Penalty Convex-Concave Procedure for Source Localization Problem

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***Abstract*—In this paper, we focus on the least-squares (LS) formulation for the localization problem, where the l_2-norm of the residual errors is minimized in a setting known as difference-of-convex-functions programming. The problem at hand is then solved by applying a penalty convex-concave procedure (PCCP) in a successive manner. Algorithmic details that are tailored to the localization problem, such as imposing additional constraints to enforce iteration path towards the LS solution and strategies to secure a good initial point, are also provided. Simulation results demonstrate promising localization performance when compared with some best known results from the literature.**

***Index Terms*—least squares; non-convex; convex-concave procedure; CCP; source localization; range measurements.**

I. INTRODUCTION

Locating a radiating source from range measurements in a passive sensor network has recently attracted an increasing amount of research interest as it finds applications in a wide range of network-based wireless systems. Least squares (LS) based algorithms for source localization problems constitute an important class of solution techniques as they are geometrically meaningful and often provide low complexity solution procedures with competitive estimation accuracy [1]-[7]. On the other hand, the error measure in an LS formulation for the localization problem of interest is shown to be highly non-convex, possessing multiple local solutions with degraded performance. This non-convexity excludes many local methods that are iterative, hence extremely sensitive to where the iteration begins. Several non-iterative *global* localization techniques are available from the literature. A global solution may be obtained by relaxing the LS model at hand to a semidefinite programming (SDP) problem which is known to be convex [10]. In doing so, however, the convexification based solution is no longer optimal in LS sense. Another representative in this class is the method proposed in [7], where localization problems for range measurements are addressed by developing solution methods for *squared* range LS (SR-LS) problems. Although these methods are efficient in terms of complexity, they remain to be suboptimal in the maximum likelihood (ML) sense because the solutions produced are merely approximations of the ML estimate.

In this paper, we focus on LS formulation for the problem of localizing a single radiating source based on range measurements. We exploit special structure of the cost function of an unconstrained LS formulation and show that it is well suited for being investigated in a setting known as difference-of-convex-functions (DC) programming. Further, we present an algorithm for solving the LS problem at hand based on a penalty convex-concave procedure (PCCP) [9] that accommodates infeasible initial points. We also provide algorithmic details that are tailored to the localization problem at hand, these include additional constraints that enforce the algorithms iteration path towards the LS solution and strategies to secure good initial points for the algorithm. Numerical results are presented to demonstrate that the proposed algorithm offers substantial performance improvement relative to some best known results from the literature.

II. PROBLEM STATEMENT AND REVIEW OF RELATED WORK

The source localization problem considered here involves a given array of m sensors specified by $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ where $\boldsymbol{a}_i \in R^n$ contains n coordinates of the ith sensor in space R^n. Each sensor measures its distance to a radiating source $\boldsymbol{x} \in R^n$. Throughout it is assumed that only noisy copies of the distance data are available, hence the *range measurements* obey the model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m. \tag{1}$$

where ε_i denotes the unknown noise that has occurred when the ith sensor measures its distance to source $\boldsymbol{x}$. Let $\boldsymbol{r} = [r_1\ r_2 \ldots r_m]^T$ and $\boldsymbol{\varepsilon} = [\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_m]^T$, the source localization problem can be stated as to estimate the exact source location $\boldsymbol{x}$ from the noisy range measurements $\boldsymbol{r}$. For the localization problem at hand, the range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \tag{2}$$

Formulation (2) is connected to the maximum-likelihood (ML) location estimation that determines $\boldsymbol{x}$ by examining the probabilistic model of the error vector $\boldsymbol{\varepsilon}$. If $\boldsymbol{\varepsilon}$ obeys a Gaussian distribution with zero mean and covariance $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then the maximum likelihood (ML) location estimator in this case is known to be

$$\boldsymbol{x}_{ML} = \arg \min_{\boldsymbol{x} \in R^n} (\boldsymbol{r} - \boldsymbol{g})^T \Sigma^{-1} (\boldsymbol{r} - \boldsymbol{g}) \tag{3}$$

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where $\boldsymbol{g} = [g_1\ g_2 \ldots\ g_m]^T$ with $g_i = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. It follows immediately that the ML solution in (3) is identical to the R-LS solution of problem (2) when covariance $\boldsymbol{\Sigma}$ is proportional to the identity matrix, i.e., $\sigma_1^2 = \ldots = \sigma_m^2 = 1$. In the literature this is known as the equal noise power case. For notation simplicity this paper focuses on the equal noise power case, however the method developed below is also applicable to the unequal noise power case by working on a weighted version of the objective in (2) with $\{\sigma_i^{-2}, i = 1, \ldots, m\}$ as the weights.

There are many methods for continuous unconstrained optimization [11], however most of them are *local* methods in the sense they are sensitive to the choice of initial point, and give no guarantee to yield global solutions when applied to non-convex objective functions. Unfortunately, the objective function in (2) is highly non-convex, possessing many local minimizers even for small-scale systems. In this paper we present an different approach to solve the positioning problem, which employs a successive convex-concave procedure.

III. Fitting the Localization Problem to the CCP Framework

A. Basic Convex-Concave Procedure

The CCP refers to an effective heuristic method to deal with a class of *nonconvex* problems of the form

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x}) - g(\boldsymbol{x}) \tag{4a}$$

$$\text{subject to:} \quad f_i(\boldsymbol{x}) \le g_i(\boldsymbol{x}) \quad \text{for: } i = 1, 2, \ldots, m \tag{4b}$$

where $f(\boldsymbol{x}), g(\boldsymbol{x}), f_i(\boldsymbol{x}), g_i(\boldsymbol{x})$ for $i = 1, 2, \ldots, m$ are convex. The basic CCP algorithm is an iterative procedure including two key steps (in the k-th iteration where iterate $\boldsymbol{x}_k$ is known):

(i) Convexification of the objective function and constraints by replacing $g(\boldsymbol{x})$ and $g_i(\boldsymbol{x})$, respectively, with their affine approximations

$$\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) = g(\boldsymbol{x}_k) + \bigtriangledown g(\boldsymbol{x}_k)^T (\boldsymbol{x} - \boldsymbol{x}_k) \tag{5a}$$

and

$$\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = g_i(\boldsymbol{x}_k) + \bigtriangledown g_i(\boldsymbol{x}_k)^T (\boldsymbol{x} - \boldsymbol{x}_k) \quad \text{for: } i = 1, 2, \ldots, m \tag{5b}$$

(ii) Solving the convex problem

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x}) - \hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) \tag{6a}$$

$$\text{subject to:} \quad f_i(\boldsymbol{x}) - \hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) \le 0 \quad \text{for: } i = 1, 2, \ldots, m \tag{6b}$$

Because of the convexity of all the functions involved, it can be shown that the basic CCP is a descent algorithm and the iterates $\boldsymbol{x}_k$ converge to the critical point of the original problem (4) [9]. The basic CCP requires a *feasible* initial point $\boldsymbol{x}_0$ (in the sense that $\boldsymbol{x}_0$ satisfies (6b) for $i = 1, 2, \ldots, m$) to start the procedure. By introducing additional slack variables, a penalty CCP has been adopted to accept infeasible initial points [12].

B. Problem Reformulation

We begin by re-writing the objective function in (2) up to a constant as:

$$F(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T \sum_{i=1}^{m} \boldsymbol{a}_i - 2\sum_{i=1}^{m} r_i \|\boldsymbol{x} - \boldsymbol{a}_i\| \tag{7}$$

The objective in (7) is not convex. This is because, for points $\boldsymbol{x}$ that are not coincided with $\boldsymbol{a}_i$ for $1 \le i \le m$, the Hessian of $F(\boldsymbol{x})$ is given by

$$\bigtriangledown^2 F(\boldsymbol{x}) = 2m\boldsymbol{I} + 2\sum_{i=1}^{m} \frac{r_i}{\|\boldsymbol{x} - \boldsymbol{a}_i\|^3} \cdot \cdot \left((\boldsymbol{x} - \boldsymbol{a}_i)(\boldsymbol{x} - \boldsymbol{a}_i)^T - \|\boldsymbol{x} - \boldsymbol{a}_i\|^2 \boldsymbol{I} \right)$$

which is not always positive semidefinite. On the other hand, by defining

$$f(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T \sum_{i=1}^{m} \boldsymbol{a}_i, \qquad g(\boldsymbol{x}) = 2\sum_{i=1}^{m} r_i \|\boldsymbol{x} - \boldsymbol{a}_i\| \tag{8}$$

the objective in (7) can be expressed as $F(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$ with both $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ convex, hence it fits naturally into (4a). Note that $g(\boldsymbol{x})$ in (8) is not differentiable at the point where $\boldsymbol{x} = \boldsymbol{a}_i$ for some $1 \le i \le m$, thus we replace the term $\bigtriangledown g(\boldsymbol{x}_k)$ in (5a) by a subgradient [13] of $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$, denoted by $\partial g(\boldsymbol{x}_k)$ as

$$\partial g(\boldsymbol{x}_k) = 2\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

where

$$\partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| = \begin{cases} \dfrac{\boldsymbol{x}_k - \boldsymbol{a}_i}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}, & \text{if } \boldsymbol{x}_k \ne \boldsymbol{a}_i \\ \boldsymbol{0}, & \text{otherwise} \end{cases}$$

Hence $\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k)$ in (5a) is given by

$$\begin{aligned} \hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) &= 2\sum_{i=1}^{m} r_i \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + 2(\boldsymbol{x} - \boldsymbol{x}_k)^T \sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| \\ &= 2\boldsymbol{x}^T \sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + c \end{aligned}$$

where c is a constant given by

$$c = -2\sum_{i=1}^{m} r_i \boldsymbol{a}_i^T \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|.$$

It follows that up to a multiplicative factor $1/m$ and an additive constant term the convex objective function in (6a) can be written as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \hat{F}(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{v}_k \tag{9}$$

where

$$\boldsymbol{v}_k = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|, \quad \bar{\boldsymbol{a}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{a}_i \tag{10}$$

It is rather straightforward to see that given $\boldsymbol{x}_k$ (in the k-th iteration) the solution of the quadratic problem (9) can be obtained as

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\| \tag{11}$$

C. Imposing Error Bounds and Penalty Terms

The algorithm being developed can be enhanced by imposing a bound on each squared measurement error, namely

$$(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i)^2 \le \delta_i^2 \tag{12}$$

which leads to

$$\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le 0 \tag{13a}$$

$$r_i - \delta_i \le \|\boldsymbol{x} - \boldsymbol{a}_i\| \tag{13b}$$

for $1 \le i \le m$. The constraints in (13a) are convex and fit into those in (6b) with $f_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i$ and $g_i(\boldsymbol{x}) = 0$, while those in (13b) are in the form of (4b) with $f_i(\boldsymbol{x}) = r_i - \delta_i$ and $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. Following CCP (see (5b)), $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$ is linearized around iterate $\boldsymbol{x}_k$ to

$$\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

and (13b) is convexified as

$$r_i - \delta_i \le \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

which now fits into (6b), or equivalently

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \le 0 \tag{14}$$

We remark that constraint (14) is not only convex but also tighter than (13b). As a matter of fact, the convexity of the norm $\|\boldsymbol{x} - \boldsymbol{a}_i\|$ implies that it obeys the property

$$\|\boldsymbol{x} - \boldsymbol{a}_i\| \ge \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

Therefore, a point $\boldsymbol{x}$ satisfying (14) automatically satisfies (13b). Summarizing, the convexified problem in the k-th iteration can be stated as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k \tag{15a}$$

$$\text{subject to:} \quad \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le 0 \tag{15b}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial\|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \le 0 \tag{15c}$$

A technical problem making the formulation in (15) difficult to implement is that it requires a feasible initial point $\boldsymbol{x}_0$. The problem can be overcome by introducing nonnegative slack variables $s_i \ge 0, \hat{s}_i \ge 0$, for $i = 1, \ldots, m$ into the constraints in (15b) and (15c) to replace their right-hand sides (which are zeros) by relaxed upper bounds (as these new bounds themselves are nonnegative variables). This leads to a *penalty* CCP (PCCP) based formulation as follows:

$$\underset{\boldsymbol{x},\boldsymbol{s},\hat{\boldsymbol{s}}}{\text{minimize}} \quad \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k + \tau_k \sum_{i=1}^{m}(s_i + \hat{s}_i) \tag{16a}$$

$$\text{subject to:} \quad \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le s_i \tag{16b}$$

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \frac{(\boldsymbol{x}_k - \boldsymbol{a}_i)^T}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}(\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \le \hat{s}_i \tag{16c}$$

$$s_i \ge 0, \hat{s}_i \ge 0, \text{ for: } i = 1, 2, \ldots, m \tag{16d}$$

where the weight $\tau_k \ge 0$ increases as iterations proceed until it reaches an upper limit τ_{max}. By using a monotonically increasing τ_k for the penalty term in (16a), the algorithm reduces the slack variables s_i and $\hat{s}_i$ very quickly. As a result, new iterates quickly become feasible as s_i and $\hat{s}_i$ vanish. The upper limit τ_{max} is imposed to avoid numerical difficulties that may occur if τ_k becomes too large and to ensure convergence if a feasible region is not found [9]. Consequently, while formulation (16) accepts *infeasible* initial points, the iterates obtained by solving (16) are practically identical to those obtained by solving (15).

D. The Algorithm

The input parameters for the algorithm include the bound δ_i on the measurement error. Setting δ_i to a lower value leads to a "tighter" solution. On the other hand, a larger δ_i would make the algorithm less sensitive to outliers. If measurement noise $\boldsymbol{\varepsilon}$ obeys a Gaussian distribution with zero mean and known covariance $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then δ_i can be expressed as $\delta_i = \gamma\sigma_i$, where γ is a parameter that determines the width of confidence interval. For example, for $\gamma = 3$ we have the probability $Pr\{|\varepsilon_i| \le 3\sigma_i\} \approx 0.99$. Other input parameters are initial point $\boldsymbol{x}_0$, maximum number of iterations K_{max}, initial weight τ_0, and upper limit of weight τ_{max} (to avoid numerical problems that may occur if τ_i becomes too large).

As mentioned in Sec. 2, the original LS objective is highly non-convex with many local minimums even for small-scale systems. Consequently, it is of critical importance to select a good initial point for the proposed PCCP-based algorithm because PCCP is essentially a local procedure. Several techniques are available, these include: (i) Select the initial point uniformly randomly over the same region as the unknown radiating source; (ii) Set the initial point to the origin; (iii) Run the algorithm from a set of candidate initial points and identify the solution as the one with lowest LS error. Typically, comparing the results from n distinct initial points shall suffice. For the planar case ($n = 2$), for example, it is sufficient to compare the two intersection points of the two circles that are associated with the two smallest distance readings as the target is very likely to be in the vicinity of these sensors; and (iv) Apply a global localization algorithm such as those in [7] to generate an approximate LS solution, then take it as the initial point to run the proposed algorithm. The algorithm can be now outlined as follows.

PCCP-based LS Algorithm for Source Localization

Step 1: Input sensor locations $\{\boldsymbol{a}_i, i = 1, \ldots, m\}$, range measurements $\{r_i, i = 1, \ldots, m\}$, $\boldsymbol{x}_0, K_{max}, \tau_0, \tau_{max}, \mu > 0, \gamma, \sigma$, and set $k = 0$.

Step 2: Form $\boldsymbol{v}_k$ as in (10) and solve (16). Denote the solution as $(\boldsymbol{s}^*, \hat{\boldsymbol{s}}^*, \boldsymbol{x}^*)$.

Step 3: Update τ_{k+1} = min $(\mu\tau_k, \tau_{max})$, set $k = k + 1$.

Step 4: If $k = K_{max}$, terminate and output $\boldsymbol{x}^*$ as the solution; otherwise, set $\boldsymbol{x}_k = \boldsymbol{x}^*$ and repeat from Step 2.

IV. Numerical Results

For illustration purposes, the proposed algorithm was applied to a network with five sensors, and its performance was evaluated and compared with existing state-of-the-art methods by Monte Carlo simulations with a set-up similar to that of [7]. SR-LS solutions were used as performance benchmarks for the PCCP-based LS Algorithm. The system consisted of 5 sensors $\{\boldsymbol{a}_i, i = 1, 2, \ldots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$, and a radiating source $\boldsymbol{x}_s$, located randomly in the region $\{\boldsymbol{x} = [x_1; x_2], -10 \leq x_1, x_2 \leq 10\}$. The coordinates of the source and sensors were generated for each dimension following a uniform distribution. Measurement noise $\{\varepsilon_i, i = 1, \ldots, m\}$ was modelled as independent and identically distributed (i.i.d) random variables with zero mean and variance σ^2, with σ being one of four possible levels $\{10^{-3}, 10^{-2}, 10^{-1}, 1\}$. The range measurements $\{r_i, i = 1, 2, \ldots, 5\}$ were calculated using (1). Accuracy of source location estimation was evaluated in terms of average of the squared position error error in the form $\|\boldsymbol{x}^* - \boldsymbol{x}_s\|^2$, where $\boldsymbol{x}_s$ denotes the exact source location and $\boldsymbol{x}^*$ is its estimation obtained by SR-LS and PCCP methods, respectively. In our simulations parameter γ was set to 3 and the number of iterations was set to 20. The proposed method was implemented by using CVX [14] and implementation of SR-LS followed [7]. The PCCP algorithm was initialized with intersection points of the two circles that are associated with the two smallest distance readings. A candidate solution point with lowest LS error in (2) was chosen as a PCCP solution. In cases when the circles did not intersect due to high noise level, the initial point was set as a midpoint between the centers of the two circles.

Table I provides comparisons of the PCCP with SR-LS and MLE, where each entry is averaged squared error over 1,000 Monte Carlo runs of the method. The MLE was implemented using Matlab function *lsqnonlin* [15], initialized with the same point as PCCP. It is observed that, comparing with SR-LS, the estimates produced by the proposed algorithm are found to be closer to the true source locations in MSE sense. The last column of the table represents relative improvement of the proposed method over SR-LS solutions in percentage.

TABLE I
Averaged MSE for SR-LS and PCCP methods

σ	MLE	SR - LS	PCCP	R.I.
1e-03	6.0159e-01	1.3394e-06	**9.5243e-07**	29%
1e-02	3.5077e-01	1.4516e-04	**9.5831e-05**	34%
1e-01	3.7866e-01	1.2058e-02	**8.7107e-03**	28%
1e+0	1.4470e+00	1.3662e+00	**1.2346e+00**	10%

V. Conclusion

In this paper, a new iterative method for locating a radiating source based on noisy range measurements have been proposed. The method is developed by transforming the original least-squares problem to a difference-of-convex-functions programming problem which is in turn relaxed to a sequential convex minimization based on PCCP that can be efficiently solved with an infeasible initial point. Along the way, we see that CCP allows a natural embedding of the LS formulation for localization into a sequential convex formulation in that no additional terms and functions are introduced into the procedure up until (15). Numerical results are presented to illustrate the proposed algorithms in comparison with the state-of-the-art results from the literature.

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