$ for all t).

- In other words, a BIBO stable system is such that it guarantees to always produce a bounded output as long as its input is bounded.

- To show that a system is *BIBO stable*, we must show that *every bounded input* leads to a *bounded output*.

- To show that a system is *not BIBO stable*, we only need to find a single *bounded input* that leads to an *unbounded output*.

- In practical terms, a BIBO stable system is *well behaved* in the sense that, as long as the system input is finite everywhere (in its domain), the output will also be finite everywhere.

- Usually, a system that is not BIBO stable will have *serious safety issues*.

- For example, a portable music player with a battery input of 3.7 volts and headset output of ∞ volts would result in one vaporized human (and likely a big lawsuit as well).

- A system $\mathcal{H}$ is said to be **time invariant (TI)** (or **shift invariant (SI)**) if, for every function x and every real constant t_0, the following condition holds:
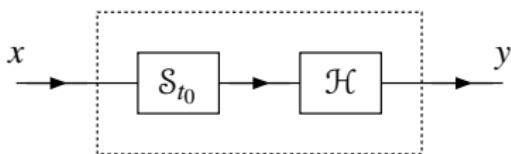
$$\mathcal{H}x(t - t_0) = \mathcal{H}x'(t) \text{ for all } t, \quad \text{where} \quad x'(t) = x(t - t_0)$$

(i.e., $\mathcal{H}$ *commutes with time shifts*).
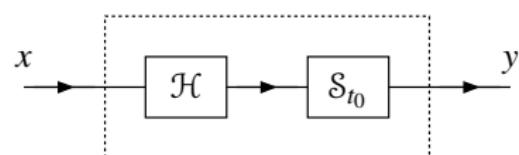
- In other words, a system is time invariant if a time shift (i.e., advance or delay) in the input always results only in an *identical time shift* in the output.
- A system that is not time invariant is said to be **time varying**.
- In simple terms, a time invariant system is a system whose behavior *does not change* with respect to time.
- Practically speaking, compared to time-varying systems, time-invariant systems are much *easier to design and analyze*, since their behavior does not change with respect to time.

- Let $\mathcal{S}_{t_0}$ denote an operator that applies a *time shift of t_0* to a function (i.e., $\mathcal{S}_{t_0}x(t) = x(t - t_0)$).
- A system $\mathcal{H}$ is *time invariant* if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ *commutes with* $\mathcal{S}_{t_0}$):

System 1: $y = \mathcal{H}\mathcal{S}_{t_0}x$
$$\begin{bmatrix} y(t) = \mathcal{H}x'(t) \\ x'(t) = \mathcal{S}_{t_0}x(t) = x(t - t_0) \end{bmatrix}$$

System 2: $y = \mathcal{S}_{t_0}\mathcal{H}x$
$$\begin{bmatrix} y(t) = \mathcal{H}x(t - t_0) \end{bmatrix}$$

Additivity, Homogeneity, and Linearity

- A system $\mathcal{H}$ is said to be **additive** if, for all functions x_1 and x_2, the following condition holds:

$$\mathcal{H}(x_1 + x_2) = \mathcal{H}x_1 + \mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with addition*).

- A system $\mathcal{H}$ is said to be **homogeneous** if, for every function x and every complex constant a, the following condition holds:

$$\mathcal{H}(ax) = a\mathcal{H}x$$

(i.e., $\mathcal{H}$ *commutes with scalar multiplication*).

- A system that is both additive and homogeneous is said to be **linear**.

- In other words, a system $\mathcal{H}$ is *linear*, if for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:
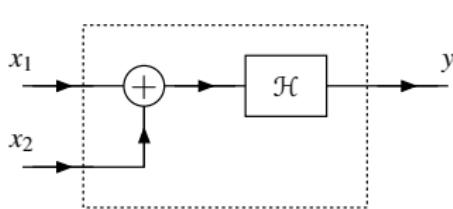
$$\mathcal{H}(a_1 x_1 + a_2 x_2) = a_1 \mathcal{H}x_1 + a_2 \mathcal{H}x_2$$

(i.e., $\mathcal{H}$ *commutes with linear combinations*).
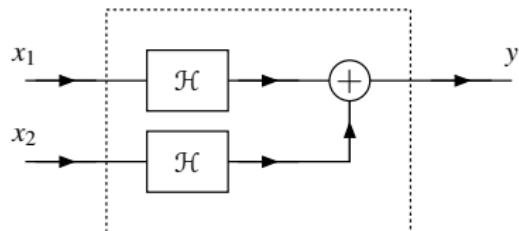
- The linearity property is also referred to as the **superposition** property.

- Practically speaking, linear systems are much *easier to design and analyze* than nonlinear systems.

■ The system $\mathcal{H}$ is *additive* if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ *commutes with addition*):
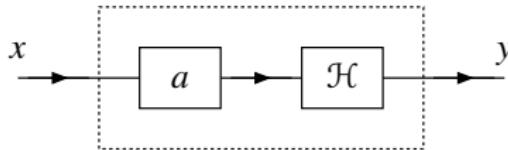
System 1: $y = \mathcal{H}(x_1 + x_2)$
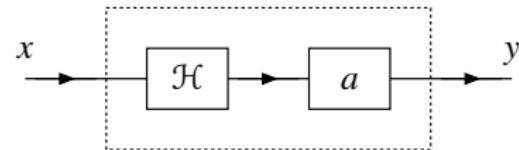
System 2: $y = \mathcal{H}x_1 + \mathcal{H}x_2$

■ The system $\mathcal{H}$ is *homogeneous* if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ *commutes with scalar multiplication*):
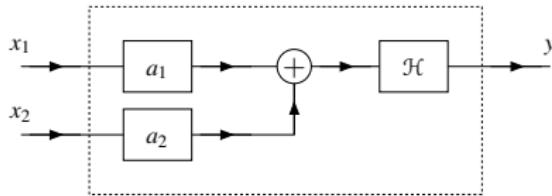
System 1: $y = \mathcal{H}(ax)$

System 2: $y = a\mathcal{H}x$

- The system $\mathcal{H}$ is **_linear_** if and only if the following two systems are equivalent (i.e., $\mathcal{H}$ **_commutes with linear combinations_**):

System 1: $y = \mathcal{H}(a_1 x_1 + a_2 x_2)$

System 2: $y = a_1 \mathcal{H} x_1 + a_2 \mathcal{H} x_2$

- A function x is said to be an **eigenfunction** of the system $\mathcal{H}$ with the **eigenvalue** λ if

$$\mathcal{H}x = \lambda x,$$

  where λ is a complex constant.

- In other words, the system $\mathcal{H}$ acts as an ideal amplifier for each of its eigenfunctions x, where the amplifier gain is given by the corresponding eigenvalue λ.

- Different systems have different eigenfunctions.

- Many of the mathematical tools developed for the study of CT systems have eigenfunctions as their basis.

Part 4

Continuous-Time Linear Time-Invariant (LTI) Systems

Why Linear Time-Invariant (LTI) Systems?

- In engineering, linear time-invariant (LTI) systems play a very important role.
- Very powerful mathematical tools have been developed for analyzing LTI systems.
- LTI systems are much easier to analyze than systems that are not LTI.
- In practice, systems that are not LTI can be well approximated using LTI models.
- So, even when dealing with systems that are not LTI, LTI systems still play an important role.

Section 4.1

Convolution

- The (CT) **convolution** of the functions x and h, denoted $x * h$, is defined as the function

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau.$$

- The convolution result $x * h$ evaluated at the point t is simply a weighted average of the function x, where the weighting is given by h time reversed and shifted by t.

- Herein, the asterisk symbol (i.e., "$*$") will always be used to denote convolution, not multiplication.

- As we shall see, convolution is used extensively in systems theory.

- In particular, convolution has a special significance in the context of LTI systems.

Practical Convolution Computation

- To compute the convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau,$$

we proceed as follows:

1. Plot $x(\tau)$ and $h(t - \tau)$ as a function of τ.
2. Initially, consider an arbitrarily large negative value for t. This will result in $h(t - \tau)$ being shifted very far to the left on the time axis.
3. Write the mathematical expression for $x * h(t)$.
4. Increase t gradually until the expression for $x * h(t)$ changes form. Record the interval over which the expression for $x * h(t)$ was valid.
5. Repeat steps 3 and 4 until t is an arbitrarily large positive value. This corresponds to $h(t - \tau)$ being shifted very far to the right on the time axis.
6. The results for the various intervals can be combined in order to obtain an expression for $x * h(t)$ for all t.

Properties of Convolution

- The convolution operation is *commutative*. That is, for any two functions x and h,

$$x * h = h * x.$$

- The convolution operation is *associative*. That is, for any functions x, h_1, and h_2,

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- The convolution operation is *distributive* with respect to addition. That is, for any functions x, h_1, and h_2,

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

- For any function x,

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t).$$

- Thus, any function x can be written in terms of an expression involving δ.
- Moreover, δ is the ***convolutional identity***. That is, for any function x,

$$x * \delta = x.$$

- The convolution of two periodic functions is usually not well defined.
- This motivates an alternative notion of convolution for periodic functions known as periodic convolution.
- The **periodic convolution** of the T-periodic functions x and h, denoted $x \circledast h$, is defined as

$$x \circledast h(t) = \int_T x(\tau)h(t-\tau)d\tau,$$

where $\int_T$ denotes integration over an interval of length T.

- The periodic convolution and (linear) convolution of the T-periodic functions x and h are related as follows:

$$x \circledast h(t) = x_0 * h(t) \quad \text{where} \quad x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT)$$

(i.e., $x_0(t)$ equals $x(t)$ over a single period of x and is zero elsewhere).

Section 4.2

Convolution and LTI Systems

Impulse Response

- The response h of a system $\mathcal{H}$ to the input δ is called the **impulse response** of the system (i.e., $h = \mathcal{H}\delta$).

- For any LTI system with input x, output y, and impulse response h, the following relationship holds:

$$y = x * h.$$

- In other words, a LTI system simply *computes a convolution*.

- Furthermore, a LTI system is *completely characterized* by its impulse response.

- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.

- Since the impulse response of a LTI system is an extremely useful quantity, we often want to determine this quantity in a practical setting.

- Unfortunately, in practice, the impulse response of a system cannot be determined directly from the definition of the impulse response.
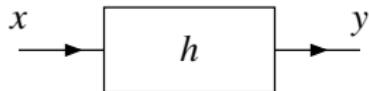
- The response s of a system $\mathcal{H}$ to the input u is called the **step response** of the system (i.e., $s = \mathcal{H}u$).
- The impulse response h and step response s of a LTI system are related as

$$h(t) = \frac{ds(t)}{dt}.$$

- Therefore, the impulse response of a system can be determined from its step response by differentiation.
- The step response provides a practical means for determining the impulse response of a system.
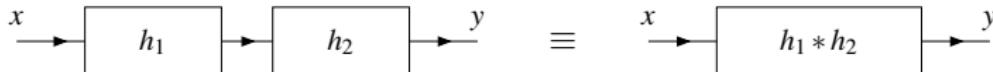
- Often, it is convenient to represent a (CT) LTI system in block diagram form.
- Since such systems are completely characterized by their impulse response, we often label a system with its impulse response.
- That is, we represent a system with input x, output y, and impulse response h, as shown below.
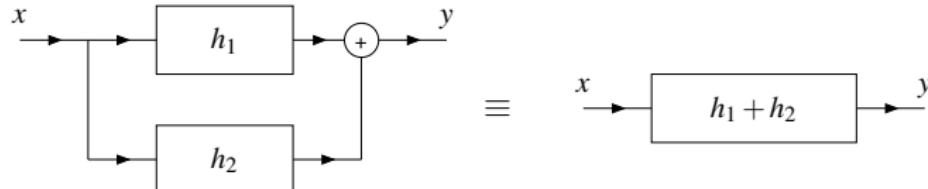
$$x \longrightarrow \boxed{h} \longrightarrow y$$

- The **_series_** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h_1 * h_2$. That is, we have the equivalence shown below.

- The **_parallel_** interconnection of the LTI systems with impulse responses h_1 and h_2 is the LTI system with impulse response $h_1 + h_2$. That is, we have the equivalence shown below.

Section 4.3

Properties of LTI Systems

Memory

- A LTI system with impulse response h is memoryless if and only if

$$h(t) = 0 \quad \text{for all } t \neq 0.$$

- That is, a LTI system is memoryless if and only if its impulse response h is of the form

$$h(t) = K\delta(t),$$

where K is a complex constant.

- Consequently, every memoryless LTI system with input x and output y is characterized by an equation of the form

$$y = x * (K\delta) = Kx$$

(i.e., the system is an ideal amplifier).

- For a LTI system, the memoryless constraint is extremely restrictive (as every memoryless LTI system is an ideal amplifier).

- A LTI system with impulse response h is causal if and only if

$$h(t) = 0 \quad \text{for all } t < 0$$

(i.e., h is a causal function).

- It is due to the above relationship that we call a function x, satisfying

$$x(t) = 0 \quad \text{for all } t < 0,$$

a causal function.

- The inverse of a LTI system, if such a system exists, is a LTI system.
- Let h and h_{inv} denote the impulse responses of a LTI system and its (LTI) inverse, respectively. Then,

$$h * h_{\text{inv}} = \delta.$$

- Consequently, a LTI system with impulse response h is invertible if and only if there exists a function h_{inv} such that

$$h * h_{\text{inv}} = \delta.$$

- Except in simple cases, the above condition is often quite difficult to test.

- A LTI system with impulse response h is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

  (i.e., h is *absolutely integrable*).

Eigenfunctions of LTI Systems

- As it turns out, every complex exponential is an eigenfunction of all LTI systems.

- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st},$$

where s is a complex constant and

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

- That is, e^{st} is an eigenfunction of a LTI system and $H(s)$ is the corresponding eigenvalue.

- We refer to H as the **system function** (or **transfer function**) of the system $\mathcal{H}$.

- From above, we can see that the response of a LTI system to a complex exponential is the same complex exponential multiplied by the complex factor $H(s)$.

Representations of Functions Using Eigenfunctions

- Consider a LTI system with input x, output y, and system function H.
- Suppose that the input x can be expressed as the linear combination of complex exponentials

$$x(t) = \sum_k a_k e^{s_k t},$$

where the a_k and s_k are complex constants.

- Using the fact that complex exponentials are eigenfunctions of LTI systems, we can conclude

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}.$$

- Thus, if an input to a LTI system can be expressed as a linear combination of complex exponentials, the output can also be expressed as a linear combination of the *same* complex exponentials.
- The above formula can be used to determine the output of a LTI system from its input in a way that does not require convolution.

Part 5

Continuous-Time Fourier Series (CTFS)

Introduction

- The (CT) Fourier series is a representation for *periodic* functions.
- With a Fourier series, a function is represented as a *linear combination of complex sinusoids*.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- For example, complex sinusoids are continuous and differentiable. They are also easy to integrate and differentiate.
- Perhaps, most importantly, complex sinusoids are *eigenfunctions* of LTI systems.

Section 5.1

Fourier Series

Harmonically-Related Complex Sinusoids

- A set of complex sinusoids is said to be **harmonically related** if there exists some constant ω_0 such that the fundamental frequency of each complex sinusoid is an integer multiple of ω_0.

- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{for all integer } k.$$

- The fundamental frequency of the kth complex sinusoid ϕ_k is $k\omega_0$, an integer multiple of ω_0.

- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of ω_0, a linear combination of these complex sinusoids must be periodic.

- More specifically, a linear combination of these complex sinusoids is periodic with period $T = \frac{2\pi}{\omega_0}$.

CT Fourier Series

- A periodic (complex-valued) function x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Such a representation is known as (the complex exponential form of) a (CT) **Fourier series**, and the c_k are called **Fourier series coefficients**.

- The above formula for x is often referred to as the **Fourier series synthesis equation**.

- The terms in the summation for $k = K$ and $k = -K$ are called the Kth **harmonic components**, and have the fundamental frequency $K\omega_0$.

- To denote that a function x has the Fourier series coefficient sequence c_k, we write

$$x(t) \xleftrightarrow{\text{CTFS}} c_k.$$

- The periodic function x with fundamental period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ has the Fourier series coefficients c_k given by

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

  where $\int_T$ denotes integration over an arbitrary interval of length T (i.e., one period of x).

- The above equation for c_k is often referred to as the **Fourier series analysis equation**.

Section 5.2

Convergence Properties of Fourier Series

Remarks on Equality of Functions

- The equality of functions can be defined in more than one way.
- Two functions x and y are said to be **equal in the pointwise sense** if $x(t) = y(t)$ for all t (i.e., x and y are equal at every point).
- Two functions x and y are said to be **equal in the mean-squared error (MSE) sense** if $\int |x(t) - y(t)|^2 dt = 0$ (i.e., the energy in $x - y$ is zero).
- Pointwise equality is a stronger condition than MSE equality (i.e., pointwise equality implies MSE equality but the converse is not true).
- Consider the functions

$$x_1(t) = 1 \text{ for all } t, \quad x_2(t) = 1 \text{ for all } t, \quad \text{and}$$

$$x_3(t) = \begin{cases} 2 & t = 0 \\ 1 & \text{otherwise.} \end{cases}$$

- The functions x_1 and x_2 are equal in both the pointwise sense and MSE sense.
- The functions x_1 and x_3 are equal in the MSE sense, but not in the pointwise sense.

Convergence of Fourier Series

- Since a Fourier series can have an infinite number of (nonzero) terms, and an infinite sum may or may not converge, we need to consider the issue of convergence.

- That is, when we claim that a periodic function x is equal to the Fourier series $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, is this claim actually correct?

- Consider a periodic function x that we wish to represent with the Fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

- Let x_N denote the Fourier series truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$

- Here, we are interested in whether $\lim_{N\to\infty} x_N$ is equal (in some sense) to x.

- Again, let x_N denote the Fourier series for the periodic function x truncated after the Nth harmonic components as given by

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t}.$$
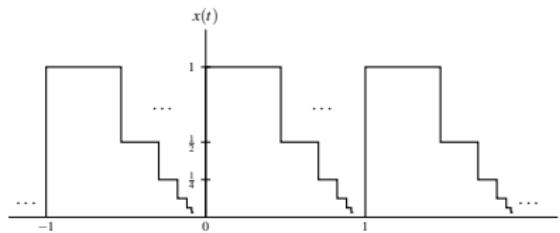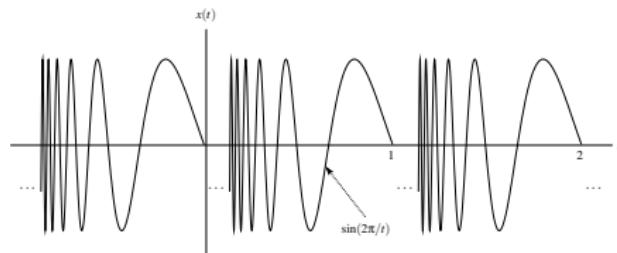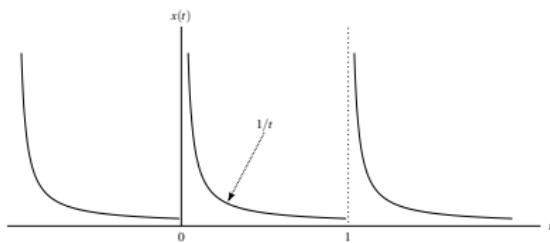
- If $\lim_{N \to \infty} x_N(t) = x(t)$ for all t (i.e., $\lim_{N \to \infty} x_N$ is equal to x in the pointwise sense), the Fourier series is said to converge **pointwise** to x.
- If convergence is pointwise and the rate of convergence is the same everywhere, the convergence is said to be **uniform**.
- If $\lim_{N \to \infty} \frac{1}{T} \int_T |x_N(t) - x(t)|^2 \, dt = 0$ (i.e., $\lim_{N \to \infty} x_N$ is equal to x in the MSE sense), the Fourier series is said to converge to x in the **MSE** sense.
- Pointwise convergence is a stronger condition than MSE convergence (i.e., pointwise convergence implies MSE convergence, but the converse is not true).

- If a periodic function x is ***continuous*** and its Fourier series coefficients c_k are ***absolutely summable*** (i.e., $\sum_{k=-\infty}^{\infty} |c_k| < \infty$), then the Fourier series representation of x converges ***uniformly*** (i.e., pointwise at the same rate everywhere).

- Since, in practice, we often encounter functions with discontinuities (e.g., a square wave), the above result is of somewhat limited value.

Convergence of Fourier Series: Finite-Energy Case

- If a periodic function x has *finite energy* in a single period (i.e., $\int_T |x(t)|^2 dt < \infty$), the Fourier series converges in the *MSE* sense.

- Since, in situations of practice interest, the finite-energy condition in the above theorem is typically satisfied, the theorem is usually applicable.

- It is important to note, however, that MSE convergence (i.e., $E = 0$) does not necessarily imply pointwise convergence (i.e., $\tilde{x}(t) = x(t)$ for all t).

- Thus, the above convergence theorem does not provide much useful information regarding the value of $\tilde{x}(t)$ for specific values of t.

- Consequently, the above theorem is typically most useful for simply determining if the Fourier series converges.

Dirichlet Conditions

- The **Dirichlet conditions** for the periodic function x are as follows:
    1. over a single period, x is ***absolutely integrable*** (i.e., $\int_T |x(t)|\, dt < \infty$);
    2. over a single period, x has a finite number of maxima and minima (i.e., x is of ***bounded variation***); and
    3. over any finite interval, x has a ***finite number of discontinuities***, each of which is ***finite***.

- Examples of functions violating the Dirichlet conditions are shown below.

- If a periodic function x satisfies the ***Dirichlet conditions***, then:

  1. the Fourier series converges pointwise everywhere to x, except at the points of discontinuity of x; and

  2. at each point t_a of discontinuity of x, the Fourier series $\tilde{x}$ converges to
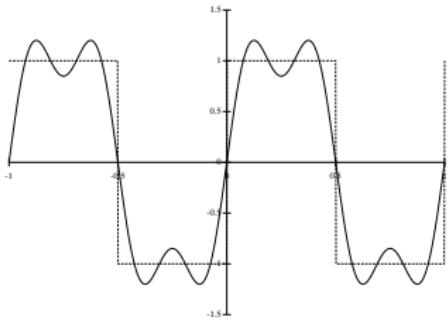
  $$\tilde{x}(t_a) = \tfrac{1}{2} \left[x(t_a^-) + x(t_a^+) \right],$$

  where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.
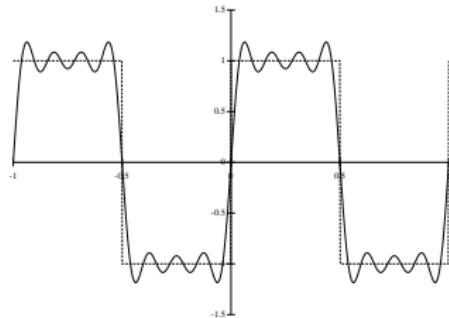
- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier series at every point, this result is often very useful in practice.

Gibbs Phenomenon

- In practice, we frequently encounter functions with discontinuities.
- When a function x has discontinuities, the Fourier series representation of x does not converge uniformly (i.e., at the same rate everywhere).
- The rate of convergence is much slower at points in the vicinity of a discontinuity.
- Furthermore, in the vicinity of a discontinuity, the truncated Fourier series x_N exhibits ripples, where the peak amplitude of the ripples does not seem to decrease with increasing N.
- As it turns out, as N increases, the ripples get compressed towards discontinuity, but, for any finite N, the peak amplitude of the ripples remains approximately constant.
- This behavior is known as **Gibbs phenomenon**.
- The above behavior is one of the weaknesses of Fourier series (i.e., Fourier series converge very slowly near discontinuities).
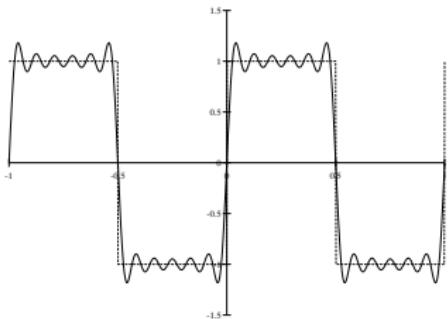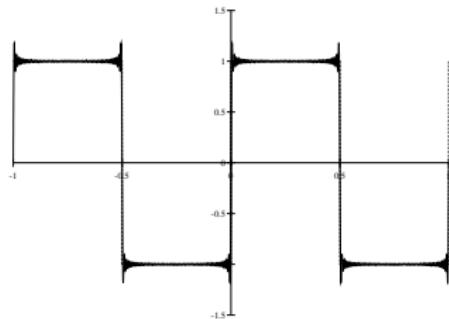
Fourier series truncated after the
3rd harmonic components

Fourier series truncated after the
7th harmonic components

Fourier series truncated after the
11th harmonic components

Fourier series truncated after the
101st harmonic components

Section 5.3

Properties of Fourier Series

Properties of (CT) Fourier Series

$$x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a_{-k}^*
Periodic Convolution	$x \circledast y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Property	
Parseval's Relation	$\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

Linearity

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(t) + \beta y(t) \overset{\text{CTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

where α and β are complex constants.

- That is, a linear combination of functions produces the same linear combination of their Fourier series coefficients.

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(t - t_0) \overset{\text{CTFS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

where t_0 is a real constant.

- In other words, time shifting a periodic function changes the argument (but not magnitude) of its Fourier series coefficients.

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$e^{jM(2\pi/T)t}x(t) = e^{jM\omega_0 t}x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_{k-M},$$

where M is an integer constant.

- In other words, multiplying a periodic function by $e^{jM\omega_0 t}$ shifts the Fourier-series coefficient sequence.

- Let x denote a periodic function with period T and the corresponding frequency $\omega_0 = 2\pi/T$. If $x(t) \stackrel{\text{CTFS}}{\longleftrightarrow} c_k$, then

$$x(-t) \stackrel{\text{CTFS}}{\longleftrightarrow} c_{-k}.$$

- That is, time reversal of a function results in a time reversal of its Fourier series coefficients.

- For a T-periodic function x with Fourier series coefficient sequence c, the following property holds:

$$x^*(t) \overset{\text{CTFS}}{\longleftrightarrow} c^*_{-k}$$

- In other words, conjugating a function has the effect of time reversing and conjugating the Fourier series coefficient sequence.

- Let x and y be two periodic functions with the same period T. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x \circledast y(t) \overset{\text{CTFS}}{\longleftrightarrow} T a_k b_k.$$

- In other words, periodic convolution of two functions corresponds to the multiplication (up to a scale factor) of their Fourier-series coefficient sequences.

- Let x and y be two periodic functions with the same period. If $x(t) \overset{\text{CTFS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{CTFS}}{\longleftrightarrow} b_k$, then

$$x(t)y(t) \overset{\text{CTFS}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

- As we shall see later, the above summation is the DT convolution of a and b.

- In other words, the multiplication of two periodic functions corresponds to the DT convolution of their corresponding Fourier-series coefficient sequences.

- A function x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

- The above relationship is simply stating that the amount of energy in x (i.e., $\frac{1}{T} \int_T |x(t)|^2 \, dt$) and the amount of energy in the Fourier series coefficient sequence a (i.e., $\sum_{k=-\infty}^{\infty} |a_k|^2$) are equal.
- In other words, the transformation between a function and its Fourier series coefficient sequence preserves energy.

Even and Odd Symmetry

■ For a periodic function x with Fourier series coefficient sequence c, the following properties hold:

$$x \text{ is even} \Leftrightarrow c \text{ is even;} \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow c \text{ is odd.}$$

■ In other words, the even/odd symmetry properties of x and c always match.

- A function x is *real* if and only if its Fourier series coefficient sequence c satisfies

$$c_k = c_{-k}^* \text{ for all } k$$

(i.e., c is *conjugate symmetric*).

- Thus, for a real-valued function, the negative-indexed Fourier series coefficients are *redundant*, as they are completely determined by the nonnegative-indexed coefficients.

- From properties of complex numbers, one can show that $c_k = c_{-k}^*$ is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}$$

(i.e., $|c_k|$ is *even* and $\arg c_k$ is *odd*).

- Note that x being real does *not* necessarily imply that c is real.

Trigonometric Forms of a Fourier Series

- Consider the periodic function x with the Fourier series coefficients c_k.
- If x is real, then its Fourier series can be rewritten in two other forms, known as the combined trigonometric and trigonometric forms.
- The **combined trigonometric form** of a Fourier series has the appearance

$$x(t) = c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(k\omega_0 t + \theta_k),$$

where $\theta_k = \arg c_k$.

- The **trigonometric form** of a Fourier series has the appearance

$$x(t) = c_0 + \sum_{k=1}^{\infty} \left[\alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t) \right],$$

where $\alpha_k = 2\operatorname{Re} c_k$ and $\beta_k = -2\operatorname{Im} c_k$.

- Note that the trigonometric forms contain only *real* quantities.

Other Properties of Fourier Series

- For a T-periodic function x with Fourier-series coefficient sequence c, the following properties hold:

  1. c_0 is the average value of x over a single period T;
  2. x is real and even $\Leftrightarrow c$ is real and even; and
  3. x is real and odd $\Leftrightarrow c$ is purely imaginary and odd.

Section 5.4

Fourier Series and Frequency Spectra

A New Perspective on Functions: The Frequency Domain
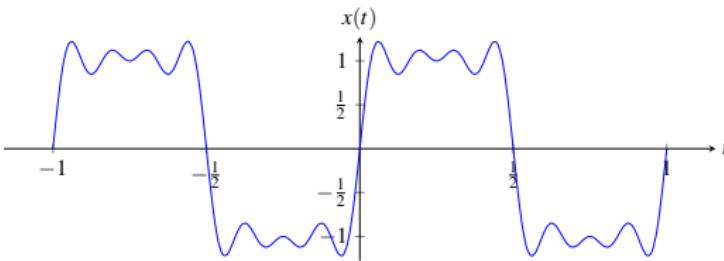
- The Fourier series provides us with an entirely new way to view functions.
- Instead of viewing a function as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a function as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved *much more easily* using the frequency domain than the time domain.
- The Fourier series coefficients of a function x provide a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a function over different frequencies is referred to as the *frequency spectrum* of the function.

Motivating Example

- Consider the real 1-periodic function x having the Fourier series representation
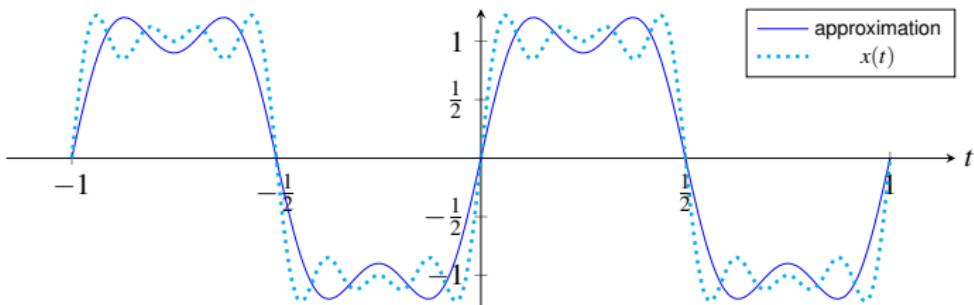
$$x(t) = -\tfrac{j}{10}e^{-j14\pi t} - \tfrac{2j}{10}e^{-j10\pi t} - \tfrac{4j}{10}e^{-j6\pi t} - \tfrac{13j}{10}e^{-j2\pi t}$$
$$+ \tfrac{13j}{10}e^{j2\pi t} + \tfrac{4j}{10}e^{j6\pi t} + \tfrac{2j}{10}e^{j10\pi t} + \tfrac{j}{10}e^{j14\pi t}.$$

- A plot of x is shown below.
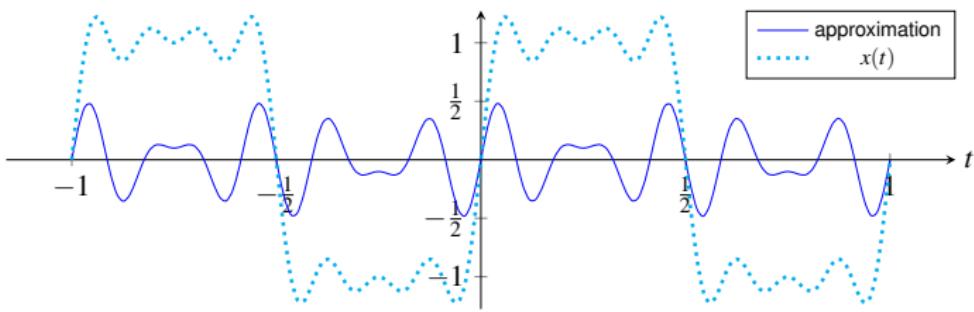
- The terms that make the most dominant contribution to the overall sum are the ones with the largest magnitude coefficients.
- To illustrate this, we consider the problem of determining the best approximation of x that keeps only 4 of the 8 terms in the Fourier series.

Approximation using the 4 terms with the
largest magnitude coefficients

Approximation using the 4 terms with the
smallest magnitude nonzero coefficients

- To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the function x, it is helpful to write the Fourier series with the c_k expressed in *polar form* as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}.$$

- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has been *amplitude scaled* by a factor of $|c_k|$ and *time shifted* by an amount that depends on $\arg c_k$.

- For a given k, the *larger* $|c_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the *larger the contribution* the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation.

- In this way, we can use $|c_k|$ as a *measure* of how much information a function x has at the frequency $k\omega_0$.

- The Fourier series coefficients c_k are referred to as the **frequency spectrum** of x.

- The magnitudes $|c_k|$ of the Fourier series coefficients are referred to as the **magnitude spectrum** of x.

- The arguments $\arg c_k$ of the Fourier series coefficients are referred to as the **phase spectrum** of x.

- Normally, the spectrum of a function is plotted against frequency $k\omega_0$ instead of k.

- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is *discrete* in the independent variable (i.e., frequency).

- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as **line spectra**.

Frequency Spectra of Real Functions

- Recall that, for a real function x, the Fourier series coefficient sequence c satisfies

$$c_k = c_{-k}^*$$

(i.e., c is *conjugate symmetric*), which is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}.$$

- Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a real function is always *even*.
- Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a real function is always *odd*.
- Due to the symmetry in the frequency spectra of real functions, we typically *ignore negative frequencies* when dealing with such functions.
- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

Section 5.5

Fourier Series and LTI Systems

Frequency Response

- Recall that a LTI system $\mathcal{H}$ with impulse response h is such that $\mathcal{H}\{e^{st}\}(t) = H_{\mathsf{L}}(s)e^{st}$, where $H_{\mathsf{L}}(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$. (That is, complex exponentials are *eigenfunctions* of LTI systems.)

- Since a complex sinusoid is a *special case* of a complex exponential, we can reuse the above result for the special case of complex sinusoids.

- For a LTI system $\mathcal{H}$ with impulse response h,

$$\mathcal{H}\{e^{j\omega t}\}(t) = H(\omega)e^{j\omega t},$$

where ω is a real constant and

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

- That is, $e^{j\omega t}$ is an *eigenfunction* of a LTI system and $H(\omega)$ is the corresponding *eigenvalue*.

- We refer to H as the **frequency response** of the system $\mathcal{H}$.

- Consider a LTI system with input x, output y, and frequency response H.
- Suppose that the T-periodic input x is expressed as the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = \tfrac{2\pi}{T}.$$

- Using our knowledge about the ***eigenfunctions*** of LTI systems, we can conclude

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}.$$

- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$ then $y(t) \overset{\text{CTFS}}{\longleftrightarrow} H(k\omega_0)c_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that ***does not require convolution***.

- In many applications, we want to *modify the spectrum* of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.
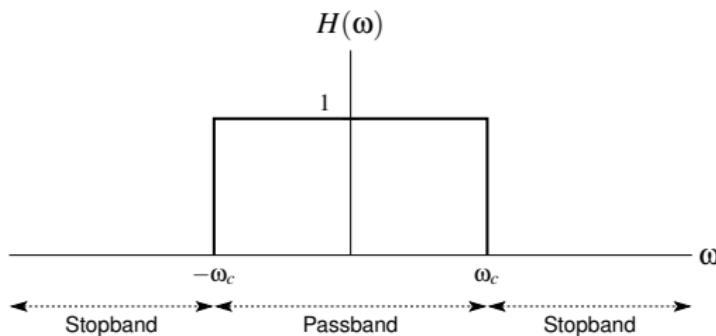
Ideal Lowpass Filter

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.

- Such a filter has a *frequency response* of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.
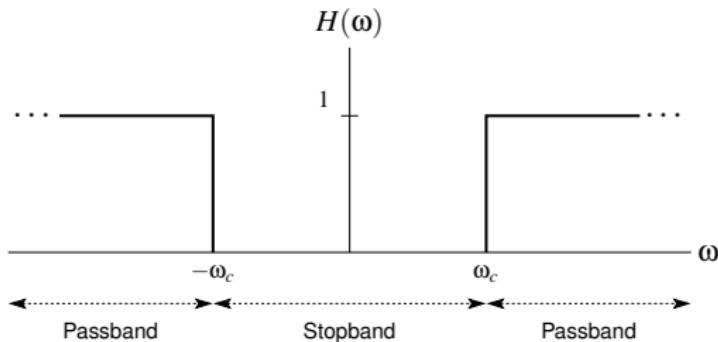
- A plot of this frequency response is given below.

Ideal Bandpass Filter

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.

- Such a filter has a *frequency response* of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise}, \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}.

- A plot of this frequency response is given below.

Part 6

Continuous-Time Fourier Transform (CTFT)

- The (CT) Fourier series provide an extremely useful representation for periodic functions.
- Often, however, we need to deal with functions that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The (CT) Fourier transform can be used to represent both periodic and aperiodic functions.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 6.1

Fourier Transform

Development of the Fourier Transform [Aperiodic Case]

- The Fourier series is an extremely useful function representation.
- Unfortunately, this function representation can only be used for periodic functions, since a Fourier series is inherently periodic.
- Many functions are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic functions.
- By viewing an aperiodic function as the limiting case of a periodic function with period T where $T \to \infty$, we can use the Fourier series to develop a function representation that can be used for aperiodic functions, known as the Fourier transform.

- Recall that the Fourier series representation of a T-periodic function x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk(2\pi/T)t} dt \right)}_{c_k} e^{jk(2\pi/T)t}.$$

- In the above representation, if we take the limit as $T \to \infty$, we obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right)}_{X(\omega)} e^{j\omega t} d\omega$$

(i.e., as $T \to \infty$, the outer summation becomes an integral, $\frac{1}{T} = \frac{\omega_0}{2\pi}$ becomes $\frac{1}{2\pi} d\omega$, and $k(2\pi/T) = k\omega_0$ becomes ω).

- This representation for aperiodic functions is known as the Fourier transform representation.

Generalized Fourier Transform

- The classical Fourier transform for aperiodic functions does not exist (i.e., $\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ fails to converge) for some functions of great practical interest, such as:
  - a nonzero constant function;
  - a periodic function (e.g., a real or complex sinusoid);
  - the unit-step function (i.e., u); and
  - the signum function (i.e., sgn).

- Fortunately, the Fourier transform can be extended to handle such functions, resulting in what is known as the **generalized Fourier transform**.

- For our purposes, we can think of the classical and generalized Fourier transforms as being defined by the same formulas.

- Therefore, in what follows, we will not typically make a distinction between the classical and generalized Fourier transforms.

- The (CT) **Fourier transform** of the function x, denoted $\mathcal{F}x$ or X, is given by

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).

- The **inverse Fourier transform** of X, denoted $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).

- As a matter of notation, to denote that a function x has the Fourier transform X, we write $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$.

- A function x and its Fourier transform X constitute what is called a **Fourier transform pair**.

- For a function x, the Fourier transform of x is denoted using operator notation as $\mathcal{F}x$.
- The Fourier transform of x evaluated at ω is denoted $\mathcal{F}x(\omega)$.
- Note that $\mathcal{F}x$ is a function, whereas $\mathcal{F}x(\omega)$ is a number.
- Similarly, for a function X, the inverse Fourier transform of X is denoted using operator notation as $\mathcal{F}^{-1}X$.
- The inverse Fourier transform of X evaluated at t is denoted $\mathcal{F}^{-1}X(t)$.
- Note that $\mathcal{F}^{-1}X$ is a function, whereas $\mathcal{F}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

- Often, we would like to write an expression for the Fourier transform of a function without explicitly naming the function.

- For example, consider writing an expression for the Fourier transform of the function $v(t) = x(5t - 3)$ but without using the name "v".

- It would be incorrect to write "$\mathcal{F}x(5t - 3)$" as this is the function $\mathcal{F}x$ evaluated at $5t - 3$, which is not the meaning that we wish to convey.

- Also, strictly speaking, it would be incorrect to write "$\mathcal{F}\{x(5t - 3)\}$" as the operand of the Fourier transform operator must be a function, and $x(5t - 3)$ is a number (i.e., the function x evaluated at $5t - 3$).

- Using dot notation, we can write the following strictly-correct expression for the desired Fourier transform: $\mathcal{F}x(5 \cdot -3)$.

- In many cases, however, it is probably advisable to avoid employing anonymous (i.e., unnamed) functions, as their use tends to be more error prone in some contexts.

- Since dot notation is less frequently used by engineers, the author has elected to minimize its use herein.
- To avoid ambiguous notation, the following conventions are followed:
    1. in the expression for the operand of a Fourier transform operator, the *independent variable is assumed to be the variable named "t"* unless otherwise indicated (i.e., in terms of dot notation, each "*t*" is treated as if it were a "·")
    2. in the expression for the operand of the inverse Fourier transform operator, the *independent variable is assumed to be the variable named "ω"* unless otherwise indicated (i.e., in terms of dot notation, each "ω" is treated as if it were a "·").
- For example, with these conventions:
    - "$\mathcal{F}\{\cos(t-\tau)\}$" denotes the function that is the Fourier transform of the function $v(t) = \cos(t-\tau)$ (not the Fourier transform of the function $v(\tau) = \cos(t-\tau)$).
    - "$\mathcal{F}^{-1}\{\delta(3\omega-\lambda)\}$" denotes the function that is the inverse Fourier transform of the function $V(\omega) = \delta(3\omega-\lambda)$ (not the inverse Fourier transform of the function $V(\lambda) = \delta(3\omega-\lambda)$).

Section 6.2

Convergence Properties of the Fourier Transform

Convergence of the Fourier Transform

- Consider an arbitrary function x.
- The function x has the Fourier transform representation $\tilde{x}$ given by

$$\tilde{x}(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\tilde{x}$ is a valid representation of x.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

- If a function x is *continuous* and *absolutely integrable* (i.e., $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |X(\omega)| \, d\omega < \infty$), then the Fourier transform representation of x converges *pointwise* (i.e., $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] e^{j\omega t} d\omega$ for all t).

- Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value.
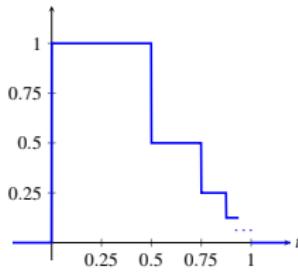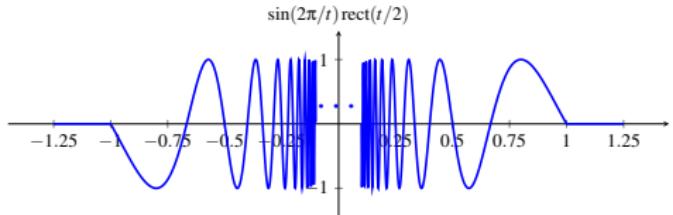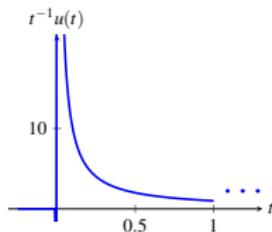
- If a function x is of *finite energy* (i.e., $\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$), then its Fourier transform representation converges in the *MSE sense*.

- In other words, if x is of finite energy, then the energy E in the difference function $\tilde{x} - x$ is zero; that is,

$$E = \int_{-\infty}^{\infty} |\tilde{x}(t) - x(t)|^2\, dt = 0.$$

- Since, in situations of practical interest, the finite-energy condition in the above theorem is often satisfied, the theorem is frequently applicable.

- It is important to note, however, that the condition $E = 0$ does not necessarily imply $\tilde{x}(t) = x(t)$ for all t.

- Thus, the above convergence result does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.

- Consequently, the above theorem is typically most useful for simply determining if the Fourier transform representation converges.

Dirichlet Conditions

- The **Dirichlet conditions** for the function x are as follows:
  1. the function x is *absolutely integrable* (i.e., $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$);
  2. on any finite interval, x has a finite number of maxima and minima (i.e., x is of *bounded variation*); and
  3. on any finite interval, x has a *finite number of discontinuities* and each discontinuity is itself *finite*.

- Examples of functions violating the Dirichlet conditions are shown below.

- If a function x satisfies the ***Dirichlet conditions***, then:
  1. the Fourier transform representation $\tilde{x}$ converges pointwise everywhere to x, except at the points of discontinuity of x; and
  2. at each point t_a of discontinuity of x, the Fourier transform representation $\tilde{x}$ converges to

  $$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

  where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.

- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier transform representation at every point, this result is often very useful in practice.

Section 6.3

Properties of the Fourier Transform

Properties of the (CT) Fourier Transform

Property	Time Domain	Frequency Domain
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
Time-Domain Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{|a|} X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega) X_2(\omega)$
Time-Domain Multiplication	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt} x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$t x(t)$	$j \frac{d}{d\omega} X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

(CT) Fourier Transform Pairs

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
6	$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
7	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
8	$\operatorname{rect}(t/T)$	$|T|\operatorname{sinc}(T\omega/2)$
9	$\frac{|B|}{\pi}\operatorname{sinc}(Bt)$	$\operatorname{rect}\left(\frac{\omega}{2B}\right)$
10	$e^{-at}u(t),\ \operatorname{Re}\{a\} > 0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t),\ \operatorname{Re}\{a\} > 0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{|T|}{2}\operatorname{sinc}^2(T\omega/4)$

■ If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$a_1 x_1(t) + a_2 x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} a_1 X_1(\omega) + a_2 X_2(\omega),$$

where a_1 and a_2 are arbitrary complex constants.

■ This is known as the **linearity property** of the Fourier transform.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0} X(\omega),$$

  where t_0 is an arbitrary real constant.

- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

■ If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\text{CTFT}} X(\omega - \omega_0),$$

where ω_0 is an arbitrary real constant.

■ This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

■ If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

■ This is known as the **dilation (or time/frequency-domain scaling) property** of the Fourier transform.

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then
$$x^*(t) \xleftrightarrow{\text{CTFT}} X^*(-\omega).$$

- This is known as the **conjugation property** of the Fourier transform.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$

- This is known as the **duality property** of the Fourier transform.

- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by

$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$

- That is, the forward and inverse Fourier transform equations are identical except for a *factor of* 2π and *different sign* in the parameter for the exponential function.

- Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively *double* the number of Fourier transform pairs that we know.

- If $x_1(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.

- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.

- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

Time-Domain Multiplication

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \tfrac{1}{2\pi} X_1 * X_2(\omega) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\theta) X_2(\omega - \theta) d\theta.$$

- This is known as the **(time-domain) multiplication (or frequency-domain convolution) property** of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\frac{dx(t)}{dt} \overset{\text{CTFT}}{\longleftrightarrow} j\omega X(\omega).$$

- This is known as the **(time-domain) differentiation property** of the Fourier transform.

- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.

- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \overset{\text{CTFT}}{\longleftrightarrow} (j\omega)^n X(\omega)$.

- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

- If $x(t) \stackrel{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$tx(t) \stackrel{\text{CTFT}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

Time-Domain Integration

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

- This is known as the **(time-domain) integration property** of the Fourier transform.

- Whereas differentiation in the time domain corresponds to *multiplication* by $j\omega$ in the frequency domain, integration in the time domain is associated with *division* by $j\omega$ in the frequency domain.

- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.

- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

Parseval's Relation

- Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$.

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

  (i.e., the energy of x and energy of X are equal up to a factor of 2π).

- This relationship is known as **Parseval's relation**.

- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform *preserves energy* (up to a scale factor).

Even/Odd Symmetry

- For a function x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow X \text{ is odd}.$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

- A function x is **real** if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X is **conjugate symmetric**).

- Thus, for a real-valued function, the portion of the graph of $X(\omega)$ for $\omega < 0$ is **completely redundant**, as it is determined by symmetry.

- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

(i.e., $|X(\omega)|$ is **even** and $\arg X(\omega)$ is **odd**).

- Note that x being real does **not** necessarily imply that X is real.

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Section 6.4

Fourier Transform of Periodic Functions

Fourier Transform of Periodic Functions

- The Fourier transform can be generalized to also handle periodic functions.
- Consider a periodic function x with period T and frequency $\omega_0 = \frac{2\pi}{T}$.
- Define the function x_T as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

  (i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).

- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_T denote the Fourier transforms of x and x_T, respectively.
- The following relationships can be shown to hold:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0),$$

$$a_k = \tfrac{1}{T}X_T(k\omega_0), \quad \text{and} \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega - k\omega_0).$$

- The Fourier transform X of a periodic function is a series of impulses that occur at integer multiples of the fundamental frequency ω_0 (i.e., $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$).
- Due to the preceding fact, the Fourier transform of a periodic function can only be nonzero at integer multiples of the fundamental frequency.
- The Fourier series coefficient sequence a is produced by sampling X_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$ (i.e., $a_k = \frac{1}{T} X_T(k\omega_0)$).

Section 6.5

Fourier Transform and Frequency Spectra of Functions

The Frequency-Domain Perspective on Functions

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on functions.

- That is, instead of viewing a function as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a function as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency).

- The Fourier transform of a function x provides a means to *quantify* how much information x has at different frequencies.

- The distribution of information in a function over different frequencies is referred to as the *frequency spectrum* of the function.

Fourier Transform and Frequency Spectra

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\omega)$ expressed in ***polar form*** as follows:

$$x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)| e^{j[\omega t + \arg X(\omega)]} d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency ω contributes to the integration result x.

- The quantity $\arg X(\omega)$ determines how the complex sinusoid at frequency ω is shifted related to complex sinusoids at other frequencies.

- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the areas of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

- Expressing the integral (from the previous slide) as the *limit of a sum*, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega |X(\omega)| e^{j[\omega t + \arg X(\omega)]},$$

where $\omega = k\Delta\omega$.

- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\omega = k\Delta\omega$ that has had its *amplitude scaled* by a factor of $|X(\omega)|$ and has been *time shifted* by an amount that depends on $\arg X(\omega)$.

- For a given $\omega = k\Delta\omega$ (which is associated with the kth term in the summation), the *larger* $|X(\omega)|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega t}$ will be, and therefore the *larger the contribution* the kth term will make to the overall summation.

- In this way, we can use $|X(\omega)|$ as a *measure* of how much information a function x has at the frequency ω.

- The Fourier transform X of the function x is referred to as the **frequency spectrum** of x.

- The magnitude $|X(\omega)|$ of the Fourier transform X is referred to as the **magnitude spectrum** of x.

- The argument $\arg X(\omega)$ of the Fourier transform X is referred to as the **phase spectrum** of x.

- Since the Fourier transform is a function of a real variable, a function can potentially have information at any real frequency.

- Since the Fourier transform X of a periodic function x with fundamental frequency ω_0 and the Fourier series coefficient sequence a is given by $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, the Fourier transform and Fourier series give consistent results for the frequency spectrum of a periodic function.

- Since the frequency spectrum is complex (in the general case), it is *usually represented using two plots*, one showing the magnitude spectrum and one showing the phase spectrum.

Frequency Spectra of Real Functions

- Recall that, for a real function x, the Fourier transform X of x satisfies

$$X(\omega) = X^*(-\omega)$$

(i.e., X is *conjugate symmetric*), which is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega).$$

- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a real function is always *even*.

- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a real function is always *odd*.

- Due to the symmetry in the frequency spectra of real functions, we typically *ignore negative frequencies* when dealing with such functions.

- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

- The relative importance of the magnitude spectrum and phase spectrum is highly dependent on the particular application of interest.

- Consider the case of the human auditory system (i.e., human hearing).

- The human auditory system tends to be quite sensitive to changes in the magnitude spectrum of a signal.

- That is, a significant change in the magnitude spectrum of an audio signal is very likely to lead to a noticeable difference in the perceived sound.

- On the other hand, the human auditory system tends to be much less sensitive to changes in the phase spectrum of a signal.

- In other words, changes to the phase spectrum of an audio signal are often only barely perceptible or not perceptible at all.

- For the above reasons, in applications involving the human auditory system, magnitude distortion (i.e., distortion of the magnitude spectrum) often tends to be more of a concern than phase distortion (i.e., distortion of the phase spectrum).

Image A

Image B

Magnitude Spectrum from Image B and
Phase Spectrum from Image A

Magnitude Spectrum from Image A and
*Phase Spectrum from Image B*
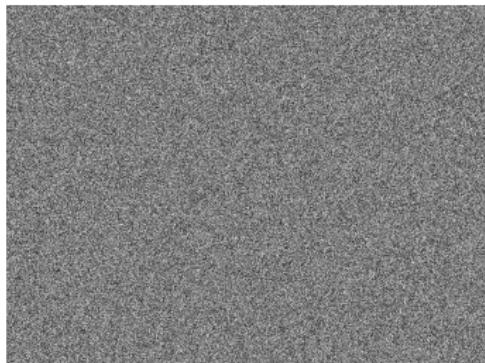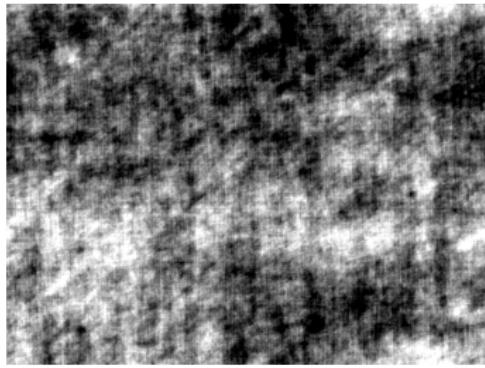
Image A

Image B (White Noise)

Magnitude Spectrum from Image B and
*Phase Spectrum from Image A*
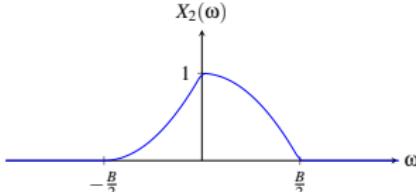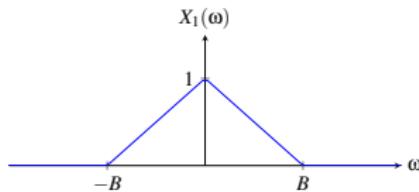
Magnitude Spectrum from Image A and
Phase Spectrum from Image B

Bandwidth

- A function with the Fourier transform X is said to be **bandlimited** if, for some (finite) nonnegative real constant B, the following condition holds:

$$X(\omega) = 0 \text{ for all } \omega \text{ satisfying } |\omega| > B.$$

- The **bandwidth** B of a function with the Fourier transform X is defined as $B = \omega_1 - \omega_0$, where $X(\omega) = 0$ for all $\omega \notin [\omega_0, \omega_1]$.

- In the case of *real-valued* functions, however, this definition of bandwidth is usually amended to consider *only nonnegative* frequencies.

- The real-valued function x_1 and complex-valued function x_2 with the respective Fourier transforms X_1 and X_2 shown below each have bandwidth B (where only nonnegative frequencies are considered in the case of x_1).

- One can show that a function *cannot be both time limited and bandlimited*.

- By Parseval's relation, the energy E in a function x with Fourier transform X is given by

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_x(\omega)d\omega,$$

where

$$E_x(\omega) = |X(\omega)|^2.$$

- We refer to E_x as the **energy-density spectrum** of the function x.

- The function E_x indicates how the energy in x is distributed with respect to frequency.

- For example, the energy contributed by frequencies in the range $[\omega_1, \omega_2]$ is given by

$$\frac{1}{2\pi} \int_{\omega_1}^{\omega_2} E_x(\omega)d\omega.$$

Section 6.6

Fourier Transform and LTI Systems

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega).$$

- The function H is called the **frequency response** of the system.
- A LTI system is *completely characterized* by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- The quantity $|H(\omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\omega)$ is called the **phase response** of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \quad \text{and} \quad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.
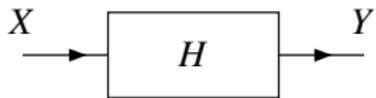
■ Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is *real*, then

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is *even* and the phase response $\arg H(\omega)$ is *odd*).

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.

$$X \longrightarrow \boxed{ H } \longrightarrow Y$$
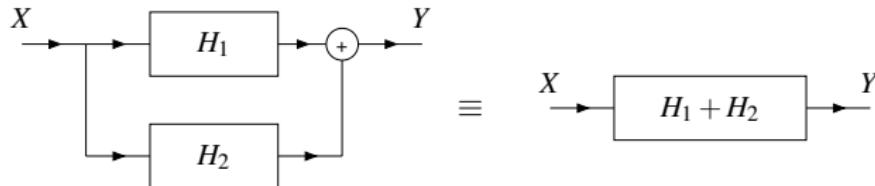
- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

- The *series* interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with frequency response $H_1 H_2$. That is, we have the equivalence shown below.

$$\xrightarrow{\ X\ } \boxed{H_1} \rightarrow \boxed{H_2} \xrightarrow{\ Y\ } \quad \equiv \quad \xrightarrow{\ X\ } \boxed{H_1 H_2} \xrightarrow{\ Y\ }$$

- The *parallel* interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with the frequency response $H_1 + H_2$. That is, we have the equivalence shown below.

$$\xrightarrow{\ X\ } \begin{array}{c} \boxed{H_1} \\ \\ \boxed{H_2} \end{array} \underset{+}{\bigoplus} \xrightarrow{\ Y\ } \quad \equiv \quad \xrightarrow{\ X\ } \boxed{H_1 + H_2} \xrightarrow{\ Y\ }$$

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.

- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t),$$

  where the a_k and b_k are complex constants and $M \leq N$.

- Let h denote the impulse response of the system, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.

- One can show that H is given by

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} a_k j^k \omega^k}{\sum_{k=0}^{N} b_k j^k \omega^k}.$$

- Observe that, for a system of the form considered above, the frequency response is a *rational function*.

Section 6.7

Application: Filtering

- In many applications, we want to *modify the spectrum* of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.

- Such a filter has a *frequency response* H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.

- A plot of this frequency response is given below.

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise}, \end{cases}$$

where ω_c is the **cutoff frequency**.
- A plot of this frequency response is given below.

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.

- Such a filter has a *frequency response* H of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise}, \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}.

- A plot of this frequency response is given below.

Section 6.8

Application: Equalization

Equalization

- Often, we find ourselves faced with a situation where we have a system with a particular frequency response that is undesirable for the application at hand.

- As a result, we would like to change the frequency response of the system to be something more desirable.

- This process of modifying the frequency response in this way is referred to as **equalization**. [Essentially, equalization is just a filtering operation.]

- Equalization is used in many applications.

- In real-world *communication systems*, equalization is used to eliminate or minimize the distortion introduced when a signal is sent over a (nonideal) communication channel.

- In *audio applications*, equalization can be employed to emphasize or de-emphasize certain ranges of frequencies. For example, equalization can be used to boost the bass (i.e., emphasize the low frequencies) in the audio output of a stereo.

Equalization (Continued)

Original System

New System with Equalization

- Let H_{orig} denote the frequency response of *original* system (i.e., without equalization).
- Let H_d denote the *desired* frequency response.
- Let H_{eq} denote the frequency response of the *equalizer*.
- The new system with equalization has frequency response

$$H_{\text{new}}(\omega) = H_{\text{eq}}(\omega) H_{\text{orig}}(\omega).$$

- By choosing $H_{\text{eq}}(\omega) = H_d(\omega)/H_{\text{orig}}(\omega)$, the new system with equalization will have the frequency response

$$H_{\text{new}}(\omega) = \left[H_d(\omega)/H_{\text{orig}}(\omega) \right] H_{\text{orig}}(\omega) = H_d(\omega).$$

- In effect, by using an equalizer, we can obtain a new system with the frequency response that we desire.

Section 6.9

Application: Circuit Analysis

Electronic Circuits

- An **electronic circuit** is a network of one or more interconnected circuit elements.
- The three most basic types of circuit elements are:
    1 resistors;
    2 inductors; and
    3 capacitors.
- Two fundamental quantities of interest in electronic circuits are current and voltage.
- **Current** is the rate at which electric charge flows through some part of a circuit, such as a circuit element, and is measured in units of amperes (A).
- **Voltage** is the difference in electric potential between two points in a circuit, such as across a circuit element, and is measured in units of volts (V).
- Voltage is essentially a force that makes electric charge (or current) flow.

Resistors

- A **resistor** is a circuit element that opposes the flow of current.
- A resistor is characterized by an equation of the form

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

  where R is a nonnegative real constant, and v and i respectively denote the voltage across and current through the resistor as a function of time.
- As a matter of terminology, the quantity R is known as the **resistance** of the resistor.
- Resistance is measured in units of ohms (Ω).
- In circuit diagrams, a resistor is denoted by the symbol shown below.

Inductors

- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor uses the energy stored in a magnetic field in order to *oppose changes in current* (through the inductor).
- An inductor is characterized by an equation of the form

$$v(t) = L\frac{d}{dt}i(t) \quad \text{(or equivalently, } i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau),$$

where L is a nonnegative real constant, and v and i respectively denote the voltage across and current through the inductor as a function of time.
- As a matter of terminology, the quantity L is known as the **inductance** of the inductor.
- Inductance is measured in units of henrys (H).
- In circuit diagrams, an inductor is denoted by the symbol shown below.

Capacitors

- A **capacitor** is a circuit element that stores electric charge.
- A capacitor uses the energy stored in an electric field in order to *oppose changes in voltage* (across the capacitor).
- A capacitor is characterized by an equation of the form

$$v(t) = \tfrac{1}{C} \int_{-\infty}^{t} i(\tau)d\tau \quad \text{(or equivalently, } i(t) = C\tfrac{d}{dt}v(t)\text{)},$$

  where C is a nonnegative real constant, and v and i respectively denote the voltage across and current through the capacitor as a function of time.
- As a matter of terminology, the quantity C is known as the **capacitance** of the capacitor.
- Capacitance is measured in units of farads (F).
- In circuit diagrams, a capacitor is denoted by the symbol shown below.

Circuit Analysis with the Fourier Transform

- The Fourier transform is a very useful tool for circuit analysis.

- The utility of the Fourier transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Fourier domain than in the time domain.

- Let v and i denote the voltage across and current through a circuit element, and let V and I denote the Fourier transforms of v and i, respectively.

- In the frequency domain, the equations characterizing a resistor, an inductor, and a capacitor respectively become:

$$V(\omega) = RI(\omega) \quad \text{(or equivalently, } I(\omega) = \tfrac{1}{R}V(\omega));$$
$$V(\omega) = j\omega LI(\omega) \quad \text{(or equivalently, } I(\omega) = \tfrac{1}{j\omega L}V(\omega)); \quad \text{and}$$
$$V(\omega) = \tfrac{1}{j\omega C}I(\omega) \quad \text{(or equivalently, } I(\omega) = j\omega CV(\omega)).$$

- Note the absence of differentiation and integration in the above equations for an inductor and a capacitor.
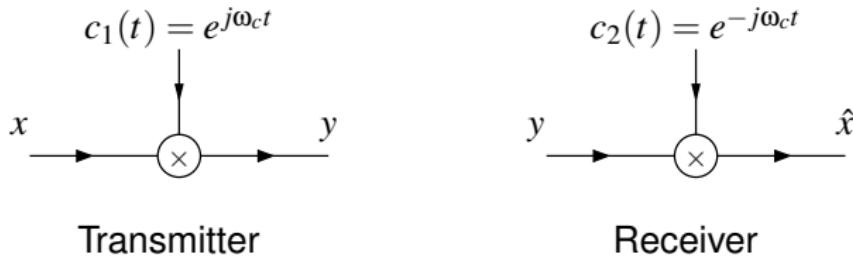
Section 6.10

Application: Amplitude Modulation (AM)

Motivation for Amplitude Modulation (AM)

- In communication systems, we often need to transmit a signal using a frequency range that is different from that of the original signal.
- For example, voice/audio signals typically have information in the range of 0 to 22 kHz.
- Often, it is not practical to transmit such a signal using its original frequency range.
- Two potential problems with such an approach are:
  1. interference; and
  2. constraints on antenna length.
- Since many signals are broadcast over the airwaves, we need to ensure that no two transmitters use the same frequency bands in order to avoid interference.
- Also, in the case of transmission via electromagnetic waves (e.g., radio waves), the length of antenna required becomes impractically large for the transmission of relatively low frequency signals.
- For the preceding reasons, we often need to change the frequency range associated with a signal before transmission.

Transmitter

Receiver

■ The transmitter is characterized by
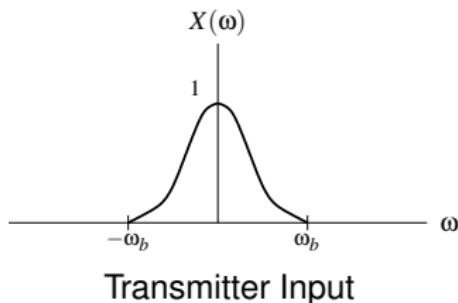
$$y(t) = e^{j\omega_c t}x(t) \quad \Longleftrightarrow \quad Y(\omega) = X(\omega - \omega_c).$$

■ The receiver is characterized by
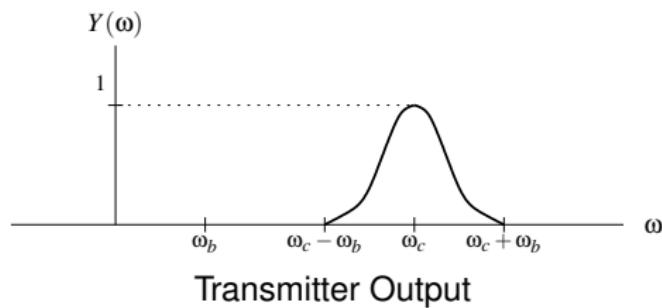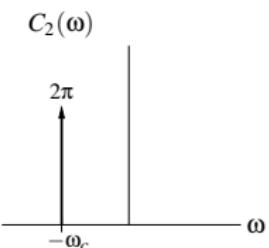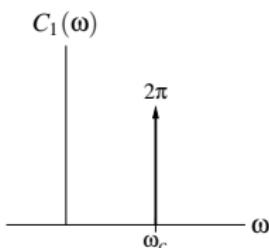
$$\hat{x}(t) = e^{-j\omega_c t}y(t) \quad \Longleftrightarrow \quad \hat{X}(\omega) = Y(\omega + \omega_c).$$

■ Clearly, $\hat{x}(t) = e^{j\omega_c t}e^{-j\omega_c t}x(t) = x(t)$.

$X(\omega)$

1

$-\omega_b$ ω_b ω

Transmitter Input

$C_1(\omega)$

2π

ω_c ω

$C_2(\omega)$

2π

$-\omega_c$ ω

$Y(\omega)$

1

ω_b $\omega_c - \omega_b$ ω_c $\omega_c + \omega_b$ ω

Transmitter Output

$\hat{X}(\omega)$

1

$-\omega_b$ ω_b ω

Receiver Output

Transmitter

Receiver

- Let $X = \mathcal{F}x$, $Y = \mathcal{F}y$, and $\hat{X} = \mathcal{F}\hat{x}$.
- Suppose that $X(\omega) = 0$ for all $\omega \notin [-\omega_b, \omega_b]$.
- The transmitter is characterized by
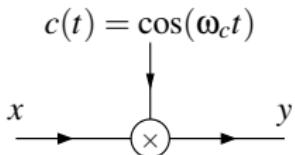
$$Y(\omega) = \tfrac{1}{2} \left[X(\omega + \omega_c) + X(\omega - \omega_c) \right].$$

- The receiver is characterized by

$$\hat{X}(\omega) = \left[Y(\omega + \omega_c) + Y(\omega - \omega_c) \right] \mathrm{rect} \left(\frac{\omega}{2\omega_{c_0}} \right).$$

- If $\omega_b < \omega_{c_0} < 2\omega_c - \omega_b$, we have $\hat{X}(\omega) = X(\omega)$ (implying $\hat{x}(t) = x(t)$).

DSB-SC AM: Transmitter

$$c(t) = \cos(\omega_c t)$$

$$
\begin{array}{c}
x \\
\xrightarrow{\hspace{1cm}} \otimes \xrightarrow{\hspace{1cm}} y
\end{array}
$$
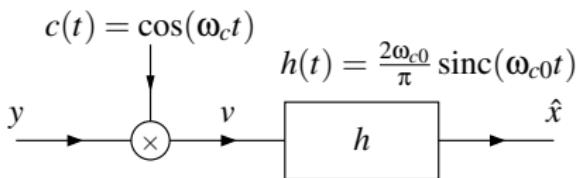
$$y(t) = \cos(\omega_c t)x(t)$$

$$X = \mathcal{F}x, \quad Y = \mathcal{F}y$$

$$
\begin{aligned}
Y(\omega) &= \mathcal{F}\{\cos(\omega_c t)x(t)\}(\omega) \\
&= \mathcal{F}\left\{ \tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)x(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega)\right] \\
&= \tfrac{1}{2}\left[X(\omega - \omega_c) + X(\omega + \omega_c)\right]
\end{aligned}
$$

DSB-SC AM: Receiver

$$c(t) = \cos(\omega_c t)$$
$$h(t) = \frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t)$$

$$y \longrightarrow \otimes \xrightarrow{v} \boxed{h} \longrightarrow \hat{x}$$

$$v(t) = \cos(\omega_c t)y(t), \quad h(t) = \frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t), \quad \hat{x}(t) = v * h(t)$$
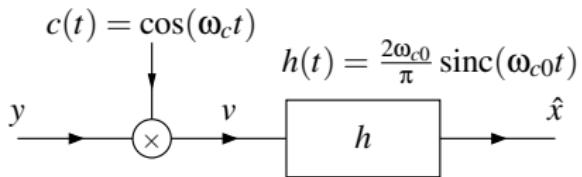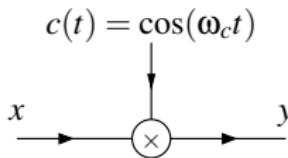
$$Y = \mathcal{F}y, \quad V = \mathcal{F}v, \quad H = \mathcal{F}h, \quad \hat{X} = \mathcal{F}\hat{x}$$

$$\begin{aligned}
V(\omega) &= \mathcal{F}\{\cos(\omega_c t)y(t)\}(\omega)\\
&= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)y(t)\right\}(\omega)\\
&= \tfrac{1}{2}\left[\mathcal{F}\left\{e^{j\omega_c t}y(t)\right\}(\omega) + \mathcal{F}\left\{e^{-j\omega_c t}y(t)\right\}(\omega)\right]\\
&= \tfrac{1}{2}[Y(\omega - \omega_c) + Y(\omega + \omega_c)]
\end{aligned}$$

$$\begin{aligned}
H(\omega) &= \mathcal{F}\left\{\frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t)\right\}(\omega)\\
&= 2\operatorname{rect}\left(\frac{\omega}{2\omega_{c0}}\right)
\end{aligned}$$

$$\hat{X}(\omega) = H(\omega)V(\omega)$$

DSB-SC AM: Complete System
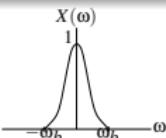
$$Y(\omega) = \tfrac{1}{2}\left[X(\omega - \omega_c) + X(\omega + \omega_c)\right]$$
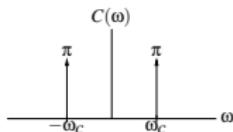
$$
\begin{aligned}
V(\omega) &= \tfrac{1}{2}\left[Y(\omega - \omega_c) + Y(\omega + \omega_c)\right] \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}\left[X([\omega - \omega_c] - \omega_c) + X([\omega - \omega_c] + \omega_c)\right] + \right.\\
&\qquad\quad \left.\tfrac{1}{2}\left[X([\omega + \omega_c] - \omega_c) + X([\omega + \omega_c] + \omega_c)\right]\right] \\
&= \tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega - 2\omega_c) + \tfrac{1}{4}X(\omega + 2\omega_c)
\end{aligned}
$$

$$
\begin{aligned}
\hat{X}(\omega) &= H(\omega)V(\omega) \\
&= H(\omega)\left[\tfrac{1}{2}X(\omega) + \tfrac{1}{4}X(\omega - 2\omega_c) + \tfrac{1}{4}X(\omega + 2\omega_c)\right] \\
&= \tfrac{1}{2}H(\omega)X(\omega) + \tfrac{1}{4}H(\omega)X(\omega - 2\omega_c) + \tfrac{1}{4}H(\omega)X(\omega + 2\omega_c) \\
&= \tfrac{1}{2}\left[2X(\omega)\right] + \tfrac{1}{4}(0) + \tfrac{1}{4}(0) \\
&= X(\omega)
\end{aligned}
$$
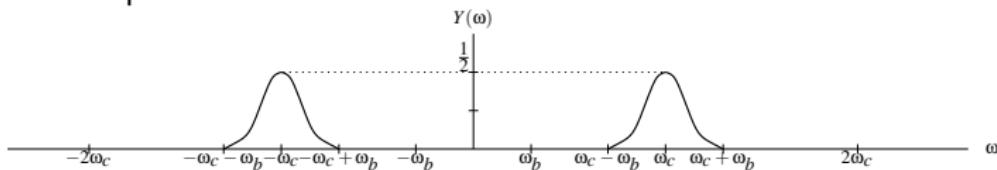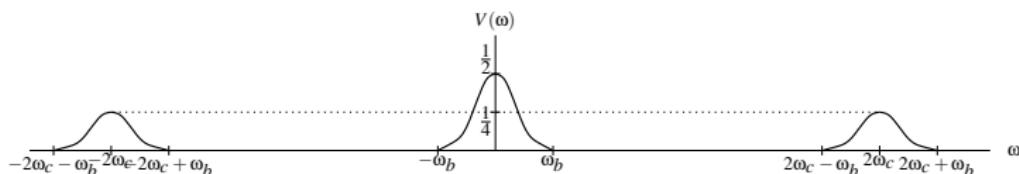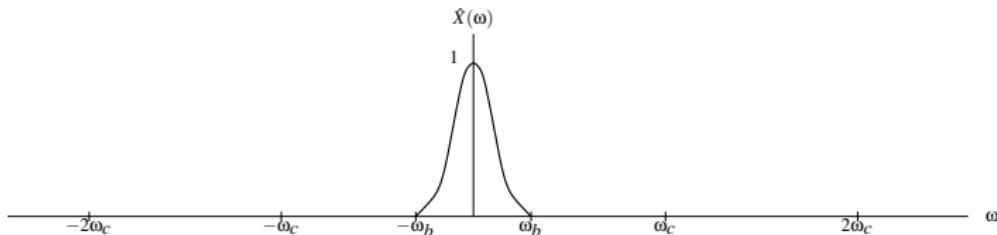
Transmitter Input
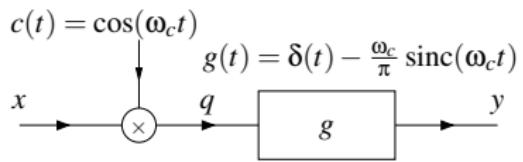
Transmitter Output
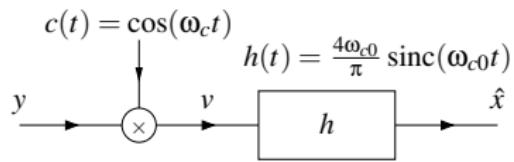
Receiver Output

## Single-Sideband Suppressed-Carrier (SSB-SC) AM
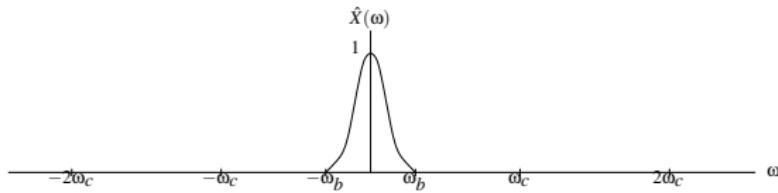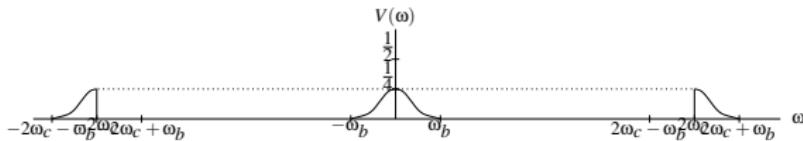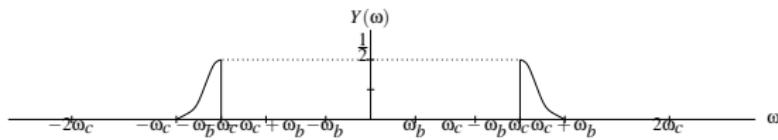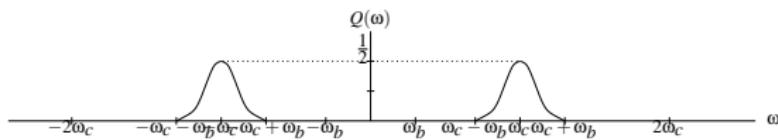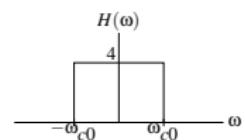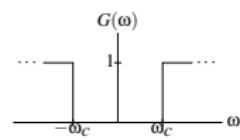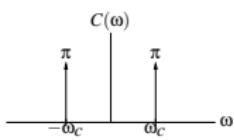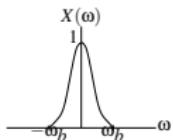
Transmitter

Receiver

- The basic analysis of the SSB-SC AM system is similar to the DSB-SC AM system.
- SSB-SC AM requires half as much bandwidth for the transmitted signal as DSB-SC AM.

SSB-SC AM: Example
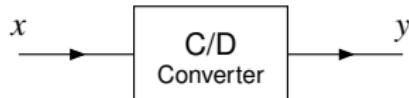
Section 6.11
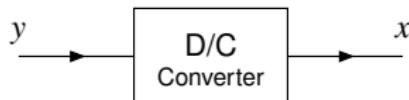
Application: Sampling and Interpolation

Sampling and Interpolation

- Often, we want to be able to transform a continuous-time signal (i.e., a function) into a discrete-time signal (i.e., a sequence) and vice versa.

- This is accomplished through processes known as *sampling* and *interpolation*.

- **Sampling**, which is performed by a **continuous-time to discrete-time (C/D) converter** shown below, transforms a function x to a sequence y.

$$x \longrightarrow \boxed{\begin{array}{c} \text{C/D} \\ \text{Converter} \end{array}} \longrightarrow y$$

- **Interpolation**, which is performed by a **discrete-time to continuous-time (D/C) converter** shown below, transforms a sequence y to a function x.

$$y \longrightarrow \boxed{\begin{array}{c} \text{D/C} \\ \text{Converter} \end{array}} \longrightarrow x$$

- Note that, unless very special conditions are met, the sampling process loses information (i.e., is *not invertible*).
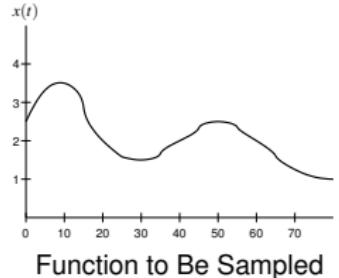
Periodic Sampling

- Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**.

- With this scheme, a sequence y of samples is obtained from a function x according to the relation

$$y(n) = x(Tn) \quad \text{for all integer } n,$$

where T is a (strictly) positive real constant.

- As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = \frac{2\pi}{T}$ as the (angular) **sampling frequency**.

- An example of periodic sampling is shown below, where the function x has been sampled with *sampling period $T = 10$*, yielding the sequence y.

Function to Be Sampled

Sequence Produced by Sampling

Invertibility of Sampling

- Unless constraints are placed on the functions being sampled, the sampling process is *not invertible*.

- In other words, in the absence of any constraints, a function cannot be uniquely determined from a sequence of its equally-spaced samples.

- Consider, for example, the functions x_1 and x_2 given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

- Sampling x_1 and x_2 with the sampling period $T = 1$ yields the respective sequences
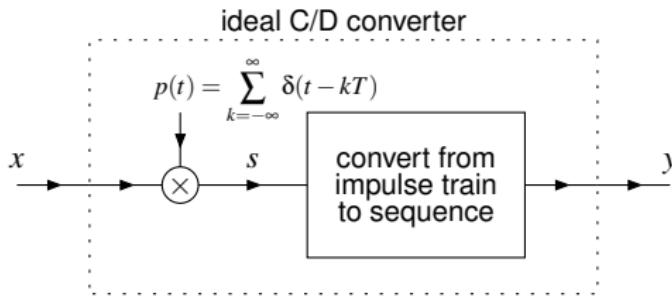
$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and}$$
$$y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

- So, although x_1 and x_2 are *distinct*, y_1 and y_2 are *identical*.

- Given the sequence y where $y = y_1 = y_2$, it is impossible to determine which function was sampled to produce y.

- Only by imposing a carefully chosen set of constraints on the functions being sampled can we ensure that a function can be exactly recovered from only its samples.
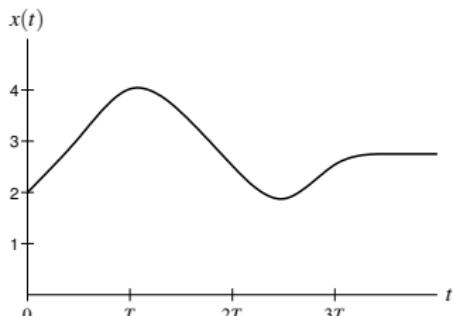
Model of Sampling

- An **impulse train** is a function of the form $v(t) = \sum_{k=-\infty}^{\infty} c_k \delta(t - kT)$, where c_k and T are real constants.
- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.

ideal C/D converter
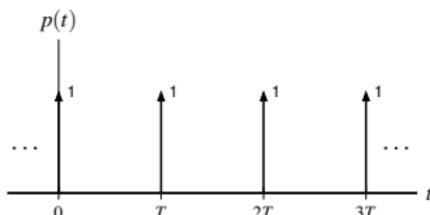
- The sampling of a function x to produce a sequence y consists of the following two steps (in order):
    1. Multiply the function x to be sampled by a periodic impulse train p, yielding the impulse train $s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$.
    2. Convert the impulse train s to a sequence y by forming y from the weights of successive impulses in s so that $y(n) = x(nT)$.
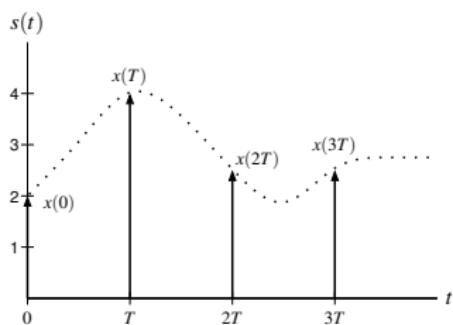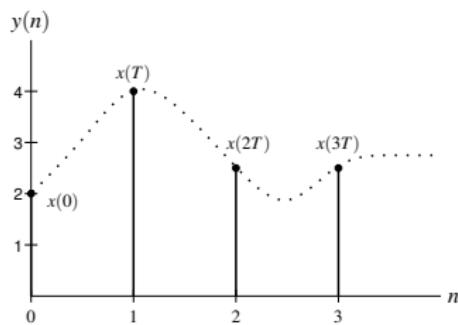
Input Function

Periodic Impulse Train
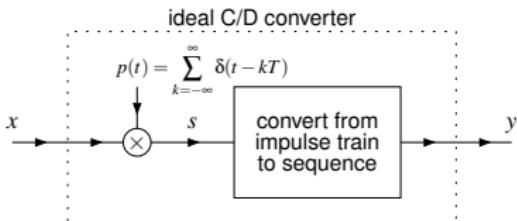
Impulse-Sampled Function
(Continuous-Time)

Output Sequence *(Discrete-Time)*

- Since sampling is not invertible and our model of sampling consists of only two steps, at least one of these two steps must not be invertible.

- Recall the two steps in our model of sampling are as follows (in order):

1. $x \longrightarrow s(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$; and

2. $s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \longrightarrow y(n) = x(nT)$.

- Step 1 cannot be undone (unless we somehow restrict which functions x can be sampled).

- Step 2 is always invertible.

- Therefore, the fact that sampling is not invertible is entirely due to step 1.

ideal C/D converter

$p(t) = \sum\limits_{k=-\infty}^{\infty} \delta(t - kT)$

x

s

convert from impulse train to sequence

y

- In the time domain, the impulse-sampled function s is given by

$$s(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

- In the Fourier domain, the preceding equation becomes

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \quad (\text{where } \omega_s = \frac{2\pi}{T}).$$

- Thus, the spectrum of the impulse-sampled function s is a scaled sum of an infinite number of *shifted copies* of the spectrum of the original function x.

Sampling: Fourier Series for a Periodic Impulse Train
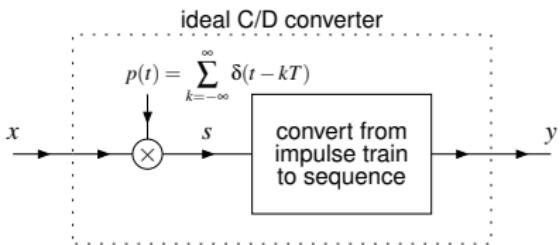
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \tfrac{2\pi}{T}$$

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$$c_k = \tfrac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt$$

$$= \tfrac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt$$

$$= \tfrac{1}{T} \int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t} dt$$

$$= \tfrac{1}{T}$$

$$= \tfrac{\omega_s}{2\pi}$$

$$p(t) = \tfrac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

Sampling: Multiplication by a Periodic Impulse Train

$$s(t) = p(t)x(t), \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT), \quad \omega_s = \frac{2\pi}{T}$$

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

$$s(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} x(t)$$

$$X = \mathcal{F}x, \quad S = \mathcal{F}s$$

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$
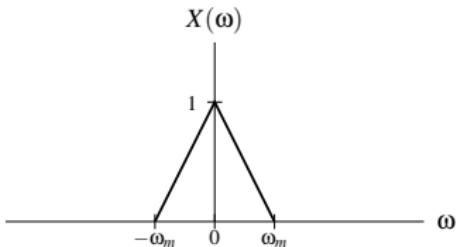
Model of Sampling: Aliasing

- Consider frequency spectrum S of the impulse-sampled function s given by
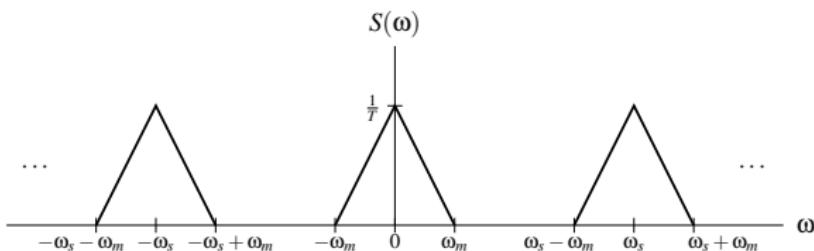
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- The function S is a scaled sum of an infinite number of ***shifted copies*** of X.

- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of x.

- In particular, the nonzero portions of the different shifted copies of X can either:
    1. overlap; or
    2. not overlap.

- In the case where overlap occurs, the various shifted copies of X add together in such a way that the original shape of X is lost. This phenomenon is known as **aliasing**.

- When aliasing occurs, the original function x cannot be recovered from its samples in y.

Spectrum of Input Function (Bandwidth ω_m)
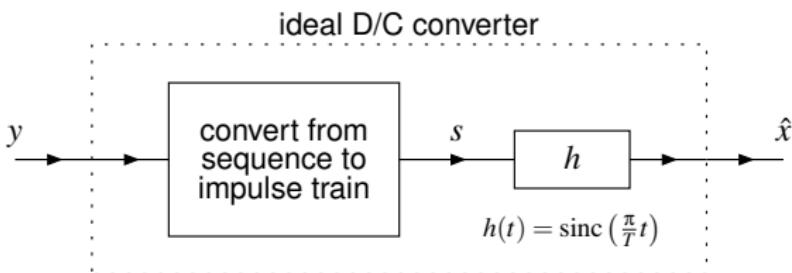
Spectrum of Impulse-Sampled Function: No Aliasing Case ($\omega_s > 2\omega_m$)

Spectrum of Impulse-Sampled Function: Aliasing Case ($\omega_s \leq 2\omega_m$)

Model of Interpolation

- For the purposes of analysis, interpolation can be modelled as shown below.

- The reconstruction of a function x from its sequence y of samples (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence y to the impulse train s by using the samples in y as the weights of successive impulses in s so that $s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn)$.
  2. Apply the lowpass filter with impulse response h to s to produce $\hat{x}$ so that $\hat{x}(t) = s * h(t) = \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}\left[\frac{\pi}{T}(t - Tn)\right]$.

- The lowpass filter is used to eliminate the extra copies of the originally-sampled function's spectrum present in the spectrum of s.

- **Sampling Theorem.** Let x be a function with Fourier transform X, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., x is bandlimited to frequencies $[-\omega_M, \omega_M]$). Then, x is uniquely determined by its samples $y(n) = x(Tn)$ for all integer n, if

$$\omega_s > 2\omega_M,$$

where $\omega_s = \frac{2\pi}{T}$. The preceding inequality is known as the **Nyquist condition**. If this condition is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc} \left[\tfrac{\pi}{T}(t - Tn) \right],$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc} \left(\tfrac{\omega_s}{2} t - \pi n \right).$$

- We call $\frac{\omega_s}{2}$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**.

Part 7

Laplace Transform (LT)

Motivation Behind the Laplace Transform

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.

- The Laplace transform can be viewed as a *generalization of the Fourier transform*.

- Due to its more general nature, the Laplace transform has a number of *advantages* over the Fourier transform.

- First, the Laplace transform representation *exists for some functions that do not have a Fourier transform representation*. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform.

- Second, since the Laplace transform is a more general tool, it can provide *additional insights* beyond those facilitated by the Fourier transform.

- Earlier, we saw that complex exponentials are eigenfunctions of LTI systems.

- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{e^{st}\}(t) = H(s)e^{st} \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

- Previously, we referred to H as the system function.

- As it turns out, H is the Laplace transform of h.

- Since the Laplace transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.

- Furthermore, as we will see, the Laplace transform has many additional uses.

Section 7.1

Laplace Transform

(Bilateral) Laplace Transform

- The (bilateral) **Laplace transform** of the function x, denoted $\mathcal{L}x$ or X, is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

- The **inverse Laplace transform** of X, denoted $\mathcal{L}^{-1}X$ or x, is then given by

$$\mathcal{L}^{-1}X(t) = x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$$

where $\mathrm{Re}(s) = \sigma$ is in the ROC of X. (Note that this is a *contour integration*, since s is complex.)

- We refer to x and X as a **Laplace transform pair** and denote this relationship as

$$x(t) \xleftrightarrow{\text{ LT }} X(s).$$

- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

Bilateral and Unilateral Laplace Transforms

- Two different versions of the Laplace transform are commonly used:
    1. the *bilateral* (or *two-sided*) Laplace transform; and
    2. the *unilateral* (or *one-sided*) Laplace transform.

- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.

- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the *lower limit of integration*.

- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0 (i.e., $\int_{-\infty}^{\infty} x(t)e^{-st}dt$ versus $\int_{0^-}^{\infty} x(t)e^{-st}dt$).

- For the most part, we will focus our attention primarily on the bilateral Laplace transform.

- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.

- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean *bilateral* Laplace transform.

Remarks on Operator Notation

- For a function x, the Laplace transform of x is denoted using operator notation as $\mathcal{L}x$.
- The Laplace transform of x evaluated at s is denoted $\mathcal{L}x(s)$.
- Note that $\mathcal{L}x$ is a function, whereas $\mathcal{L}x(s)$ is a number.
- Similarly, for a function X, the inverse Laplace transform of X is denoted using operator notation as $\mathcal{L}^{-1}X$.
- The inverse Laplace transform of X evaluated at t is denoted $\mathcal{L}^{-1}X(t)$.
- Note that $\mathcal{L}^{-1}X$ is a function, whereas $\mathcal{L}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

Remarks on Dot Notation

- Often, we would like to write an expression for the Laplace transform of a function without explicitly naming the function.
- For example, consider writing an expression for the Laplace transform of the function $v(t) = x(5t - 3)$ but without using the name "v".
- It would be incorrect to write "$\mathcal{L}x(5t - 3)$" as this is the function $\mathcal{L}x$ evaluated at $5t - 3$, which is not the meaning that we wish to convey.
- Also, strictly speaking, it would be incorrect to write "$\mathcal{L}\{x(5t - 3)\}$" as the operand of the Laplace transform operator must be a function, and $x(5t - 3)$ is a number (i.e., the function x evaluated at $5t - 3$).
- Using dot notation, we can write the following strictly-correct expression for the desired Laplace transform: $\mathcal{L}\{x(5 \cdot -3)\}$.
- In many cases, however, it is probably advisable to avoid employing anonymous (i.e., unnamed) functions, as their use tends to be more error prone in some contexts.

Remarks on Notational Conventions

- Since dot notation is less frequently used by engineers, the author has elected to minimize its use herein.
- To avoid ambiguous notation, the following conventions are followed:
    1. in the expression for the operand of a Laplace transform operator, the *independent variable is assumed to be the variable named "t"* unless otherwise indicated (i.e., in terms of dot notation, each "*t*" is treated as if it were a "·")
    2. in the expression for the operand of the inverse Laplace transform operator, the *independent variable is assumed to be the variable named "s"* unless otherwise indicated (i.e., in terms of dot notation, each "*s*" is treated as if it were a "·").
- For example, with these conventions:
    - "$\mathcal{L}\{(t-\tau)u(t-\tau)\}$" denotes the function that is the Laplace transform of the function $v(t) = (t-\tau)u(t-\tau)$ (not the Laplace transform of the function $v(\tau) = (t-\tau)u(t-\tau)$).
    - "$\mathcal{L}^{-1}\{\frac{1}{s^2-\lambda}\}$" denotes the function that is the inverse Laplace transform of the function $V(s) = \{\frac{1}{s^2-\lambda}\}$ (not the inverse Laplace transform of the function $V(\lambda) = \{\frac{1}{s^2-\lambda}\}$).

Relationship Between Laplace and Fourier Transforms

- Let X and X_F denote the Laplace and (CT) Fourier transforms of x, respectively.

- The function X evaluated at $j\omega$ (where ω is real) yields $X_\mathsf{F}(\omega)$. That is,

$$X(j\omega) = X_\mathsf{F}(\omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(j\omega)$.

- The function X evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \mathrm{Re}(s)$ and $\omega = \mathrm{Im}(s)$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(\sigma + j\omega) = X_\mathsf{F}'(\omega),$$

where X_F' is the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$.

- So, in general, the Laplace transform of x is the Fourier transform of an exponentially-weighted version of x.

- Due to this weighting, the Laplace transform of a function may exist when the Fourier transform of the same function does not.

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Section 7.2

Region of Convergence (ROC)
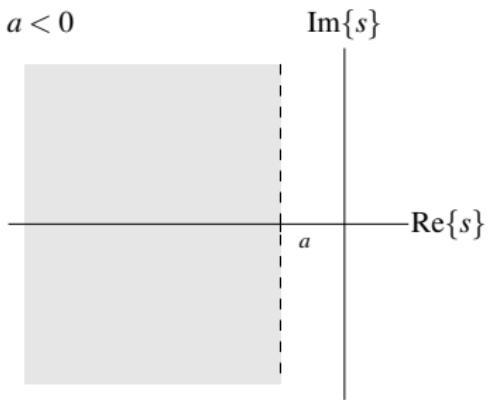
■ The set R of all complex numbers s satisfying
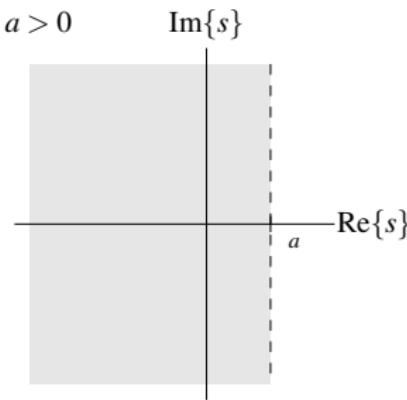
$$\text{Re}(s) < a$$

for some real constant a is said to be a **left-half plane (LHP)**.

■ Some examples of LHPs are shown below.
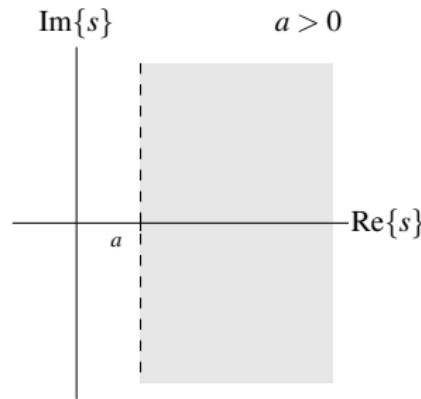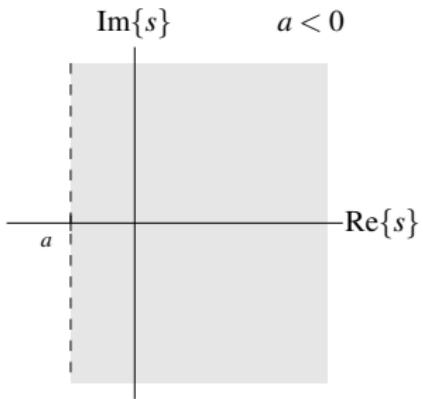
Right-Half Plane (RHP)

■ The set R of all complex numbers s satisfying

$$\mathrm{Re}(s) > a$$
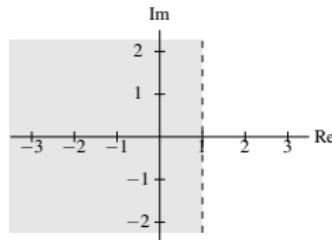
for some real constant a is said to be a **right-half plane (RHP)**.

■ Some examples of RHPs are shown below.
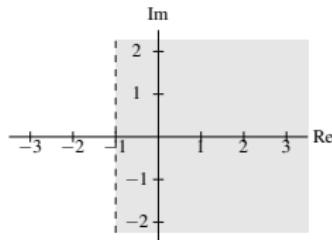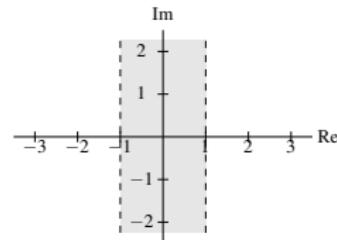
- For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.

R_1 R_2 $R_1 \cap R_2$
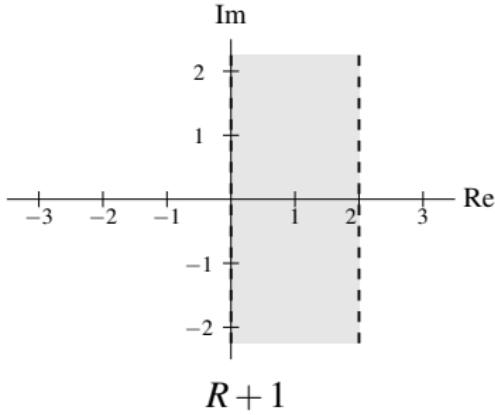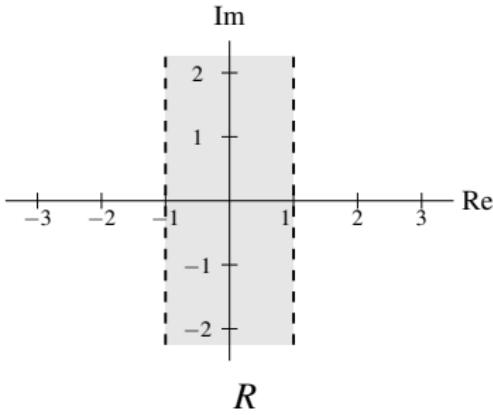
Adding a Scalar to a Set

■ For a set S and a scalar constant a, $S + a$ denotes the set given by

$$S + a = \{z + a : z \in S\}$$

(i.e., $S + a$ is the set formed by adding a to each element of S).

■ Effectively, adding a scalar to a set applies a translation (i.e., shift) to the region associated with the set.

■ An illustrative example is given below.

R $\qquad\qquad\qquad$ $R + 1$

- For a set S and a scalar constant a, aS denotes the set given by
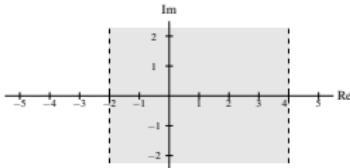
$$aS = \{az : z \in S\}$$

(i..e, aS is the set formed by multiplying each element of S by a).
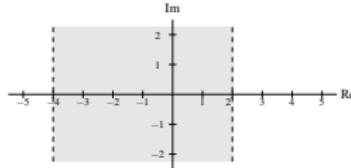
- Multiplying z by a effects z by: scaling by $|a|$ and rotating about the origin by $\arg a$.
- So, effectively, multiplying a set by a scalar applies a scaling and/or rotation to the region associated with the set.
- An illustrative example is given below.

R

$2R$
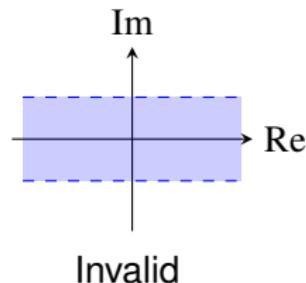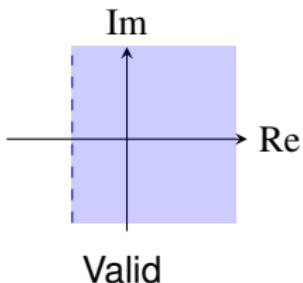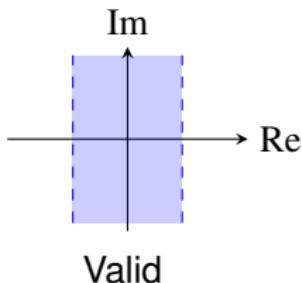
$-2R$

Region of Convergence (ROC)

- As we saw earlier, for a function x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X.

- Two very different functions can have the same algebraic expressions for X.

- On the slides that follow, we will examine a number of key properties of the ROC of the Laplace transform.
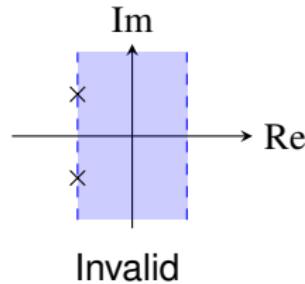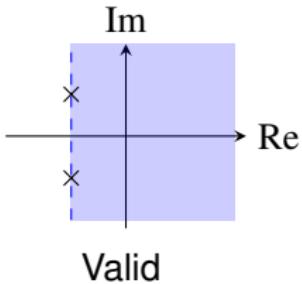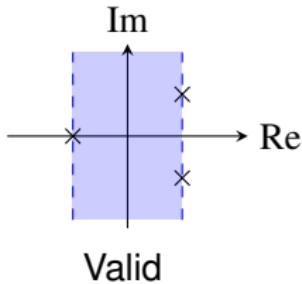
- The ROC of a Laplace transform consists of *strips parallel to the imaginary axis* in the complex plane.
- That is, if a point s_0 is in the ROC, then the vertical line through s_0 (i.e., $\operatorname{Re}(s) = \operatorname{Re}(s_0)$) is also in the ROC.
- Some examples of sets that would be either valid or invalid as ROCs are shown below.
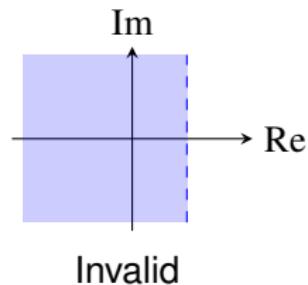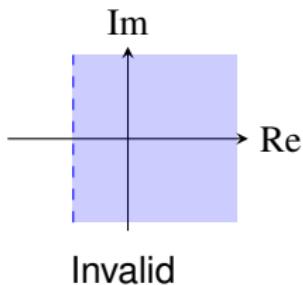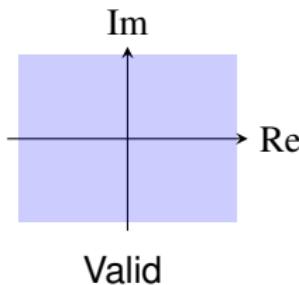
- If a Laplace transform X is a *rational* function, the ROC of X *does not contain any poles* and is *bounded by poles or extends to infinity*.
- Some examples of sets that would be either valid or invalid as ROCs of rational Laplace transforms are shown below.
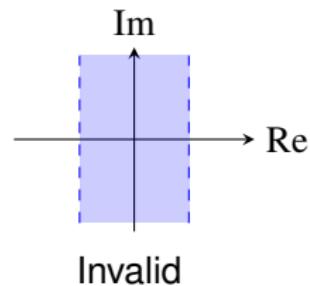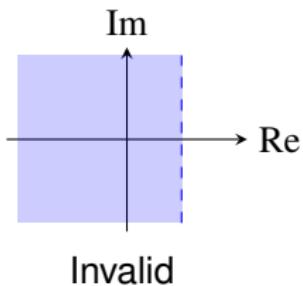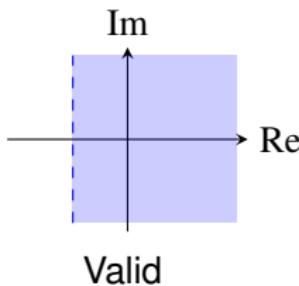
- If a function x is *finite duration* and its Laplace transform X converges for at least one point, then X converges for *all* points in the complex plane (i.e., the ROC is the entire complex plane).
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown below.

Valid   Invalid   Invalid
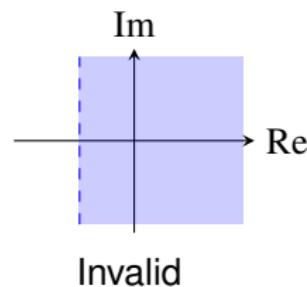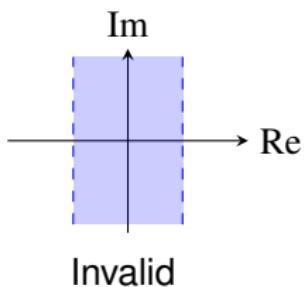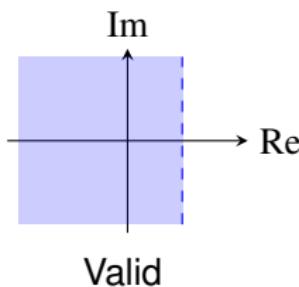
- If a function x is **right sided** and the (vertical) line $\operatorname{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\operatorname{Re}(s) > \sigma_0$ must also be in the ROC (i.e., the ROC is a **RHP** including $\operatorname{Re}(s) = \sigma_0$).

- Some examples of sets that would be either valid or invalid as ROCs for X, if x is right sided, are shown below.

Valid   Invalid   Invalid
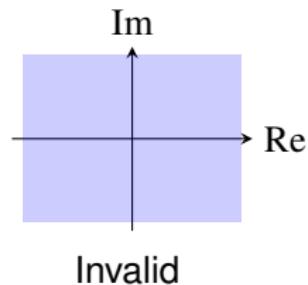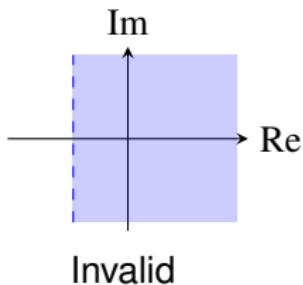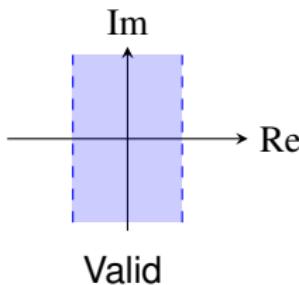
- If a function x is *left sided* and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) < \sigma_0$ must also be in the ROC (i.e., the ROC is a *LHP* including $\mathrm{Re}(s) = \sigma_0$).

- Some examples of sets that would be either valid or invalid as ROCs for X, if x is left sided, are shown below.

Valid   Invalid   Invalid
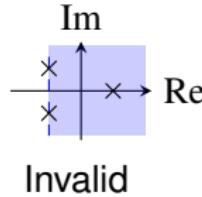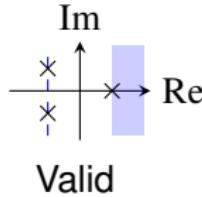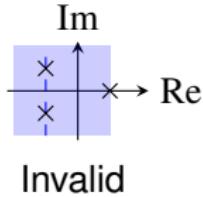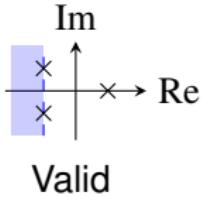
- If a function x is *two sided* and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then the ROC will consist of a *strip* in the complex plane that includes the line $\mathrm{Re}(s) = \sigma_0$.

- Some examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown below.

Valid   Invalid   Invalid

- If the Laplace transform X of a function x is *rational* (with at least one pole), then:
    1. If x is *right sided*, the ROC of X is to the right of the rightmost pole of X (i.e., the *RHP* to the *right of the rightmost pole*).
    2. If x is *left sided*, the ROC of X is to the left of the leftmost pole of X (i.e., the *LHP* to the *left of the leftmost pole*).

- This property is implied by properties 1, 2, 4, and 5.

- Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are given below.

| Valid | Invalid | Valid | Invalid |
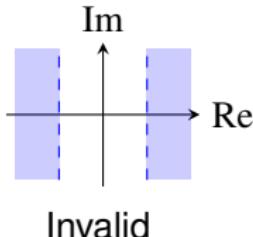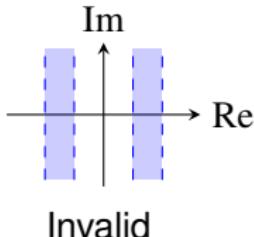
General Form of the ROC

- To summarize the results of properties 3, 4, 5, and 6, if the Laplace transform X of the function x exists, the ROC of X depends on the left- and right-sidedness of x as follows:

x		ROC of X
left sided	right sided	
no	no	strip
no	yes	RHP
yes	no	LHP
yes	yes	everywhere

- Thus, we can infer that, if X exists, its ROC can only be of the form of a LHP, a RHP, a vertical strip, or the entire complex plane.
- For example, the sets shown below would not be valid as ROCs.

Invalid   Invalid

Section 7.3

Properties of the Laplace Transform

Properties of the Laplace Transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(s) + a_2 X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
Laplace-Domain Shifting	$e^{s_0 t} x(t)$	$X(s - s_0)$	$R + \operatorname{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{|a|} X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s) X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt} x(t)$	$s X(s)$	At least R
Laplace-Domain Differentiation	$-t x(t)$	$\frac{d}{ds} X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{s} X(s)$	At least $R \cap \{\operatorname{Re}(s) > 0\}$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s \to \infty} s X(s)$
Final Value Theorem	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)$

Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) < 0$
6	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
7	$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -a$
8	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) > -a$
9	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) < -a$
10	$\cos(\omega_0 t) u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
11	$\sin(\omega_0 t) u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
12	$e^{-at} \cos(\omega_0 t) u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$
13	$e^{-at} \sin(\omega_0 t) u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

  $$a_1 x_1(t) + a_2 x_2(t) \overset{\text{LT}}{\longleftrightarrow} a_1 X_1(s) + a_2 X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Laplace transform.
- The ROC R always contains $R_1 \cap R_2$ but can be larger (in the case that pole-zero cancellation occurs).

- If $x(t) \xleftrightarrow{\text{LT}} X(s)$ with ROC R, then

$$x(t - t_0) \xleftrightarrow{\text{LT}} e^{-st_0} X(s) \text{ with ROC } R,$$

where t_0 is an arbitrary real constant.

- This is known as the **time-domain shifting property** of the Laplace transform.
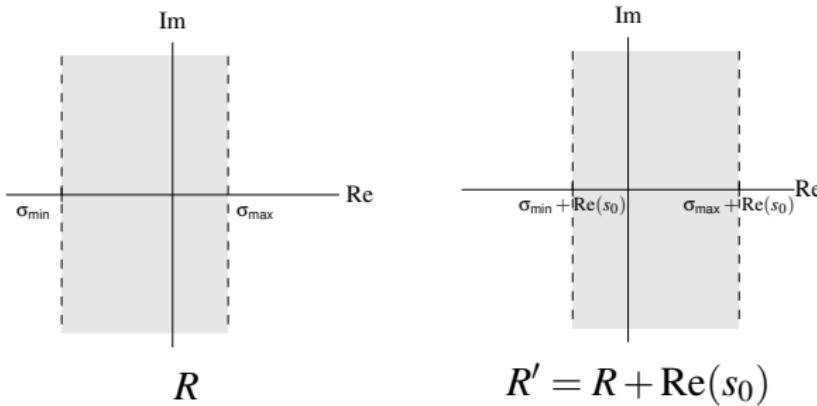
- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$e^{s_0 t} x(t) \overset{\text{LT}}{\longleftrightarrow} X(s - s_0) \text{ with ROC } R' = R + \text{Re}(s_0),$$

where s_0 is an arbitrary complex constant.

- This is known as the **Laplace-domain shifting property** of the Laplace transform.

- As illustrated below, the ROC R is *shifted* right by $\text{Re}(s_0)$.
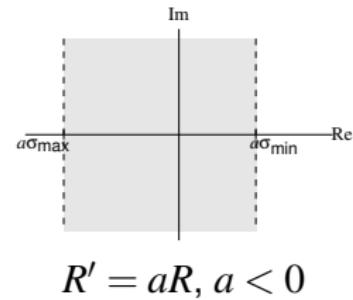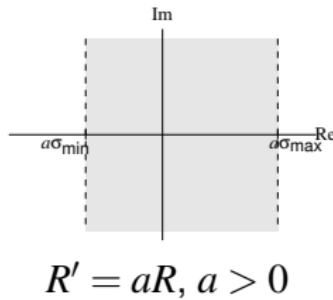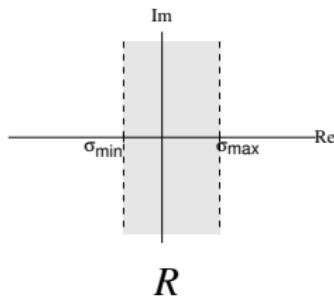
- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x(at) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \text{ with ROC } R' = aR,$$

  where a is a nonzero real constant.

- This is known as the **(time-domain/Laplace-domain) scaling property** of the Laplace transform.

- As illustrated below, the ROC R is *scaled* and *possibly flipped* left to right.

R $\qquad\qquad$ $R' = aR, a > 0$ $\qquad\qquad$ $R' = aR, a < 0$

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x^*(t) \overset{\text{LT}}{\longleftrightarrow} X^*(s^*) \text{ with ROC } R.$$

- This is known as the **conjugation property** of the Laplace transform.

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

  $$x_1 * x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2.$$

- This is known as the **time-domain convolution property** of the Laplace transform.
- The ROC R always contains $R_1 \cap R_2$ but can be larger than this intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes *multiplication* in the Laplace domain.
- Consequently, it is often much easier to work with LTI systems in the Laplace domain, rather than the time domain.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\frac{dx(t)}{dt} \overset{\text{LT}}{\longleftrightarrow} sX(s) \text{ with ROC } R' \text{ containing } R.$$

- This is known as the **time-domain differentiation property** of the Laplace transform.

- The ROC R' always contains R but can be larger than R (if pole-zero cancellation occurs).

- Differentiation in the time domain becomes *multiplication by s* in the Laplace domain.

- Consequently, it can often be much easier to work with differential equations in the Laplace domain, rather than the time domain.

If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$-tx(t) \overset{\text{LT}}{\longleftrightarrow} \frac{dX(s)}{ds} \text{ with ROC } R.$$

■ This is known as the **Laplace-domain differentiation property** of the Laplace transform.

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s}X(s) \text{ with ROC } R' \text{ containing } R \cap \{\text{Re}(s) > 0\}.$$

- This is known as the **time-domain integration property** of the Laplace transform.
- The ROC R' always contains at least $R \cap \{\text{Re}(s) > 0\}$ but can be larger (if pole-zero cancellation occurs).
- Integration in the time domain becomes *division by s* in the Laplace domain.
- Consequently, it is often much easier to work with integral equations in the Laplace domain, rather than the time domain.

Initial Value Theorem

- For a function x with Laplace transform X, if x is *causal* and contains *no impulses or higher order singularities at the origin*, then

$$x(0^+) = \lim_{s \to \infty} sX(s),$$

where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.

- This result is known as the **initial value theorem**.

- In situations where X is known but x is not, the initial value theorem eliminates the need to explicitly find x by an inverse Laplace transform calculation in order to evaluate $x(0^+)$.

- In practice, the values of functions at the origin are frequently of interest, as such values often convey information about the initial state of systems.

- The initial value theorem can sometimes also be helpful in checking for errors in Laplace transform calculations.

Final Value Theorem

- For a function x with Laplace transform X, if x is *causal* and $x(t)$ has a *finite limit* as $t \to \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s).$$

- This result is known as the **final value theorem**.

- In situations where X is known but x is not, the final value theorem eliminates the need to explicitly find x by an inverse Laplace transform calculation in order to evaluate $\lim_{t \to \infty} x(t)$.

- In practice, the values of functions at infinity are frequently of interest, as such values often convey information about the steady-state behavior of systems.

- The final value theorem can sometimes also be helpful in checking for errors in Laplace transform calculations.

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Section 7.4

Determination of Inverse Laplace Transform

Finding Inverse Laplace Transform

- Recall that the inverse Laplace transform x of X is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds,$$

where $\operatorname{Re}(s) = \sigma$ is in the ROC of X.

- Unfortunately, the above contour integration can often be *quite tedious* to compute.

- Consequently, we do not usually compute the inverse Laplace transform directly using the above equation.

- For rational functions, the inverse Laplace transform can be more easily computed using *partial fraction expansions*.

- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse Laplace transforms can typically be found in tables.

Section 7.5

Laplace Transform and LTI Systems
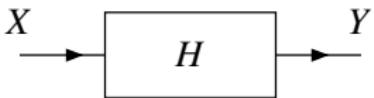
- Consider a LTI system with input x, output y, and impulse response h. Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

- As a matter of terminology, we refer to H as the **system function** (or **transfer function**) of the system (i.e., the system function is the Laplace transform of the impulse response).
- A LTI system is *completely characterized* by its system function H.
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- If the ROC of H includes the imaginary axis, then $H(j\omega)$ is the *frequency response* of the LTI system.

Block Diagram Representations of LTI Systems
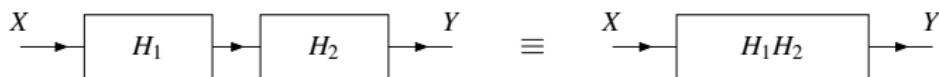
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.

$$X \longrightarrow \boxed{H} \longrightarrow Y$$
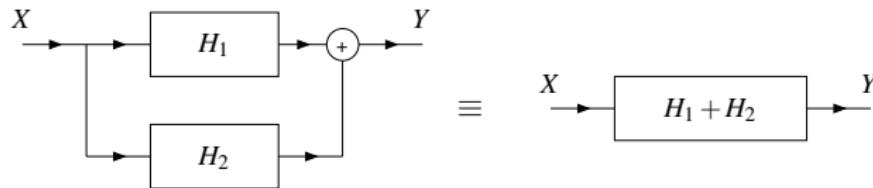
- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

■ The *series* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function $H_1 H_2$. That is, we have the equivalence shown below.

■ The *parallel* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with the system function $H_1 + H_2$. That is, we have the equivalence shown below.

- If a LTI system is *causal*, its impulse response is causal, and therefore *right sided*. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a *causal* LTI system is a *RHP* or the *entire complex plane*.
- In general, the *converse* of the above theorem is *not necessarily true*. That is, if the ROC of the system function is a RHP or the entire complex plane, it is not necessarily true that the system is causal.
- If the system function is *rational*, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a LTI system with a *rational* system function H, *causality* of the system is *equivalent* to the ROC of H being the *RHP to the right of the rightmost pole* or, if H has no poles, the entire complex plane.

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is *BIBO stable* if and only if the ROC of its system function H contains the *imaginary axis* (i.e., $\mathrm{Re}(s) = 0$).
- **Theorem.** A *causal* LTI system with a (proper) *rational* system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have *negative real parts*).

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(s)H_{\text{inv}}(s) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(s) = \frac{1}{H(s)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is *not necessarily unique*.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in *one specific choice* of inverse system (due to these additional constraints of stability and/or causality).

- Many LTI systems of practical interest can be represented using an *Nth-order linear differential equation with constant coefficients*.

- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\tfrac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\tfrac{d}{dt}\right)^k x(t),$$

where the a_k and b_k are complex constants and $M \leq N$.

- Let h denote the impulse response of the system, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.

- One can show that H is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always *rational*.

Section 7.6

Application: Circuit Analysis

Electronic Circuits

- An **electronic circuit** is a network of one or more interconnected circuit elements.
- The three most basic types of circuit elements are:
    1. resistors;
    2. inductors; and
    3. capacitors.
- Two fundamental quantities of interest in electronic circuits are current and voltage.
- **Current** is the rate at which electric charge flows through some part of a circuit, such as a circuit element, and is measured in units of amperes (A).
- **Voltage** is the difference in electric potential between two points in a circuit, such as across a circuit element, and is measured in units of volts (V).
- Voltage is essentially a force that makes electric charge (or current) flow.

Resistors

- A **resistor** is a circuit element that opposes the flow of current.
- A resistor is characterized by an equation of the form

$$v(t) = Ri(t) \quad \left(\text{or equivalently, } i(t) = \tfrac{1}{R}v(t)\right),$$

  where R is a nonnegative real constant, and v and i respectively denote
  the voltage across and current through the resistor as a function of time.

- As a matter of terminology, the quantity R is known as the **resistance** of
  the resistor.
- Resistance is measured in units of ohms (Ω).
- In circuit diagrams, a resistor is denoted by the symbol shown below.

Inductors

- An **inductor** is a circuit element that converts an electric current into a magnetic field and vice versa.
- An inductor uses the energy stored in a magnetic field in order to *oppose changes in current* (through the inductor).
- An inductor is characterized by an equation of the form

$$v(t) = L\frac{d}{dt}i(t) \quad \text{(or equivalently, } i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau)d\tau\text{)},$$

  where L is a nonnegative real constant, and v and i respectively denote the voltage across and current through the inductor as a function of time.
- As a matter of terminology, the quantity L is known as the **inductance** of the inductor.
- Inductance is measured in units of henrys (H).
- In circuit diagrams, an inductor is denoted by the symbol shown below.

Capacitors

- A **capacitor** is a circuit element that stores electric charge.
- A capacitor uses the energy stored in an electric field in order to *oppose changes in voltage* (across the capacitor).
- A capacitor is characterized by an equation of the form

$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)d\tau \quad \text{(or equivalently, } i(t) = C\frac{d}{dt}v(t)),$$

  where C is a nonnegative real constant, and v and i respectively denote the voltage across and current through the capacitor as a function of time.
- As a matter of terminology, the quantity C is known as the **capacitance** of the capacitor.
- Capacitance is measured in units of farads (F).
- In circuit diagrams, a capacitor is denoted by the symbol shown below.

Circuit Analysis with the Laplace Transform

- The Laplace transform is a very useful tool for circuit analysis.
- The utility of the Laplace transform is partly due to the fact that the *differential/integral* equations that describe inductors and capacitors are much simpler to express in the Laplace domain than in the time domain.
- Let v and i denote the voltage across and current through a circuit element, and let V and I denote the Laplace transforms of v and i, respectively.
- In the Laplace domain, the equations characterizing a resistor, an inductor, and a capacitor respectively become:

$$V(s) = RI(s) \quad \text{(or equivalently, } I(s) = \tfrac{1}{R}V(s)\text{)};$$
$$V(s) = sLI(s) \quad \text{(or equivalently, } I(s) = \tfrac{1}{sL}V(s)\text{)}; \quad \text{and}$$
$$V(s) = \tfrac{1}{sC}I(s) \quad \text{(or equivalently, } I(s) = sCV(s)\text{)}.$$

- Note the absence of differentiation and integration in the above equations for an inductor and a capacitor.
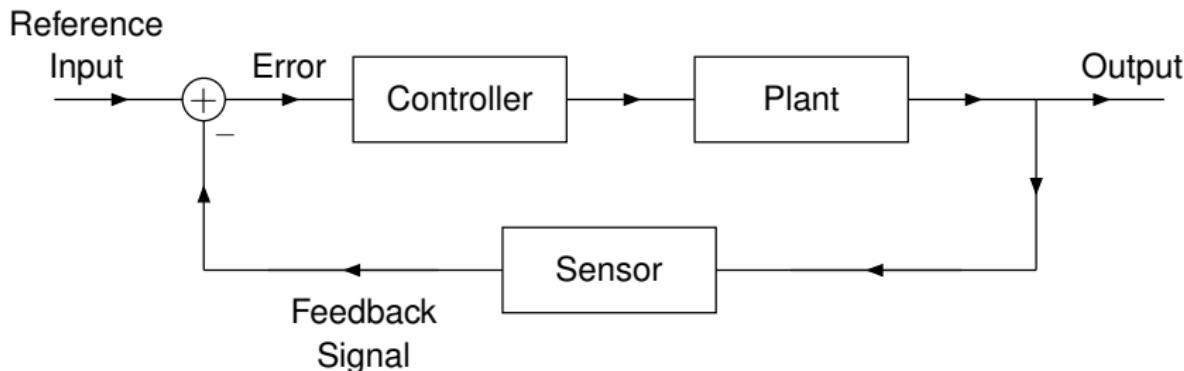
Section 7.7

Application: Design and Analysis of Control Systems

Control Systems

- A control system manages the behavior of one or more other systems with some specific goal.
- Typically, the goal is to force one or more physical quantities to assume particular desired values, where such quantities might include: positions, velocities, accelerations, forces, torques, temperatures, or pressures.
- The *desired* values of the quantities being controlled are collectively viewed as the *input* of the control system.
- The *actual* values of the quantities being controlled are collectively viewed as the *output* of the control system.
- A control system whose behavior is not influenced by the actual values of the quantities being controlled is called an **open loop** (or **non-feedback**) system.
- A control system whose behavior is influenced by the actual values of the quantities being controlled is called a **closed loop** (or **feedback**) system.
- An example of a simple control system would be a thermostat system, which controls the temperature in a room or building.
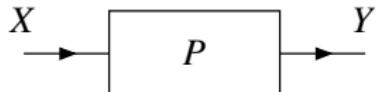
- **input**: *desired value* of the quantity to be controlled
- **output**: *actual value* of the quantity to be controlled
- **error**: *difference* between the desired and actual values
- **plant**: system to be controlled
- **sensor**: device used to measure the actual output
- **controller**: device that monitors the error and changes the input of the plant with the goal of forcing the error to zero

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.
- Therefore, the Laplace domain is extremely useful for the stability analysis of systems.

- causal LTI plant:

$$P(s) = \frac{10}{s-1}$$

- ROC of P:

- system is not BIBO stable

■ system formed by series interconnection of plant and causal LTI compensator:
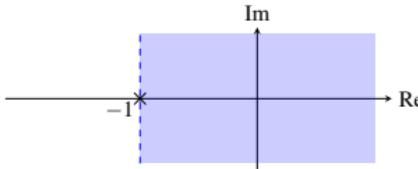
$$P(s) = \frac{10}{s-1}, \quad W(s) = \frac{s-1}{10(s+1)}$$

■ system function H of overall system:

$$H(s) = W(s)P(s) = \left(\frac{s-1}{10(s+1)} \right) \left(\frac{10}{s-1} \right) = \frac{1}{s+1}$$
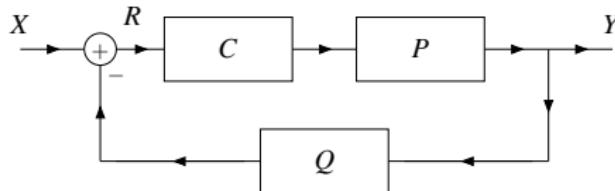
■ ROC of H:

■ overall system is BIBO stable

Stabilization Example: Using Feedback (1)

■ feedback system (with causal LTI compensator and sensor):
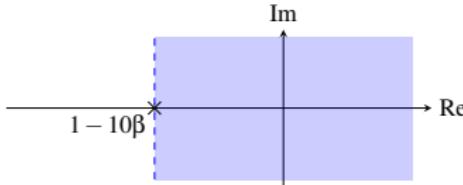
$$P(s) = \tfrac{10}{s-1}, \quad C(s) = \beta, \quad Q(s) = 1$$

■ system function H of feedback system:
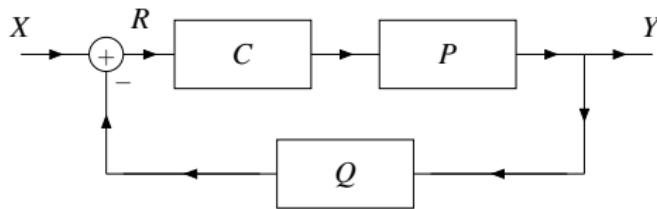
$$H(s) = \tfrac{C(s)P(s)}{1+C(s)P(s)Q(s)} = \tfrac{10\beta}{s-(1-10\beta)}$$

■ ROC of H:

■ feedback system is BIBO stable if and only if $1 - 10\beta < 0$ or equivalently $\beta > \tfrac{1}{10}$

$$R(s) = X(s) - Q(s)Y(s)$$
$$Y(s) = C(s)P(s)R(s)$$

$$
\begin{aligned}
Y(s) &= C(s)P(s)R(s) \\
&= C(s)P(s)[X(s) - Q(s)Y(s)] \\
&= C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s)
\end{aligned}
$$

$$[1 + C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s)$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}$$

$$P(s) = \frac{10}{s-1}, \quad C(s) = \beta, \quad Q(s) = 1$$

$$
\begin{aligned}
H(s) &= \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)} \\
&= \frac{\beta\left(\frac{10}{s-1}\right)}{1 + \beta\left(\frac{10}{s-1}\right)(1)} \\
&= \frac{10\beta}{s - 1 + 10\beta} \\
&= \frac{10\beta}{s - (1 - 10\beta)}
\end{aligned}
$$

Remarks on Stabilization Via Pole-Zero Cancellation

- Pole-zero cancellation is not achievable in practice, and therefore it cannot be used to stabilize real-world systems.
- The theoretical models used to represent real-world systems are only approximations due to many factors, including the following:
    - Determining the system function of a system involves measurement, which always has some error.
    - A system cannot be built with such precision that it will have exactly some prescribed system function.
    - The system function of most systems will vary at least slightly with changes in the physical environment.
    - Although a LTI model is used to represent a system, the likely reality is that the system is not exactly LTI, which introduces error.
- Due to approximation error, the effective poles and zeros of the system function will only be approximately where they are expected to be.
- Since pole-zero cancellation requires that a pole and zero be placed at exactly the same location, any error will prevent this cancellation from being achieved.

Section 7.8

Unilateral Laplace Transform

Unilateral Laplace Transform

- The **unilateral Laplace transform** of the function x, denoted $\mathcal{L}_u x$ or X, is defined as

$$\mathcal{L}_u x(s) = X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt.$$

- The unilateral Laplace transform is related to the bilateral Laplace transform as follows:

$$\mathcal{L}_u x(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} x(t) u(t) e^{-st} dt = \mathcal{L}\{xu\}(s).$$

- In other words, the unilateral Laplace transform of the function x is simply the bilateral Laplace transform of the function xu.

- Since $\mathcal{L}_u x = \mathcal{L}\{xu\}$ and xu is always a *right-sided* function, the ROC associated with $\mathcal{L}_u x$ is always either a *RHP* or the *entire complex plane*.

- For this reason, we often *do not explicitly indicate the ROC* when working with the unilateral Laplace transform.

- With the unilateral Laplace transform, the same inverse transform equation is used as in the bilateral case.
- The unilateral Laplace transform is *only invertible for causal functions*.
- In particular, we have

$$\mathcal{L}_{\mathsf{u}}^{-1}\{\mathcal{L}_{\mathsf{u}}x\}(t) = \mathcal{L}_{\mathsf{u}}^{-1}\{\mathcal{L}\{xu\}\}(t)$$
$$= \mathcal{L}^{-1}\{\mathcal{L}\{xu\}\}(t)$$
$$= x(t)u(t)$$
$$= \begin{cases} x(t) & t \geq 0 \\ 0 & t < 0. \end{cases}$$

- For a noncausal function x, we can only recover $x(t)$ for $t \geq 0$.

Unilateral Versus Bilateral Laplace Transform

- Due to the close relationship between the unilateral and bilateral Laplace transforms, these two transforms have some similarities in their properties.
- Since these two transforms are not identical, however, their properties differ in some cases, often in subtle ways.
- In the unilateral case, we have that:
    1. the time-domain convolution property has the additional requirement that the functions being convolved must be *causal*;
    2. the time/Laplace-domain scaling property has the additional constraint that the scaling factor must be *positive*;
    3. the time-domain differentiation property has an *extra term* in the expression for $\mathcal{L}_u\{\mathcal{D}x\}(t)$, where $\mathcal{D}$ denotes the derivative operator (namely, $-x(0^-)$);
    4. the time-domain integration property has a *different lower limit* in the time-domain integral (namely, 0^- instead of $-\infty$); and
    5. the time-domain shifting property *does not hold* (except in special circumstances).

Properties of the Unilateral Laplace Transform

Property	Time Domain	Laplace Domain
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(s) + a_2 X_2(s)$
Laplace-Domain Shifting	$e^{s_0 t} x(t)$	$X(s - s_0)$
Time/Laplace-Domain Scaling	$x(at), a > 0$	$\frac{1}{a} X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1 * x_2(t)$, x_1 and x_2 are causal	$X_1(s) X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt} x(t)$	$sX(s) - x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds} X(s)$
Time-Domain Integration	$\int_{0^-}^{t} x(\tau) d\tau$	$\frac{1}{s} X(s)$

Property	
Initial Value Theorem	$x(0^+) = \lim_{s \to \infty} sX(s)$
Final Value Theorem	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$

Unilateral Laplace Transform Pairs

Pair	$x(t), t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos(\omega_0 t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin(\omega_0 t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

- Many systems of interest in engineering applications can be characterized by constant-coefficient linear differential equations.

- One common use of the unilateral Laplace transform is in solving constant-coefficient linear differential equations with nonzero initial conditions.

Part 8

Complex Analysis

Complex Numbers

- A **complex number** is a number of the form $z = x + jy$ where x and y are real numbers and j is the constant defined by $j^2 = -1$ (i.e., $j = \sqrt{-1}$).

- The **Cartesian form** of the complex number z expresses z in the form

$$z = x + jy,$$

where x and y are real numbers. The quantities x and y are called the **real part** and **imaginary part** of z, and are denoted as $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.

- The **polar form** of the complex number z expresses z in the form
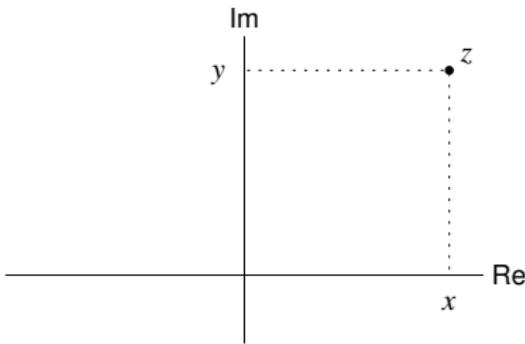
$$z = r(\cos\theta + j\sin\theta) \quad \text{or equivalently} \quad z = re^{j\theta},$$

where r and θ are real numbers and $r \geq 0$. The quantities r and θ are called the **magnitude** and **argument** of z, and are denoted as $|z|$ and $\arg z$, respectively. [Note: $e^{j\theta} = \cos\theta + j\sin\theta$.]

- Since $e^{j\theta} = e^{j(\theta + 2\pi k)}$ for all real θ and all integer k, the argument of a complex number is only uniquely determined to within an additive multiple of 2π.

- The **principal argument** of a complex number z, denoted $\text{Arg}\, z$, is the particular value θ of $\arg z$ that satisfies $-\pi < \theta \leq \pi$.

- The principal argument of a complex number (excluding zero) is *unique*.
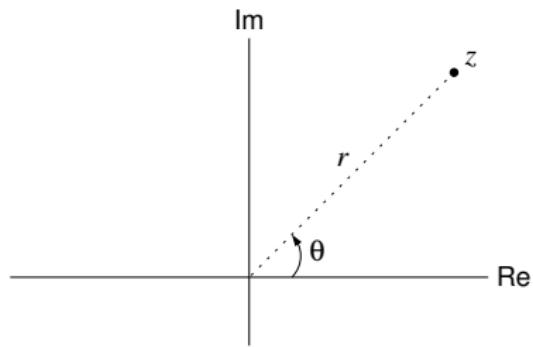
Cartesian form:

$z = x + jy$

where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$
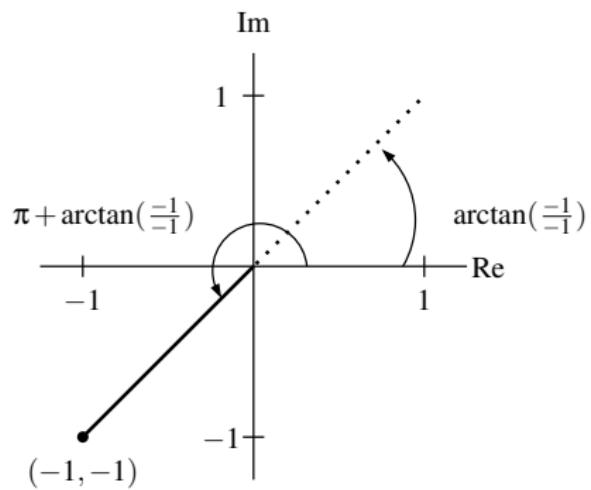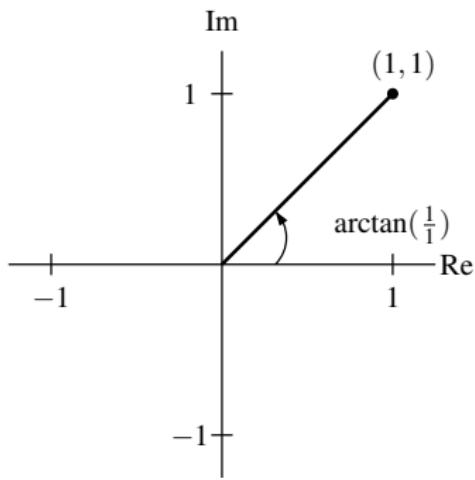
Polar form:

$z = r(\cos\theta + j\sin\theta) = re^{j\theta}$

where $r = |z|$ and $\theta = \arg z$

The arctan Function

- The range of the arctan function is $-\pi/2$ (exclusive) to $\pi/2$ (exclusive).
- Consequently, the arctan function always yields an angle in either the first or fourth quadrant.

The atan2 Function

- The angle θ that a vector from the origin to the point (x, y) makes with the positive x axis is given by $\theta = \text{atan2}(y, x)$, where

$$\text{atan2}(y, x) \triangleq \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \arctan(y/x) + \pi & x < 0 \text{ and } y \geq 0 \\ \arctan(y/x) - \pi & x < 0 \text{ and } y < 0. \end{cases}$$

- The range of the atan2 function is from $-\pi$ (exclusive) to π (inclusive).
- For the complex number z expressed in Cartesian form $x + jy$, $\text{Arg}\, z = \text{atan2}(y, x)$.
- Although the atan2 function is quite useful for computing the principal argument (or argument) of a complex number, it is *not advisable to memorize* the definition of this function. It is better to simply understand what this function is doing (namely, intelligently applying the arctan function).

Conversion Between Cartesian and Polar Form

- Let z be a complex number with the Cartesian and polar form representations given respectively by

$$z = x + jy \quad \text{and} \quad z = re^{j\theta}.$$

- To convert from *polar to Cartesian* form, we use the following identities:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

- To convert from *Cartesian to polar* form, we use the following identities:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \operatorname{atan2}(y, x) + 2\pi k,$$

where k is an arbitrary integer.

- Since the atan2 function simply amounts to the intelligent application of the arctan function, instead of memorizing the definition of the atan2 function, one should simply *understand* how to use the arctan function to achieve the same result.

Properties of Complex Numbers

- For complex numbers, addition and multiplication are *commutative*. That is, for any two complex numbers z_1 and z_2,

$$z_1 + z_2 = z_2 + z_1 \quad \text{and}$$
$$z_1 z_2 = z_2 z_1.$$

- For complex numbers, addition and multiplication are *associative*. That is, for any three complex numbers z_1, z_2, and z_3,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \text{and}$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- For complex numbers, the *distributive* property holds. That is, for any three complex numbers z_1, z_2, and z_3,
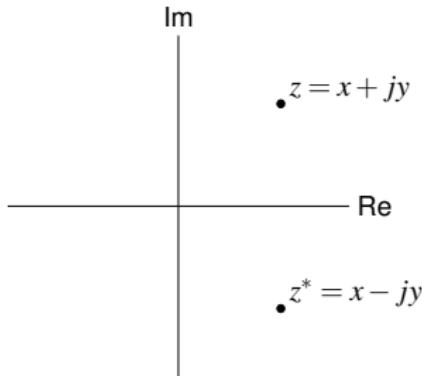
$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Conjugation

- The **conjugate** of the complex number $z = x + jy$ is denoted as z^* and defined as

$$z^* = x - jy.$$

- Geometrically, the conjugation operation reflects a point in the complex plane about the real axis.

- The geometric interpretation of the conjugate is illustrated below.

Properties of Conjugation

■ For every complex number z, the following identities hold:

$$|z^*| = |z|,$$
$$\arg z^* = -\arg z,$$
$$zz^* = |z|^2,$$
$$\operatorname{Re} z = \tfrac{1}{2}(z + z^*), \quad \text{and}$$
$$\operatorname{Im} z = \tfrac{1}{2j}(z - z^*).$$

■ For all complex numbers z_1 and z_2, the following identities hold:

$$(z_1 + z_2)^* = z_1^* + z_2^*,$$
$$(z_1 z_2)^* = z_1^* z_2^*, \quad \text{and}$$
$$(z_1/z_2)^* = z_1^*/z_2^*.$$

Addition

- *Cartesian form:* Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$
$$= (x_1 + x_2) + j(y_1 + y_2).$$

- That is, to add complex numbers expressed in Cartesian form, we simply add their real parts and add their imaginary parts.

- *Polar form:* Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$z_1 + z_2 = r_1 e^{j\theta_1} + r_2 e^{j\theta_2}$$
$$= (r_1 \cos\theta_1 + jr_1 \sin\theta_1) + (r_2 \cos\theta_2 + jr_2 \sin\theta_2)$$
$$= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + j(r_1 \sin\theta_1 + r_2 \sin\theta_2).$$

- That is, to add complex numbers expressed in polar form, we first rewrite them in Cartesian form, and then add their real parts and add their imaginary parts.

- For the purposes of addition, it is easier to work with complex numbers expressed in Cartesian form.

Multiplication

- *Cartesian form:* Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$$
$$= x_1 x_2 + jx_1 y_2 + jx_2 y_1 - y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1).$$

- That is, to multiply two complex numbers expressed in Cartesian form, we use the distributive law along with the fact that $j^2 = -1$.

- *Polar form:* Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$z_1 z_2 = \left(r_1 e^{j\theta_1} \right)\left(r_2 e^{j\theta_2} \right) = r_1 r_2 e^{j(\theta_1 + \theta_2)}.$$

- That is, to multiply two complex numbers expressed in polar form, we use exponent rules.

- For the purposes of multiplication, it is easier to work with complex numbers expressed in polar form.

Division

- **Cartesian form:** Let $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. Then,

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1 x_2 - jx_1 y_2 + jx_2 y_1 + y_1 y_2}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2 + j(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}.$$

- That is, to compute the quotient of two complex numbers expressed in Cartesian form, we convert the problem into one of division by a real number.

- **Polar form:** Let $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$. Then,

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}.$$

- That is, to compute the quotient of two complex numbers expressed in polar form, we use exponent rules.

- For the purposes of division, it is easier to work with complex numbers expressed in polar form.

Properties of the Magnitude and Argument

- For any complex numbers z_1 and z_2, the following identities hold:

$$|z_1 z_2| = |z_1| |z_2|,$$
$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{for } z_2 \neq 0,$$
$$\arg z_1 z_2 = \arg z_1 + \arg z_2, \quad \text{and}$$
$$\arg \left(\frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2 \quad \text{for } z_2 \neq 0.$$

- The above properties trivially follow from the polar representation of complex numbers.

■ **Euler's relation.** For all real θ,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

■ From Euler's relation, we can deduce the following useful identities:

$$\cos\theta = \tfrac{1}{2}(e^{j\theta} + e^{-j\theta}) \quad \text{and}$$
$$\sin\theta = \tfrac{1}{2j}(e^{j\theta} - e^{-j\theta}).$$

■ **De Moivre's theorem.** For all real θ and all *integer* n,

$$e^{jn\theta} = \left(e^{j\theta}\right)^n.$$

[Note: This relationship does not necessarily hold for *real* n.]

- Every complex number $z = re^{j\theta}$ (where $r = |z|$ and $\theta = \arg z$) has n distinct *nth roots* given by

$$\sqrt[n]{r}e^{j(\theta + 2\pi k)/n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

- For example, 1 has the two distinct square roots 1 and -1.

Quadratic Formula

- Consider the equation

$$az^2 + bz + c = 0,$$

where a, b, and c are real, z is complex, and $a \neq 0$.

- The roots of this equation are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- This formula is often useful in factoring quadratic polynomials.

- The quadratic $az^2 + bz + c$ can be factored as $a(z - z_0)(z - z_1)$, where

$$z_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Complex Functions

- A **complex function** maps complex numbers to complex numbers. For example, the function $F(z) = z^2 + 2z + 1$, where z is complex, is a complex function.
- A complex **polynomial function** is a mapping of the form

$$F(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where $z, a_0, a_1, \ldots, a_n$ are complex.

- A complex **rational function** is a mapping of the form

$$F(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m},$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ and z are complex.

- Observe that a polynomial function is a special case of a rational function.
- Herein, we will mostly focus our attention on polynomial and rational functions.

- A function F is said to be **continuous at a point** z_0 if $F(z_0)$ is defined and given by

$$F(z_0) = \lim_{z \to z_0} F(z).$$

- A function that is continuous at every point in its domain is said to be **continuous**.

- Polynomial functions are continuous everywhere.

- Rational functions are continuous everywhere except at points where the denominator polynomial becomes zero.

Differentiability

- A function F is said to be **differentiable at a point** $z = z_0$ if the limit

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0}$$

  exists. This limit is called the **derivative** of F at the point $z = z_0$.

- A function is said to be **differentiable** if it is differentiable at every point in its domain.

- The rules for differentiating sums, products, and quotients are the same for complex functions as for real functions. If $F'(z_0)$ and $G'(z_0)$ exist, then

  1. $(aF)'(z_0) = aF'(z_0)$ for any complex constant a;
  2. $(F + G)'(z_0) = F'(z_0) + G'(z_0)$;
  3. $(FG)'(z_0) = F'(z_0)G(z_0) + F(z_0)G'(z_0)$;
  4. $(F/G)'(z_0) = \frac{G(z_0)F'(z_0) - F(z_0)G'(z_0)}{G(z_0)^2}$; and
  5. if $z_0 = G(w_0)$ and $G'(w_0)$ exists, then the derivative of $F(G(z))$ at w_0 is $F'(z_0)G'(w_0)$ (i.e., the chain rule).

- A polynomial function is differentiable everywhere.

- A rational function is differentiable everywhere except at the points where its denominator polynomial becomes zero.
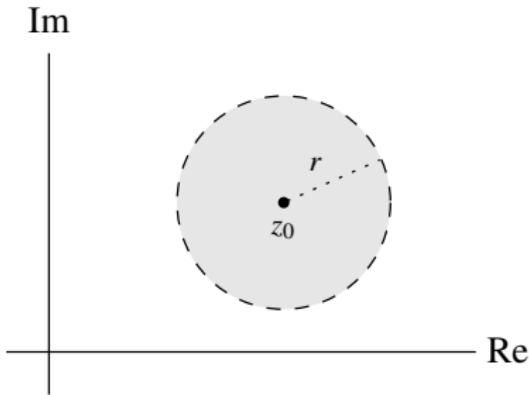
Open Disks

- An **open disk** in the complex plane with center z_0 and radius r is the set of complex numbers z satisfying

$$|z - z_0| < r,$$

where r is a strictly positive real number.

- A plot of an open disk is shown below.

- A function is said to be **analytic at a point** z_0 if it is differentiable at every point in an open disk about z_0.
- A function is said to be **analytic** if it is analytic at every point in its domain.
- A polynomial function is analytic everywhere.
- A rational function is analytic everywhere, except at the points where its denominator polynomial becomes zero.

## Zeros and Singularities

- If a function F is zero at the point z_0 (i.e., $F(z_0) = 0$), F is said to have a **zero** at z_0.
- If a function F is such that $F(z_0) = 0, F^{(1)}(z_0) = 0, \ldots, F^{(n-1)}(z_0) = 0$ (where $F^{(k)}$ denotes the kth order derivative of F), F is said to have an **nth order zero** at z_0.
- A point at which a function fails to be analytic is called a **singularity**.
- Polynomials do not have singularities.
- Rational functions can have a type of singularity called a pole.
- If a function F is such that $G(z) = 1/F(z)$ has an nth order zero at z_0, F is said to have an **nth order pole** at z_0.
- A pole of first order is said to be **simple**, whereas a pole of order two or greater is said to be **repeated**. A similar terminology can also be applied to zeros (i.e., **simple zero** and **repeated zero**).

- Given a rational function F, we can always express F in factored form as

$$F(z) = \frac{K(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_M)^{\alpha_M}}{(z-b_1)^{\beta_1}(z-b_2)^{\beta_2}\cdots(z-b_N)^{\beta_N}},$$

where K is complex, $a_1, a_2, \ldots, a_M, b_1, b_2, \ldots, b_N$ are distinct complex numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_M$ and $\beta_1, \beta_2, \ldots, \beta_N$ are strictly positive integers.

- One can show that F has *poles* at $b_1, b_2, \ldots, b_N$ and *zeros* at $a_1, a_2, \ldots, a_M$.

- Furthermore, the kth pole (i.e., b_k) is of *order* β_k, and the kth zero (i.e., a_k) is of *order* α_k.

- When plotting zeros and poles in the complex plane, the symbols "o" and "x" are used to denote zeros and poles, respectively.

Part 9

Partial Fraction Expansions (PFEs)

- Sometimes it is beneficial to be able to express a rational function as a sum of *lower-order* rational functions.
- This can be accomplished using a type of decomposition known as a partial fraction expansion.
- Partial fraction expansions are often useful in the calculation of inverse Laplace transforms, inverse z transforms, and inverse CT/DT Fourier transforms.

Strictly-Proper Rational Functions

- Consider a rational function

$$F(v) = \frac{\alpha_m v^m + \alpha_{m-1} v^{m-1} + \ldots + \alpha_1 v + \alpha_0}{\beta_n v^n + \beta_{n-1} v^{n-1} + \ldots + \beta_1 v + \beta_0}.$$

- The function F is said to be **strictly proper** if $m < n$ (i.e., the order of the numerator polynomial is strictly less than the order of the denominator polynomial).

- Through polynomial long division, any rational function can be written as the sum of a polynomial and a strictly-proper rational function.

- A *strictly-proper* rational function can be expressed as a sum of lower-order rational functions, with such an expression being called a partial fraction expansion.

Section 9.1

PFEs for First Form of Rational Functions

- Any rational function F can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_0}{v^n + b_{n-1} v^{n-1} + \ldots + b_0}.$$

- Furthermore, the denominator polynomial $D(v) = v^n + b_{n-1} v^{n-1} + \ldots + b_0$ in the above expression for $F(v)$ can be factored to obtain

$$D(v) = (v - p_1)^{q_1} (v - p_2)^{q_2} \cdots (v - p_n)^{q_n},$$

where the p_k are distinct and the q_k are integers.

- If F has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that F is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of F, there are *two cases* to consider:
    1. F has *only simple poles*; and
    2. F has *at least one repeated pole*.

- Suppose that the (rational) function F has only simple poles.
- Then, the denominator polynomial D for F is of the form

$$D(v) = (v - p_1)(v - p_2) \cdots (v - p_n),$$

where the p_k are distinct.

- In this case, F has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{v - p_1} + \frac{A_2}{v - p_2} + \ldots + \frac{A_{n-1}}{v - p_{n-1}} + \frac{A_n}{v - p_n},$$

where

$$A_k = (v - p_k)F(v)|_{v = p_k}.$$

- Note that the (simple) pole p_k contributes a single term to the partial fraction expansion.

- Suppose that the (rational) function F has at least one repeated pole.
- In this case, F has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{1,1}}{v - p_1} + \frac{A_{1,2}}{(v - p_1)^2} + \ldots + \frac{A_{1,q_1}}{(v - p_1)^{q_1}} \right]$$
$$+ \left[\frac{A_{2,1}}{v - p_2} + \ldots + \frac{A_{2,q_2}}{(v - p_2)^{q_2}} \right]$$
$$+ \ldots + \left[\frac{A_{P,1}}{v - p_P} + \ldots + \frac{A_{P,q_P}}{(v - p_P)^{q_P}} \right],$$

where

$$A_{k,\ell} = \frac{1}{(q_k - \ell)!} \left[\left[\tfrac{d}{dv} \right]^{q_k - \ell} [(v - p_k)^{q_k} F(v)] \right] \Big|_{v = p_k}.$$

- Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.
- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Section 9.2

PFEs for Second Form of Rational Functions

Partial Fraction Expansions (PFEs) [DT Context]

- Any rational function F can be expressed in the form of

$$F(v) = \frac{a_m v^m + a_{m-1} v^{m-1} + \ldots + a_1 v + a_0}{b_n v^n + b_{n-1} v^{n-1} + \ldots + b_1 v + 1}.$$

- Furthermore, the denominator polynomial
  $D(v) = b_n v^n + b_{n-1} v^{n-1} + \ldots + b_1 v + 1$ in the above expression for $F(v)$
  can be factored to obtain

$$D(v) = (1 - p_1^{-1} v)^{q_1} (1 - p_2^{-1} v)^{q_2} \cdots (1 - p_n^{-1} v)^{q_n},$$

  where the p_k are distinct and the q_k are integers.

- If F has only simple poles, $q_1 = q_2 = \cdots = q_n = 1$.
- Suppose that F is strictly proper (i.e., $m < n$).
- In the determination of a partial fraction expansion of F, there are *two cases* to consider:
  1. F has *only simple poles*; and
  2. F has *at least one repeated pole*.

- Suppose that the (rational) function F has only simple poles.
- Then, the denominator polynomial D for F is of the form

$$D(v) = (1 - p_1^{-1}v)(1 - p_2^{-1}v)\cdots(1 - p_n^{-1}v),$$

where the p_k are distinct.

- In this case, F has a partial fraction expansion of the form

$$F(v) = \frac{A_1}{1 - p_1^{-1}v} + \frac{A_2}{1 - p_2^{-1}v} + \ldots + \frac{A_{n-1}}{1 - p_{n-1}^{-1}v} + \frac{A_n}{1 - p_n^{-1}v},$$

where

$$A_k = (1 - p_k^{-1}v)F(v)\big|_{v=p_k}.$$

- Note that the (simple) pole p_k contributes a single term to the partial fraction expansion.

Repeated-Pole Case [DT Context]

- Suppose that the (rational) function F has at least one repeated pole.

- In this case, F has a partial fraction expansion of the form

$$F(v) = \left[\frac{A_{1,1}}{1 - p_1^{-1}v} + \frac{A_{1,2}}{(1 - p_1^{-1}v)^2} + \ldots + \frac{A_{1,q_1}}{(1 - p_1^{-1}v)^{q_1}} \right]$$
$$+ \left[\frac{A_{2,1}}{1 - p_2^{-1}v} + \ldots + \frac{A_{2,q_2}}{(1 - p_2^{-1}v)^{q_2}} \right]$$
$$+ \ldots + \left[\frac{A_{P,1}}{1 - p_P^{-1}v} + \ldots + \frac{A_{P,q_P}}{(1 - p_P^{-1}v)^{q_P}} \right],$$

where

$$A_{k,\ell} = \frac{1}{(q_k - \ell)!} (-p_k)^{q_k - \ell} \left[\left[\tfrac{d}{dv} \right]^{q_k - \ell} [(1 - p_k^{-1}v)^{q_k} F(v)] \right] \Bigg|_{v = p_k}.$$

- Note that the q_kth-order pole p_k contributes q_k terms to the partial fraction expansion.

- Note that $n! = (n)(n-1)(n-2)\cdots(1)$ and $0! = 1$.

Part 10

Miscellany

Sum of Arithmetic and Geometric Sequences

- The sum of the arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$ is given by

$$\sum_{k=0}^{n-1} (a+kd) = \frac{n[2a+d(n-1)]}{2}.$$

- The sum of the geometric sequence $a, ra, r^2a, \ldots, r^{n-1}a$ is given by

$$\sum_{k=0}^{n-1} r^k a = a\frac{r^n - 1}{r - 1} \quad \text{for } r \neq 1.$$

- The sum of the infinite geometric sequence $a, ra, r^2a, \ldots$ is given by
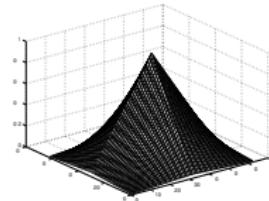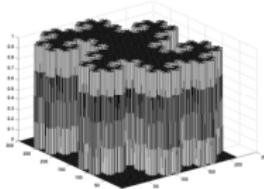
$$\sum_{k=0}^{\infty} r^k a = \frac{a}{1-r} \quad \text{for } |r| < 1.$$

Part 11

Epilogue

Other Courses Offered by the Author of These Lecture Slides

- If you did not suffer permanent emotional scarring as a result of using these lecture slides and you happen to be a student at the University of Victoria, you might wish to consider taking another one of the courses developed by the author of these lecture slides:
  - *ECE 486: Multiresolution Signal and Geometry Processing with C++*
  - *SENG 475: Advanced Programming Techniques for Robust Efficient Computing*
- For further information about the above courses (including the URLs for web sites of these courses), please refer to the slides that follow.

ECE 486/586:
Multiresolution Signal and Geometry Processing with C++

- normally offered in Summer (May-August) term; only prerequisite ECE 310
- subdivision surfaces and subdivision wavelets
    - 3D computer graphics, animation, gaming (Toy Story, Blender software)
    - geometric modelling, visualization, computer-aided design
- multirate signal processing and wavelet systems
    - sampling rate conversion (audio processing, video transcoding)
    - signal compression (JPEG 2000, FBI fingerprint compression)
    - communication systems (transmultiplexers for CDMA, FDMA, TDMA)
- C++ (classes, templates, standard library), OpenGL, GLUT, CGAL
- software applications (using C++)
- for more information, visit course web page:
  http://www.ece.uvic.ca/~mdadams/courses/wavelets

SENG 475:
Advanced Programming Techniques for Robust Efficient Computing (With C++)

- advanced programming techniques for robust efficient computing explored in context of C++ programming language
- topics covered may include:
  - concurrency, multithreading, transactional memory, parallelism, vectorization; cache-efficient coding; compile-time versus run-time computation; compile-time versus run-time polymorphism; generic programming techniques; resource/memory management; copy and move semantics; exception-safe coding
- applications areas considered may include:
  - geometry processing, computer graphics, signal processing, and numerical analysis
- open to any student with necessary prerequisites, which are:
  - SENG 265 or CENG 255 or CSC 230 or CSC 349A or ECE 255 or permission of Department
- for more information, see course web site:
    http://www.ece.uvic.ca/~mdadams/courses/cpp

Part 12

References

Online Resources I

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3. Iman Moazzen. YouTube Channel for Signal Processing Hacks With Iman. https://www.youtube.com/channel/UCVkatNMgkEdpWLhH0kBqqLw.

4. Wolfram Alpha Derivative Calculator. https://www.wolframalpha.com/input/?i=derivative+.

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6. Wolfram Alpha Unilateral Laplace Transform Calculator. https://www.wolframalpha.com/input/?i=laplace+transform+calculator.

7. Wolfram Alpha Unilateral Z Transform Calculator. https://www.wolframalpha.com/input/?i=Z+transform+calculator.

8. DSP Stack Exchange. https://dsp.stackexchange.com.

9. Math Stack Exchange. https://math.stackexchange.com.