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Balanced approximation of two-dimensional and delay-differential systems

W.-S. LU†, E. B. LEE† and Q.-T. ZHANG†

A generalized balanced approximation method for reducing two-dimensional (2-D) and delay-differential models is given. This is possible because of the form the gramians of such systems take. In particular it is shown that the reachability and observability gramians can be expressed as integrals of a certain system-related function along the imaginary axis or unit circle of the complex plane. The gramians provide two sets of system invariants which lead in a natural way to the concept of a generalized balanced realization, which in turn leads to a way to accomplish model reduction.

1. Introduction

Approximating a model of a linear dynamical system by a lower-order one can considerably simplify various analysis and synthesis procedures. The appearance of the balanced approximation method of model reduction initiated by Moore (1978) has proved to be most significant because of its desirable properties such as good error bounds, computational simplicity, stability, and its close connection to robust multivariable control (Laub 1980, Pernebo and Silverman 1982, Glover 1984, Glover and Limebeer 1983, Doyle 1984).

However, for infinite-dimensional systems such as two-dimensional (2-D) systems and delay-differential systems, results on model reduction are only just starting to appear (see Paraskevopoulos 1983, Lee *et al.* 1985 and Jury and Premaratne 1986, for example). Recently, some work on balanced representations for certain infinite dimensional systems has been reported (Curtain and Glover 1986) with a finite dimensional approximation scheme to provide the reduced-order model. In our approach, we try to retain the original form of the model (if we have a differential-difference equation (DDE) we try to obtain a DDE model approximation) in the reduction, but one that involves fewer parameters.

In this paper, we give a generalized balanced approximation method for reducing 2-D and delay-differential models. To this end, the concept of a balanced realization is extended to these systems in a natural manner, which is made possible through the use of certain complex integral representations of the gramians of the systems considered. Some preliminaries are given in the next section. In particular, we show that the reachability and the observability gramians of a linear time-invariant system can be expressed as complex integrals of certain system-related functions along the imaginary axis or the unit circle, depending on whether the system considered is continuous or discrete-time. In § 3, for the Roesser model of a stable 2-D discrete system, generalized gramians are defined in terms of double integrals of similar system-related functions on the unit bi-circle. These gramians provide two sets of system invariants,

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leading naturally to the concept of a generalized balanced realization. To obtain a 2-D balancing transformation, the upper-left and the lower-right submatrices of the gramians should be positive definite. Several results regarding their computation and positive definiteness are given. Based on these observations, a balanced approximation method is proposed. Section 4 is devoted to an analogous study for delay-differential systems. A key point for reducing 2-D and delay system models using the balanced approximation method is to compute the upper-left and the lower-right parts of the associated gramians. In § 5, a detailed analysis on this issue is carried out using a Lyapunov approach. Two examples are provided in § 6 to illustrate the main results.

2. Preliminaries

Consider a minimal realization of a stable discrete-time system as represented by the difference equations

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}\right\} \quad k = 0, 1, \ldots \tag{1}$$

with $x(k) \in R^n$, $u(k) \in R^m$ and $y(k) \in R^p$.

Define the reachability and observability gramians by

$$K_d = \sum_{i=0}^{\infty} A^i BB^{\mathrm{T}} (A^{\mathrm{T}})^i \quad \text{and} \quad W_d = \sum_{i=0}^{\infty} (A^{\mathrm{T}})^i C^{\mathrm{T}} C A^i \tag{2}$$

respectively. An asymptotically stable system is said to be balanced whenever

$$K_d = W_d = \Sigma = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\}, \quad \sigma_1 \geqslant \ldots \geqslant \sigma_n > 0 \tag{3}$$

Assume that System (1) has been balanced. We may regard the gramians as certain measures of input-to-state and state-to-output couplings, and the balanced realization provides the system a coordinate setting where these couplings are equally weighted so that those state components which are weakly coupled may be discarded (Harshavardhana *et al.* 1983). Actually, when

$$\sigma_1 \geqslant \ldots \geqslant \sigma_r \gg \sigma_{r+1} \geqslant \ldots \geqslant \sigma_n > 0$$

we partition A, B and C of (1) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \text{and} \quad C = [C_1 \quad C_2]$$

with $A_{11} \in R^{r \times r}$, $B_1 \in R^{r \times m}$ and $C_1 \in R^{p \times r}$, and thereby obtain a reduced system (A_{11}, B_1, C_1). It turns out (Glover 1984) that such a lower-order system represents a good approximation of the original one in terms of the L^{∞}-norm.

In the continuous case, let (F, G, H) be a minimal realization of a stable linear continuous system of dimension n. Define the reachability and the observability gramians by

$$K_c = \int_0^{\infty} \exp(Ft)\, GG^{\mathrm{T}} \exp(F^{\mathrm{T}} t)\, dt \quad \text{and} \quad W_c = \int_0^{\infty} \exp(F^{\mathrm{T}} t)\, H^{\mathrm{T}} H \exp(Ft)\, dt \tag{4}$$

respectively. The balanced realization of (F, G, H) can be defined similarly, and the above reduction procedure also applies.

Given a minimal realization of a linear time-invariant system, a non-singular coordinate (1-D similarity) transformation in the state space is said to be a balancing transformation if the transformed realization is balanced. As is seen in the works of Moore (1978) and Laub (1980), a key step for obtaining a balancing transformation is to compute the gramians of the system considered. One way of doing so is to solve the following Lyapunov equations

$$AK_dA^{\mathrm{T}} - K_d = -BB^{\mathrm{T}} \tag{5 a}$$

$$A^{\mathrm{T}}W_dA - W_d = -C^{\mathrm{T}}C \tag{5 b}$$

$$FK_c + K_cE^{\mathrm{T}} = -GG^{\mathrm{T}} \tag{6 a}$$

$$F^{\mathrm{T}}W_c + W_cF = -H^{\mathrm{T}}H \tag{6 b}$$

An efficient and reliable algorithm for solving (6) can be found in the work of Bartels and Stewart (1972). In Appendix A, an algorithm for computing a balancing transformation given by Laub (1980) is listed.

Alternatively these gramians can also be obtained through certain complex integrals. To do this, let us define

$$f_d(z) = (zI - A)^{-1}B \tag{7 a}$$

$$g_d(z) = C(zI - A)^{-1} \tag{7 b}$$

$$f_c(s) = (sI - F)^{-1}G \tag{8 a}$$

$$g_c(s) = H(sI - F)^{-1} \tag{8 b}$$

and then the gramians can be found as follows.

Theorem 1

Let K_d, W_d, K_c and W_c be given as in (2) and (4), then

$$K_d = \frac{1}{2\pi j}\int_{|z|=1} f_d(z)f_d^*(z)\frac{dz}{z} \tag{9 a}$$

$$W_d = \frac{1}{2\pi j}\int_{|z|=1} g_d^*(z)g_d(z)\frac{dz}{z} \tag{9 b}$$

$$K_c = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} f_c(s)f_c^*(s)\,ds \tag{10 a}$$

and

$$W_c = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} g_c^*(s)g_c(s)\,ds \tag{10 b}$$

where * denotes the conjugate transpose.

Proof

This is simply an application of Parseval's theorem.

It will be seen that the above integral representations of the gramians make it possible to extend the concept of a 1-D balanced realization to 2-D and delay-differential systems in a natural way.

3. Approximations of 2-D systems

Consider a 2-D discrete system realized as a Roesser model:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j) \equiv Ax + Bu \tag{11 a}$$

$$y(i,j) = [C_1 \quad C_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \equiv Cx \tag{11 b}$$

with $A_1 \in R^{n_1 \times n_1}$, $A_4 \in R^{n_2 \times n_2}$, $B_1 \in R^{n_1 \times m}$ and $C_1 \in R^{p \times n_1}$. The 2-D z-transform of (11) yields the transfer function

$$Q(z_1, z_2) = C[I(z_1, z_2) - A]^{-1} B \equiv \frac{N(z_1, z_2)}{p(z_1, z_2)} \tag{12}$$

where

$$I(z_1, z_2) = z_1 I_{n_1} \oplus z_2 I_{n_2} \tag{13}$$

$$p(z_1, z_2) = \det [I(z_1, z_2) - A] \tag{14}$$

and $\oplus$ denotes the direct sum. Throughout this section we assume that

$$p(z_1, z_2) \neq 0 \quad \text{for} \quad (z_1, z_2) \in \{(z_1, z_2) \,|\, |z_1| \geqslant 1, |z_2| \geqslant 1\} \tag{15}$$

to guarantee the BIBO stability of the system.

Similarly as in (7), let

$$F_2(z_1, z_2) = [I(z_1, z_2) - A]^{-1} B \tag{16 a}$$

and

$$G_2(z_1, z_2) = C[I(z_1, z_2) - A]^{-1} \tag{16 b}$$

and then the generalized reachability and observability gramians of (11) are defined as

$$K_2 = \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} F_2(z_1, z_2) F_2^*(z_1, z_2) \frac{dz_2}{z_2} \frac{dz_1}{z_1} \tag{17 a}$$

and

$$W_2 = \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} G_2^*(z_1, z_2) G_2(z_1, z_2) \frac{dz_2}{z_2} \frac{dz_1}{z_1} \tag{17 b}$$

respectively.

To see that (17) provides a natural generalization for the 2-D case, one may recall that in the 1-D case, in a digital filtering content, the observability and reachability gramians are closely related to the roundoff noise power and the dynamic range constraints on the state variable (Mullis and Roberts 1976 and Hwang 1977). Lu and Antoniou (1986) showed that an analogous relationship holds whenever the definitions (17 a) and (17 b) are adopted. Moreover, just as in the 1-D case, the generalized gramians provide certain system invariants which play an essential role in extending the concept of balanced realization to the 2-D case. To show this, observe first that a 2-D similarity transformation should be of the form $T = T_1 \oplus T_2$ where $T_1 \in R^{n_1 \times n_1}$ and $T_2 \in R^{n_2 \times n_2}$ are non-singular, and using this transformation leads to an equivalent realization of $Q(z_1, z_2)$ as $(\hat{A}, \hat{B}, \hat{C}) = (T^{-1}AT, T^{-1}B, CT)$. Further, denote the reachability and observability matrices associated with $(\hat{A}, \hat{B}, \hat{C})$ by $\hat{K}_2$ and $\hat{W}_2$, respectively, and partition K_2, W_2, $\hat{K}_2$ and $\hat{W}_2$ as

$$K_2 = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathrm{T}} & K_{22} \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^{\mathrm{T}} & W_{22} \end{bmatrix}, \quad \hat{K}_2 = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{12}^{\mathrm{T}} & \hat{K}_{22} \end{bmatrix}, \quad \text{and}$$

$$\hat{W}_2 = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{12}^{\mathrm{T}} & \hat{W}_{22} \end{bmatrix} \tag{18}$$

where K_{11}, W_{11}, $\hat{K}_{11}$, $\hat{W}_{11}$ are of dimension n_1. We then have the following result concerning invariants.

Theorem 2

The eigenvalues of $K_{11}W_{11}$ and $K_{22}W_{22}$ are invariant under 2-D similarity transformations.

Proof

By (17), we have

$$\hat{K}_2 = T^{-1}K_2T^{-\mathrm{T}} \quad \text{and} \quad \hat{W}_2 = T^{\mathrm{T}}W_2T \tag{19}$$

It follows that

$$\hat{K}_{11} = T_1^{-1}K_{11}T_1^{-\mathrm{T}}$$
$$\hat{K}_{22} = T_2^{-1}K_{22}T_2^{-\mathrm{T}}$$
$$\hat{W}_{11} = T_1^{\mathrm{T}}W_{11}T_1$$

and

$$\hat{W}_{22} = T_2^{\mathrm{T}}W_{22}T_2$$

which imply that

$$\hat{K}_{11}\hat{W}_{11} = T_1^{-1}K_{11}W_{11}T_1 \quad \text{and} \quad \hat{K}_{22}\hat{W}_{22} = T_2^{-1}K_{22}W_{22}T_2 \tag{20}$$

□

Remark

By (19),

$$\hat{K}_2\hat{W}_2 = T^{-1}K_2W_2T$$

meaning that the eigenvalues of K_2W_2 are also invariant under 2-D similarity transformations. It should be noted, however, that since a 2-D similarity transformation must be block-diagonal, it is in general impossible to find a transformation $T = T_1 \oplus T_2$ such that both $\hat{K}_2$ and $\hat{W}_2$ are diagonal and equal. Theorem 2 combined with the above observation leads to the following definition.

Definition 1

A realization $(\hat{A}, \hat{B}, \hat{C})$ of a 2-D transfer function is said to be balanced if

$$\hat{K}_{11} = \hat{W}_{11} = \mathrm{diag}\,(\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n_1}) \tag{21 a}$$

and

$$\hat{K}_{22} = \hat{W}_{22} = \mathrm{diag}\,(\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n_2}) \tag{21 b}$$

with $\sigma_{11} \geqslant \sigma_{12} \geqslant \ldots \geqslant \sigma_{1n_1} \geqslant 0$ and $\sigma_{21} \geqslant \sigma_{22} \geqslant \ldots \geqslant \sigma_{2n_2} \geqslant 0$. □

Williamson (1986) related the 1-D balanced realization to an optimization problem. That is, with K_d and W_d defined by (2), he proved that among all possible similarity transformations associated with discrete-time, a system as represented by (1):

$$J_1 = \mathrm{tr}\,(K_d + W_d)$$

will be minimized if and only if (3) holds, i.e. (A, B, C) in (1) is a balanced realization. Using Definition 1, one can extend Williamson's result to the 2-D case in a straightforward manner claiming that among all possible 2-D similarity transformations associated with system (11),

$$J_2 = \mathrm{tr}\,(K_2 + W_2)$$

will be minimized if and only if (A, B, C) in (11) represents a 2-D balanced realization.

To obtain a 2-D balancing transformation, K_{ii} and W_{ii} $(i = 1, 2)$ need to be positive definite. We have the following result regarding this issue.

Theorem 3

K_{11} and K_{22} are positive definite if (A, B) is locally reachable, namely

$$\mathrm{rank}\,[M(1,0) \quad M(0,1) \quad \ldots \quad M(n_1, n_2)] = n_1 + n_2 \tag{22}$$

W_{11} and W_{22} are positive definite if (A, C) is locally observable, that is

$$\mathrm{rank}\begin{bmatrix} C \\ CA_{01} \\ \vdots \\ CA_{0n_2} \\ CA_{10} \\ \vdots \\ CA_{1n_2} \\ \vdots \\ CA_{n_1,n_2-1} \end{bmatrix} = n_1 + n_2 \tag{23}$$

where A_{ij} are recursively defined as (Roesser 1975)

$$A_{ij} = A_{10}A_{i-1,j} + A_{01}A_{i,j-1} \quad \text{for} \quad i \geqslant 0, j \geqslant 0 \tag{24}$$

with

$$A_{00} = I_{n_1+n_2}, \quad A_{10} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

$$A_{-i,j} = A_{i,-j} = 0 \quad \text{for} \quad i > 0, \quad j > 0$$

and

$$M(i,j) = A_{i-1,j}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A_{i,j-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

Proof

See Appendix B.

It has become clear that under the conditions of Theorem 3 a 2-D balancing transformation T can be found by using K_{ii} and W_{ii} $(i = 1, 2)$ as matrices K and W of Algorithm 1 (see Appendix A) to obtain two 1-D balancing transformations T_1 and T_2, and then form $T = T_1 \oplus T_2$.

A crucial step for obtaining such a T is to compute K_{ii} and W_{ii} $(i = 1, 2)$. This computation will be detailed in § 5. At this point it is also of interest to note the following result.

Theorem 4

The gramians K_2 and W_2 defined in (17) can be computed as

$$K_2 = \sum_{i,j=0}^{\infty} M(i,j)M^{\mathrm{T}}(i,j) \tag{25 a}$$

and

$$W_2 = \begin{bmatrix} \left(\sum_{i,j=0}^{\infty} A_{ij}^{\mathrm{T}}C^{\mathrm{T}}CA_{ij}\right)_{11} & \left(\sum_{i,j=0}^{\infty} A_{ij}^{\mathrm{T}}C^{\mathrm{T}}CA_{i-1,j+1}\right)_{12} \\ \left(\sum_{i,j=0}^{\infty} A_{ij}^{\mathrm{T}}C^{\mathrm{T}}CA_{i+1,j-1}\right)_{21} & \left(\sum_{i,j=0}^{\infty} A_{ij}^{\mathrm{T}}C^{\mathrm{T}}CA_{ij}\right)_{22} \end{bmatrix} \tag{25 b}$$

where $(\cdot)_{11}$ denotes the upper-left submatrix of dimension n_1 of the matrix involved, and $(\cdot)_{12}$, $(\cdot)_{21}$ are defined similarly.

Proof

See Appendix B.

Based on these observations, it is now a simple matter to describe a 2-D balanced approximation approach, that is, once a 2-D balanced realization (say $(\hat{A}, \hat{B}, \hat{C})$) is found, a reduced 2-D model $(\hat{A}_r, \hat{B}_r, \hat{C}_r)$ of order (r_1, r_2) can be obtained by a further subpartitioning of $(\hat{A}, \hat{B}, \hat{C})$ as

$$\hat{A} = \begin{array}{c} \\ r_1\{ \\ r_2\{ \end{array}\!\!\begin{array}{c} \begin{array}{cc} r_1 & r_2 \end{array} \\ \left[\begin{array}{c|c} \hat{A}_{1r} & \hat{A}_{2r} \\ \hline \hat{A}_{3r} & \hat{A}_{4r} \end{array}\right] \end{array}, \quad \hat{B} = \begin{array}{c} r_1\{ \\ r_2\{ \end{array}\!\!\left[\begin{array}{c} \hat{B}_{1r} \\ \hline \hat{B}_{2r} \end{array}\right], \quad \text{and} \quad \hat{C} = [\overbrace{\hat{C}_{1r}}^{r_1} \mid \overbrace{\hat{C}_{2r}}^{r_2}] \tag{26}$$

and then taking as the approximation the triple

$$\hat{A}_r = \begin{bmatrix} \hat{A}_{1r} & \hat{A}_{2r} \\ \hat{A}_{3r} & \hat{A}_{4r} \end{bmatrix}, \quad \hat{B}_r = \begin{bmatrix} \hat{B}_{1r} \\ \hat{B}_{2r} \end{bmatrix}, \quad \text{and} \quad \hat{C}_r = [\hat{C}_{1r} \quad \hat{C}_{2r}] \tag{27}$$

4. Approximation of delay-differential systems

Realization of retarded and neutral delay-differential systems has been a subject of study for some time. It was demonstrated by Sontag (1978), Lèvy (1981) and Żak *et al.* (1986) that, given a transfer function $Q(z, s)$ of a retarded or a neutral delay-differential system, there exists a neutral type of realization described as

$$\begin{bmatrix} h(t+1) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} h(t) \\ x(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \equiv A \begin{bmatrix} h \\ x \end{bmatrix} + Bu \tag{28 a}$$

$$y(t) = [C_1 \quad C_2] \begin{bmatrix} h(t) \\ x(t) \end{bmatrix} \equiv C \begin{bmatrix} h \\ x \end{bmatrix} \tag{28 b}$$

with $A_1 \in R^{n_1 \times n_1}$, $A_4 \in R^{n_2 \times n_2}$, $B_1 \in R^{n_1 \times m}$ and $C_1 \in R^{p \times n_1}$. Throughout this section, it is assumed that

$$\det [I(z, s) - A] \neq 0, \quad \text{for} \quad |z| \geqslant 1 \quad \text{and} \quad \mathrm{Re}\, s \geqslant 0 \tag{29}$$

with $I(z, s) = zI_{n_1} \oplus sI_{n_2}$. Condition (29) implies stability independent of delay (i.o.d.) of (28) (Kamen 1982).

Now let

$$F_d(z, s) = [I(z, s) - A]^{-1} B \tag{30 a}$$

and

$$G_d(z, s) = C[I(z, s) - A]^{-1} \tag{30 b}$$

with the generalized reachability and observability gramians of (28) defined as

$$K_{dd} = \frac{1}{2\pi} \int_{-j\infty}^{+j\infty} F_d(\exp sh, s) F_d^*(\exp sh, s)\, ds \tag{31 a}$$

and

$$W_{dd} = \frac{1}{2\pi} \int_{-j\infty}^{+j\infty} G_d^*(\exp sh, s) G_d(\exp sh, s)\, ds \tag{31 b}$$

respectively. Lu and Lee (1986) have shown that the above-defined W_{dd} and K_{dd} can be related to an L_2 sensitivity measure with respect to coefficient variations of a delay-differential system and the dynamic range constraints on the state variables, respectively.

One of the advantages of employing model (28) is that it enables us to reduce the system's order in s and the number of the delay elements used *independently*. This is particularly appropriate in the present case since (28) is assumed to be stable i.o.d. so that z and s can be treated as two almost independent variables (Kamen 1982, 1983).

Analogous to the 2-D case, the similarity transformations of (28) should be of the form $T = T_1 \oplus T_2$, and an equivalent realization of (28) under transformation T is $(\hat{A}, \hat{B}, \hat{C}) = (T^{-1}AT, T^{-1}B, CT)$. By (31), the gramians of the transformed realization can be computed as

$$\hat{K}_{dd} = T^{-1} K_{dd} T^{-\mathrm{T}} \tag{32 a}$$

$$\hat{W}_{dd} = T^{\mathrm{T}} W_{dd}^{\mathrm{T}} \tag{32 b}$$

Now partition K_{dd}, W_{dd}, $\hat{K}_{dd}$ and $\hat{W}_{dd}$ as

$$K_{dd} = \begin{bmatrix} K_{11d} & K_{12d} \\ K_{12d}^{\mathrm{T}} & K_{22d} \end{bmatrix}, \quad W_{dd} = \begin{bmatrix} W_{11d} & W_{12d} \\ W_{12d}^{\mathrm{T}} & W_{22d} \end{bmatrix}$$

$$\hat{K}_{dd} = \begin{bmatrix} \hat{K}_{11d} & \hat{K}_{12d} \\ \hat{K}_{12d}^{\mathrm{T}} & \hat{K}_{22d} \end{bmatrix}, \quad \text{and} \quad \hat{W}_{dd} = \begin{bmatrix} \hat{W}_{11d} & \hat{W}_{12d} \\ \hat{W}_{12d}^{\mathrm{T}} & \hat{W}_{22d} \end{bmatrix}$$

where K_{iid}, W_{iid}, $\hat{K}_{iid}$ and $\hat{W}_{iid}$ are of dimension n_i for $i = 1, 2$. An argument analogous to the proof of Theorem 2 leads to the following result.

Theorem 5

The eigenvalues of $K_{11d}W_{11d}$ and $K_{22d}W_{22d}$ are invariant under similarity transformations.

We now give a definition for balancing a delay-differential-type system.

Definition 2

Neutral realization of a delay-differential system, say $(\hat{A}, \hat{B}, \hat{C})$, is called a balanced realization if

$$\hat{K}_{11d} = \hat{W}_{11d} = \operatorname{diag}(\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n_1})$$

and

$$\hat{K}_{22d} = \hat{W}_{22d} = \operatorname{diag}(\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n_2})$$

with $\sigma_{11} \geqslant \sigma_{12} \geqslant \ldots \geqslant \sigma_{1n_1} \geqslant 0$ and $\sigma_{21} \geqslant \sigma_{22} \geqslant \ldots \geqslant \sigma_{2n_2} \geqslant 0$.

Concerning the positive definiteness of K_{iid} and W_{iid} $(i = 1, 2)$, sufficient conditions are provided next.

Theorem 6

K_{iid} $(i = 1, 2)$ are positive definite if

$$\operatorname{rank}[M(1,0) \quad M(0,1) \quad \ldots \quad M(n_1, n_2)] = n_1 + n_2 \tag{33 a}$$

W_{iid} $(i = 1, 2)$ are positive definite if

$$\operatorname{rank}\begin{bmatrix} C \\ CA_{01} \\ \vdots \\ CA_{0n_2} \\ CA_{10} \\ \vdots \\ CA_{1n_2} \\ \vdots \\ CA_{n_1,n_2-1} \end{bmatrix} = n_1 + n_2 \tag{33 b}$$

where $M(i,j)$ and $A_{i,j}$ are defined as for the 2-D case.

Proof

See Appendix B.

We now conclude that upon the availability of positive definite K_{iid} and W_{iid} $(i = 1, 2)$, a balancing transformation of (27) can be obtained through the use of Algorithm 1 where K and W are substituted by K_{iid} and W_{iid} $(i = 1, 2)$ respectively, resulting in two 1-D balancing transformations T_1 and T_2, and then forming $T = T_1 \oplus T_2$. Furthermore, a reduced model of order (r_1, r_2) can be found by the further subpartitioning procedure described by (26), (27).

The computation of K_{iid} and W_{iid} will be considered in the next section.

5. Computational issues

5.1. *2-D Systems: computation of K_{ii} and W_{ii} ($i = 1, 2$)*

5.1.1. *Truncation method.* By Theorem 4, a straightforward way of computing K_{ii} and W_{ii} ($i = 1, 2$) is to use truncated double sums, i.e.

$$K_{11} \approx [I_{n_1} \quad 0]\tilde{K}_2 \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \tag{34 a}$$

$$K_{22} \approx [0 \quad I_{n_2}]\tilde{K}_2 \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \tag{34 b}$$

$$W_{ii} \approx \left(\sum_{l=0}^{L} \sum_{m=0}^{M} A_{lm}^{\mathrm{T}} C^{\mathrm{T}} C A_{lm} \right)_{ii}, \quad i = 1, 2 \tag{35}$$

where

$$\tilde{K}_2 = \sum_{l=0}^{L} \sum_{m=0}^{M} \left(A_{l-1,m} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A_{l,m-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right) \times \left(A_{l-1,m} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A_{lm-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right)^{\mathrm{T}} \tag{36}$$

and L and M are positive integers. Through the use of the recursive formula (24), the computation procedure of the above finite sums can readily be programmed. In general, when L and M are large enough, (34) and (35) will represent good approximations of K_{ii} and W_{ii}, respectively. On the other hand, experience has shown that even for small n_1 and n_2, this approach needs a considerable amount of computational time.

5.1.2. *Lyapunov approach.* It is easy to verify that a 2-D transfer function associated with the realization (A, B, C) can be written as

$$\begin{aligned} C[I(z_1, z_2) - A]^{-1} B &= C_2(z_2 I - A_4)^{-1} B_2 + [C_1 + C_2(z_2 I - A_4)^{-1} A_3] \\ &\quad \times [z_1 I - A_1 - A_2(z_2 I - A_4)^{-1} A_3]^{-1} \times [B_1 + A_2(z_2 I - A_4)^{-1} B_2] \\ &\equiv D_1(z_2) + C_1(z_2)(z_1 I - A_1(z_2))^{-1} B_1(z_2) \end{aligned} \tag{37 a}$$

or

$$\begin{aligned} C[I(z_1, z_2) - A]^{-1} B &= C_1(z_1 I - A_1)^{-1} B_1 + [C_2 + C_1(z_1 I - A_1)^{-1} A_2] \\ &\quad \times [z_2 I - A_4 - A_3(z_1 I - A_1)^{-1} A_2]^{-1} \times [B_2 + A_3(z_1 I - A_1)^{-1} B_1] \\ &\equiv D_2(z_1) + C_2(z_1)(z_2 I - A_2(z_1))^{-1} B_2(z_1) \end{aligned} \tag{37 b}$$

Further, notice that

$$K_{11} = [I_{n_1} \quad 0] K \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} = \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} ([I_{n_1} \quad 0] F_2(z_1, z_2)) \times ([I_{n_1} \quad 0] F_2(z_1, z_2))^* \frac{dz_1}{z_1} \frac{dz_2}{z_2}$$

which, by (37 *a*), implies that

$$K_{11} = \frac{1}{2\pi j}\oint_{|z_2|=1} \frac{dz_2}{z_2}\frac{1}{2\pi j}\oint_{|z_1|=1} (z_1 I - A_1(z_2))^{-1} B_1(z_2) B_1^*(z_2)$$
$$\times (\bar{z}_1 I - A_1^*(z_2))^{-1} \frac{dz_1}{z_1} \tag{38}$$

Regarding z_2 as a complex parameter, the second integral of (38) is the positive-definite hermitian solution of the Lyapunov equation

$$A_1(z_2)K_1(z_2)A_1^*(z_2) - K_1(z_2) = -B_1(z_2)B_1^*(z_2) \tag{39}$$

Thus

$$K_{11} = \frac{1}{2\pi j}\oint_{|z_2|=1} K_1(z_2)\frac{dz_2}{z_2} \tag{40}$$

The integral in (40) can be computed through the use of the residue theorem. Alternatively, one may notice that $K_1(z_2)$ can be factorized as (see Youla 1961)

$$K_1(z_2) = \tilde{K}_1(z_2)\tilde{K}_1^*(z_2) \tag{41}$$

where $\tilde{K}_1(z_2)$ is a proper, stable rational matrix. Thus, if $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1)$ is a minimal realization of $\tilde{K}_1(z_1)$, then (40) becomes

$$\begin{aligned} K_{11} &= \tilde{C}_1\left(\frac{1}{2\pi j}\oint_{|z_2|=1} (z_2 I - \tilde{A}_1)^{-1}\tilde{B}_1\tilde{B}_1^{\mathrm{T}}(\bar{z}_2 I - \tilde{A}_1)^{-T}\frac{dz_2}{z_2}\right)\tilde{C}_1^{\mathrm{T}} \\ &= \tilde{C}_1\tilde{K}_1\tilde{C}_1^{\mathrm{T}} \end{aligned} \tag{42}$$

where $\tilde{K}_1$ is the positive-definite solution of the Lyapunov equation

$$\tilde{A}_1\tilde{K}_1\tilde{A}_1^{\mathrm{T}} - \tilde{K}_1 = -\tilde{B}_1\tilde{B}_1^{\mathrm{T}} \tag{43}$$

Similarly, by (37 *b*) we have

$$K_{22} = \frac{1}{2\pi j}\oint_{|z_2|=1} \frac{dz_1}{z_1}\frac{1}{2\pi j}\oint_{|z_1|=1} (z_2 I - A_2(z_1))^{-1} B_2(z_1) B_2^*(z_1)$$
$$\times (\bar{z}_2 I - A_2^*(z_1))^{-1}\frac{dz_2}{z_2} = \frac{1}{2\pi j}\oint_{|z_1|=1} K_2(z_1)\frac{dz_1}{z_1} \tag{44}$$

where $A_2(z_1)$ and $B_2(z_1)$ are defined in (37 *b*) and $K_2(z_1)$ is the positive hermitian solution of the Lyapunov equation

$$A_2(z_1)K_2(z_1)A_2^*(z_1) - K_2(z_1) = -B_2(z_1)B_2^*(z_1) \tag{45}$$

Analogous to (40), the last integral in (44) can be evaluated by either applying the residue theorem or factorizing $K_2(z_1)$ as $\tilde{K}_2(z_1)\tilde{K}_2^*(z_1)$ with $\tilde{K}_2(z_1)$ proper and stable and then computing K_{22} as

$$K_{22} = \tilde{C}_2\tilde{K}_2\tilde{C}_2^{\mathrm{T}}$$

where $\tilde{K}_2$ is the positive-definite solution of the Lyapunov equation

$$\tilde{A}_2\tilde{K}_2\tilde{A}_2^{\mathrm{T}} - \tilde{K}_2 = -\tilde{B}_2\tilde{B}_2^{\mathrm{T}}$$

and $(\tilde{A}_2, \tilde{B}_2, \tilde{C}_2)$ is a minimal realization of $K_2(z_1)$.

It is now obvious that W_{ii} $(i = 1, 2)$ can also be calculated by applying the above Lyapunov approach with certain straightforward modifications.

5.2. *Delay-differential systems: computation of K_{iid} and W_{iid} ($i = 1, 2$)*

Since variables exp (sh) and s in (31) are dependent, the Lyapunov approach used for the 2-D case will not work. A possible way of computing K_{iid} and W_{iid} ($i = 1, 2$) is first write (31) as

$$K_{dd} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_d(\exp(j\omega h), j\omega) F_d^*(\exp(j\omega h), j\omega)\, d\omega \tag{46 a}$$

and

$$W_{dd} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_d^*(\exp(j\omega h), j\omega) G_d(\exp(j\omega h), j\omega)\, d\omega \tag{46 b}$$

and then used the residue theorem or some numerical integration method to compute K_{dd} and W_{dd} in (46).

6. Examples and concluding remarks

Example 1 (Jury and Premaratne 1986)

Let us consider a 2-D system having model (11) with

$$A = \left[\begin{array}{cc|cc} -0.5 & 0.75 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0.2625 & 0 & -0.5 & 0.75 \\ 0 & 0 & -0.05 & -0.025 \end{array}\right], \quad b = \left[\begin{array}{c} 1 \\ 0 \\ \hline 1 \\ 0 \end{array}\right] \tag{47}$$

and

$$c = [0.1 \quad 0.2 \;\vdots\; 0.3 \quad 0]$$

The approach proposed by Jury and Premaratne (1986) yields a reduced system $(\tilde{A}, \tilde{b}, \tilde{c})$ or order (2,1) where

$$\tilde{A} = \left[\begin{array}{cc|c} -0.5 & 0.75 & 1 \\ 0 & 0 & 0 \\ \hline 0.2625 & 0 & 0.5366 \end{array}\right], \quad \tilde{b} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 1 \end{array}\right], \quad \text{and} \quad \tilde{c} = [0.1 \quad 0.2 \;\vdots\; 0.3]$$

that was proved to be unstable.

We use the truncated double sums (34) and (35) to approximate K_{ii} and W_{ii} ($i = 1, 2$). Taking $L = M = 240$, numerical computation gives

$$K_{11} = \begin{bmatrix} 3.887426 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_{22} = \begin{bmatrix} 0.960192 & 0.056943 \\ 0.056943 & 0.005045 \end{bmatrix} \tag{48 a}$$

$$W_{11} = \begin{bmatrix} 0.020792 & -0.012830 \\ -0.012830 & 0.017320 \end{bmatrix}, \quad \text{and}$$

$$W_{22} = \begin{bmatrix} 0.075208 & -0.065187 \\ -0.065187 & 0.095434 \end{bmatrix} \tag{48 b}$$

Notice that K_{11} of (48 a) is not positive definite, so that we can only reduce the order of the system in the vertical direction.

Using K_{22} and W_{22} of (48) as K and W in Algorithm 1 (described in Appendix A) yields a 1-D balancing transformation:

$$T_2 = \begin{bmatrix} 1{\cdot}937878 & 0{\cdot}389475 \\ 0{\cdot}111965 & 0{\cdot}466287 \end{bmatrix}$$

Letting $T = I_2 \oplus T_2$, an equivalent realization balanced in the vertical direction can be computed as

$$\hat{A} = T^{-1}AT = \left[\begin{array}{cc|cc} -0{\cdot}5 & 0{\cdot}75 & 1{\cdot}937878 & 0{\cdot}389475 \\ 0 & 0 & 0 & 0 \\ \hline 0{\cdot}142326 & 0 & -0{\cdot}434674 & 0{\cdot}098126 \\ -0{\cdot}034175 & 0 & -0{\cdot}109428 & -0{\cdot}090326 \end{array}\right] \tag{49}$$

$$\hat{b} = T^{-1}b = \left[\begin{array}{c} 1{\cdot}0 \\ 0 \\ \hline 0{\cdot}542194 \\ -0{\cdot}130192 \end{array}\right], \quad \text{and} \quad \hat{C} = CT = [0{\cdot}1 \quad 0{\cdot}2 \;\vdots\; 0{\cdot}581363 \quad 0{\cdot}116843]$$

A reduced system of order (2,1) can now be obtained from (49) as $(\hat{A}_r, \hat{b}_r, c_r)$ with

$$\hat{A}_r = \left[\begin{array}{cc|c} -0{\cdot}5 & 0{\cdot}75 & 1{\cdot}937878 \\ 0 & 0 & 0 \\ \hline 0{\cdot}142326 & 0 & -0{\cdot}434674 \end{array}\right], \quad \hat{b}_r = \left[\begin{array}{c} 1{\cdot}0 \\ 0{\cdot}0 \\ \hline 0{\cdot}542194 \end{array}\right], \quad \text{and}$$

$$\hat{c}_r = [0{\cdot}1 \quad 0{\cdot}2 \;\vdots\; 0{\cdot}581363] \tag{50}$$

It is easy to check (Lu and Lee 1983) that the reduced system given by (50) is BIBO stable. The magnitude of the frequency responses of (47) and (50) are shown in Figs. 1 (*a*) and (*b*), respectively.

Example 2

We now consider reducing a retarded delay differential system described by

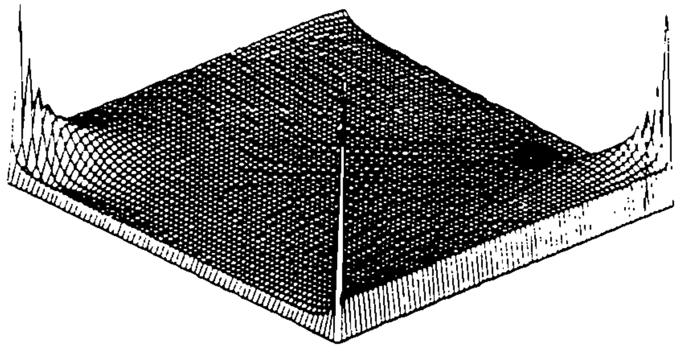
$$Q(z, s) = \frac{0{\cdot}5z}{z(s^2 + s + 0{\cdot}25) + 0{\cdot}5s} \tag{51}$$

This system was found stable i.o.d. (Driver 1977). The neutral realization of (51) can be obtained using the method suggested by Sontag (1978):

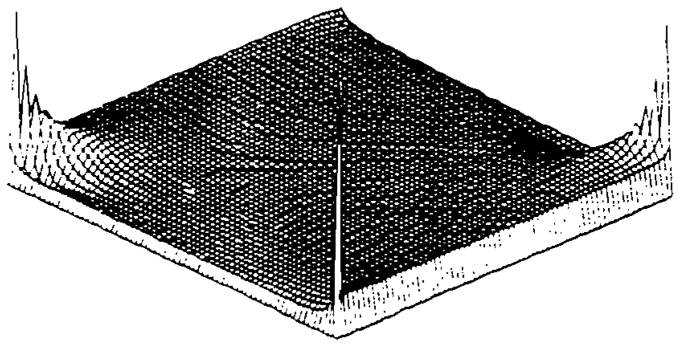
$$\begin{bmatrix} h(t+1) \\ \dot{x}(t) \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & -0{\cdot}5 \\ \hline 0 & 0 & 0{\cdot}5 \\ 1 & -0{\cdot}5 & -1 \end{array}\right] \begin{bmatrix} h(t) \\ x(t) \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline -1 \\ 0 \end{array}\right] u(t) \equiv A \begin{bmatrix} h \\ x \end{bmatrix} + bu \tag{52 a}$$

$$y(t) = [0 \;\vdots\; 0 \quad 1] \begin{bmatrix} h(t) \\ x(t) \end{bmatrix} \equiv c \begin{bmatrix} h \\ x \end{bmatrix} \tag{52 b}$$

By (46 *a*), we use the numerical integration method to compute K_{22d} (with $h = 0{\cdot}5$) as



(*a*)



(*b*)

Figure 1.

$$K_{22d} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \times \frac{\begin{bmatrix} \exp(j\omega/2)(1+j\omega)+0{\cdot}5 \\ -0{\cdot}5\exp(j\omega/2) \end{bmatrix}[(-\exp -(j\omega/2))(1-j\omega)+0{\cdot}5 \quad -0{\cdot}5\exp(-j\omega/2)]}{\exp(j\omega/2)\,|[(0{\cdot}25-\omega^2)+j\omega]+0{\cdot}5j\omega|^2}\,d\omega$$

$$= \begin{bmatrix} 10{\cdot}2057 & -3{\cdot}1715 \\ -3{\cdot}1715 & 1{\cdot}0680 \end{bmatrix} \tag{53 a}$$

Similarly, we have

$$W_{22d} = \begin{bmatrix} 0{\cdot}3399 & 0 \\ 0 & 0{\cdot}4076 \end{bmatrix} \tag{53 b}$$

Algorithm 1 where K and W are replaced by K_{22d} and W_{22d} of (53) gives a 1-D balancing transformation as

$$T_2 = \begin{bmatrix} 1{\cdot}71454 & 0{\cdot}17776 \\ -0{\cdot}53276 & 0{\cdot}47704 \end{bmatrix}$$

Forming $T = I \oplus T_2$, an equivalent realization of (52), which is balanced in the s-direction, is given by

$$\hat{A} = T^{-1}AT = \left[\begin{array}{c|cc} 0 & 0.26638 & -0.23852 \\ \hline -0.19478 & -0.07603 & 0.23491 \\ 1.87875 & -0.76518 & -0.92397 \end{array}\right] \tag{54}$$

$$\hat{b} = T^{-1}b = \left[\begin{array}{c} 0 \\ \hline -0.52272 \\ -0.58379 \end{array}\right], \quad \text{and} \quad \hat{c} = cT = [0 \ \vdots \ -0.53276 \quad 0.47704]$$

From (54), a reduced system of order (1, 1) is obtained as $(\hat{A}_r, \hat{b}_r, \hat{c}_r)$ with

$$\hat{A}_r = \left[\begin{array}{cc} 0 & 0.26638 \\ \hline -0.19478 & -0.07603 \end{array}\right], \quad \hat{b}_r = \left[\begin{array}{c} 0 \\ \hline -0.52272 \end{array}\right], \quad \text{and} \quad \hat{c}_r = [0 \ \vdots \ -0.53276] \tag{55}$$

The transfer function of (55) is

$$Q_r(z, s) = \frac{0.27849z}{z(s + 0.07603) + 0.05189} \tag{56}$$

Clearly, (56) represents a reduced-delay system which is stable i.o.d. The magnitude of the frequency responses of both original and reduced systems are shown in Fig. 2.

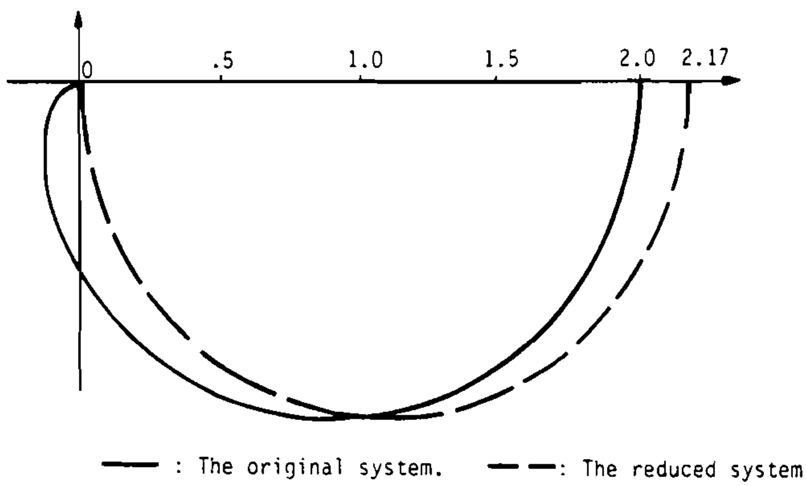


Figure 2.

In conclusion it can be stated that a natural extension of the concept of the balanced realization to 2-D and delay-differential systems has been obtained through the use of the complex integral representations of the gramians for the systems considered. The generalized balanced realization is then used in approximating 2-D and delay systems. Several relevant issues, such as the positive definiteness of the various gramians and computation methods have also been considered. While the numerical examples show that the resulting approximations maintain the stability property in both the 2-D and delay cases, and the reduction error appears to be acceptable, a general analysis of the stability property and the reduction error associated with the suggested balanced approximation approach is still under way. Finally, it should be pointed out that the results can readily be extended to N-dimensional ($N \geqslant 3$) and non-commensurate delay systems.

ACKNOWLEDGMENT

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Appendix A

An efficient algorithm suggested by Laub (1980) for obtaining a balanced realization is listed below. It is worthwhile to note that the algorithm is valid for both continuous and discrete-time systems that are assumed to be stable.

Algorithm 1 (Laub 1980)

Step 1. Determination of the reachability gramian K and the observability gramian W via the Bartels–Stewart algorithm. If the system is of discrete-time, then K and W are the solutions of (5 *a*) and (5 *b*), respectively.

Step 2. Cholesky factorization of K

$$K = LL^{\mathrm{T}}$$

where L is lower triangular.

Step 3. Formation of $L^{\mathrm{T}}WL$.

Step 4. Solution of the symmetric eigenvalue/eigenvector problem

$$U^{\mathrm{T}}(L^{\mathrm{T}}WL)U = \Lambda^2$$

Step 5. Formation of T

$$T = LU\Lambda^{-1/2}$$

Step 6. Formation of the balanced realization $(\hat{A}, \hat{B}, \hat{C})$ with

$$\hat{A} = T^{-1}AT, \quad \hat{B} = T^{-1}B, \quad \text{and} \quad \hat{C} = CT$$

whose reachability gramian and observability gramian are the same and equal to Λ.

Appendix B

Proof of Theorem 3

Clearly, both K_2 and W_2 defined in (17) are semi-positive definite. Now assume that (22) holds and there exists an (n_1+n_2)-dimensional vector x such that $x^{\mathrm{T}}K_2x = 0$. It can be shown that

$$x^{\mathrm{T}}K_2x = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \|x^{\mathrm{T}}M(i,j)\|^2$$

(see Theorem 4), thus $x^{\mathrm{T}}K_2x = 0$ implies that

$$x^{\mathrm{T}}M(i,j) = 0, \quad \text{for} \quad i \geqslant 0, \quad j \geqslant 0 \tag{B 1}$$

Therefore,

$$x^{\mathrm{T}}[M(1,0) \quad M(0,1) \quad \dots \quad M(n_1,n_2)] = 0$$

Condition (22) now implies $x = 0$ and hence K_2 is positive definite. It follows that K_{ii} $(i = 1, 2)$ are positive definite.

Next we show that condition (23) implies the positive definiteness of

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij}^{\mathrm{T}} C^{\mathrm{T}} C A_{ij}$$

Let $x \in R^{n_1+n_2}$ be such that

$$x^{\mathrm{T}}\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij}^{\mathrm{T}} C^{\mathrm{T}} C A_{ij}\right)x = 0$$

which means

$$CA_{ij}x = 0 \quad \text{for} \quad i \geqslant 0, \quad j \geqslant 0$$

Condition (23) now leads to $x = 0$, and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij}^{\mathrm{T}} C^{\mathrm{T}} C A_{ij}$$

is positive definite. It follows from Theorem 4 that W_{ii} $(i = 1, 2)$ are positive definite. □

Proof of Theorem 4

Write $F_2(z_1, z_2)$ as

$$\begin{aligned} F_2(z_1,z_2) &= [I(z_1,z_2) - A]^{-1}B = [I - (A_{10}z_1^{-1} + A_{01}z_2^{-1})]^{-1}\begin{bmatrix} B_1 z_1^{-1} \\ B_2 z_2^{-1}\end{bmatrix} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij} z_1^{-i} z_2^{-j}\begin{bmatrix} B_1 z_1^{-1} \\ B_2 z_2^{-1}\end{bmatrix} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(A_{ij}\begin{bmatrix} B_1 \\ 0\end{bmatrix} z_1^{-i-1} z_2^{-j} + A_{ij}\begin{bmatrix} 0 \\ B_2\end{bmatrix} z_1^{-i} z_2^{-j-1}\right)\end{aligned}$$

Note that on the unit circles $|z_1| = 1$ and $|z_2| = 1$, we have $\bar{z}_1 = z_1^{-1}$ and $\bar{z}_2 = z_2^{-1}$. The use of the residue theorem then gives (25 *a*). To prove (25 *b*), write $G_2(z_1, z_2)$ as

$$G_2(z_1,z_2) = C[I(z_1,z_2) - A]^{-1} = \left(C\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij} z_1^{-i} z_2^{-j}\right) I^{-1}(z_1,z_2)$$

hence

$$\begin{aligned} W_2 = \frac{1}{(2\pi j)^2}\oint_{|z_1|=1}\oint_{|z_2|=1} I(z_1,z_2)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} A_{ij}^{\mathrm{T}} C^{\mathrm{T}} C A_{lk} z_1^{i-l} z_2^{j-k}\right) \\ \times I^{-1}(z_1,z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \end{aligned} \qquad \text{(B 2)}$$

Set $A_{ij}^{\mathrm{T}} C^{\mathrm{T}} C A_{lk} = W(i,j,l,k)$ and partition it as

$$W(i,j,l,k) = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}, \quad U_1 \in R^{n_1 \times n_1}, \quad U_4 \in R^{n_2 \times n_2}$$

(B 2) then gives

$$W_2 = \frac{1}{(2\pi j)^2} \oint_{|z_1|=1} \oint_{|z_2|=1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \times \begin{bmatrix} U_1 z_1^{i-l} z_2^{j-k} & U_2 z_1^{i+1-l} z_2^{j-k-1} \\ U_3 z_1^{i-l-1} z_2^{j-k+1} & U_4 z_1^{i-l} z_2^{j-k} \end{bmatrix} \frac{dz_1}{z_1} \frac{dz_2}{z_2} \tag{B 3}$$

The residue theorem now leads (B 3) to (25 *b*) immediately. □

Proof of Theorem 6

By (31), it is seen that $K_{iid} \geqslant 0$ and $W_{iid} \geqslant 0$ ($i = 1, 2$).

To show $K_{iid} > 0$, let us assume that Condition (33 *a*) holds and there exists an $x \in R^{n_1+n_2}$ such that $x^{\mathrm{T}} K_{iid} = 0$. Equation (31 *a*) then gives $x^{\mathrm{T}} F_d(j\omega) = 0$ for $\omega \in (-\infty, \infty)$ almost everywhere (a.e.). That is, when $m = 1$,

$$x^{\mathrm{T}} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left(A_{lk} \begin{bmatrix} b_1 \\ 0 \end{bmatrix} (j\omega)^{-(l+1)} \exp(-j\omega k) + A_{lk} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} (j\omega)^{-l} \exp[-j\omega(k+1)] \right) = 0$$

$$\text{for} \quad \omega \in (-\infty, \infty) \text{ a.e.}$$

which implies that

$$x^{\mathrm{T}} \sum_{k=0}^{\infty} \left\{ A_{0k} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \exp[-j\omega(k+1)] \right\} = 0 \tag{B 4 a}$$

$$x^{\mathrm{T}} \sum_{k=0}^{\infty} \left\{ A_{0k} \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \exp(-j\omega k) + A_{1k} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \exp[-j\omega(k+1)] \right\} = 0 \tag{B 4 b}$$

$$x^{\mathrm{T}} \sum_{k=0}^{\infty} \left\{ A_{1k} \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \exp(-j\omega k) + A_{2k} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \exp[-j\omega(k+1)] \right\} = 0 \tag{B 4 c}$$

It follows from (B 4 *a*) that

$$\sum_{k=0}^{\infty} \left\| x^{\mathrm{T}} A_{0k} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \right\|^2 = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} x^{\mathrm{T}} \left[\frac{1}{2\pi} \int_0^{2\pi} \left\{ A_{0k} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \exp[-j\omega(k+1)] \right\} \times \left\{ A_{0l} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \exp[-j\omega(l+1)^* \, d\omega] \right\} \right] x = 0$$

i.e.

$$x^{\mathrm{T}} A_{0k} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} = 0 \quad \text{for } k \geqslant 0$$

Similarly by (B 4 *b*) and (B 4 *c*) we have

$$x^{\mathrm{T}}\left(A_{0k}\begin{bmatrix} b_1 \\ 0 \end{bmatrix} + A_{1,k-1}\begin{bmatrix} 0 \\ b_2 \end{bmatrix}\right) = 0 \quad \text{for} \quad k \geqslant 0$$

and

$$x^{\mathrm{T}}\left(A_{1k}\begin{bmatrix} b_1 \\ 0 \end{bmatrix} + A_{2,k-1}\begin{bmatrix} 0 \\ b_2 \end{bmatrix}\right) = 0 \quad \text{for} \quad k \geqslant 0$$

respectively. Repeating this argument gives

$$x^{\mathrm{T}} M(l, k) = 0 \quad \text{for} \quad l \geqslant 0, \quad k \geqslant 0$$

and thus

$$x^{\mathrm{T}}[M(1, 0) \quad M(0, 1) \quad \dots \quad M(n_1, n_2)] = 0$$

Condition (33 *a*) then implies $x = 0$. Therefore $K_{dd} > 0$ and so are $K_{iid} (i = 1, 2)$.

To prove $W_{iid} > 0$, assume that (33 *b*) holds and there exists an $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^{n_1+n_2}$ such that $W_{dd} = 0$. This, by (31 *b*), means that

$$\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} C\left\{A_{lk}\begin{bmatrix} x_1 \\ 0 \end{bmatrix}(j\omega)^{-(l+1)} \exp(-j\omega k) + A_{lk}\begin{bmatrix} 0 \\ x_2 \end{bmatrix}(j\omega)^{-l} \exp[j\omega(k+1)]\right\} = 0$$

for $\omega \in (-\infty, \infty)$ a.e.

Using the same type of argument, we may conclude that

$$Cx = 0, \quad CA_{01}x = 0, \quad \dots, \quad CA_{n_1, n_2 - 1}x = 0$$

Condition (33 *b*) then implies $x = 0$. Therefore $W_{dd} > 0$ and thus $W_{iid} > 0$. □

References

Bartels, R. H., and Stewart, G. W., 1972, *Comm. Assoc. Comput. Mach.*, **15**, 820.

Curtain, R. F., and Glover, K., 1986, *Operator Th. Adv. Applic.*, **19**, 87.

Doyle, J. C., 1984, Lecture Notes, ONR/Honeywell Workshop on Advances in Multivariable Control, Minneapolis, MN.

Driver, R. D., 1977, *Ordinary and Delay Differential Equations* (New York: Springer-Verlag).

Glover, K., 1984, *Int. J. Control*, **39**, 1115.

Glover, K., and Limebeer, D., 1983, *Proc. American Control Conf.*

Harshavardhana, P., Jonckheere, E. A., and Silverman, L. M., 1983, *Proc. I.E.E.E. Int. Symp. on Circuits and Systems*, p. 126.

Hwang, S. Y., 1977, *I.E.E.E. Trans. Acoust. Speech Signal Process.*, **25**, 256.

Jury, E. I., and Premaratne, K., 1986, *I.E.E.E. Trans. Circuits Syst.*, **33**, 558.

Kamen, E. W., 1982, *I.E.E.E. Trans. autom. Control*, **27**, 367; 1983, *Ibid.*, **28**, 248.

Laub, A. J., 1980, *Proc. Joint Automatic Control Conference*, FA8-E.

Lee, E. B., Lu, W.-S., and Wu, W. E., 1985, *Proc. 19th Conf. on Information Science and Systems*, p. 468.

Lèvy, B. C., 1981, 2-D polynomial and rational matrices, and their applications for the modeling of 2-D dynamical systems. Technical Report No. M735-11, Information Systems Laboratory, Stanford University.

Lu, W.-S., and Antoniou, A., 1986, *I.E.E.E. Trans. Circuits Syst.*, **33**, 965.

LU, W.-S., and LEE, E. B., 1983, *I.E.E.E. Trans. Circuits Syst.*, **30**, 455; 1986, Optimal structure sensitivity of linear systems with time-delays. Research Report, University of Minnesota.

MOORE, B. C., 1978, *Proc. I.E.E.E. Conf. on Decision and Control*, p. 66.

MULLIS, C. T., and ROBERTS, T., 1976, *I.E.E.E. Trans. Circuits Syst.*, **23**, 551.

PARASKEVOPOULOS, P. N., 1983, *Proc. 4th Polish–English Seminar on Real Time Process Control*, p. 270.

PERNEBO, L., and SILVERMAN, L. M., 1982, *I.E.E.E. Trans. autom. Control*, **27**, 382.

ROESSER, R. P., 1975, *I.E.E.E. Trans. autom. Control*, **20**, 1.

SONTAG, E. D., 1978, *I.E.E.E. Trans. Acoust. Speech Signal Process.*, **26**, 480.

WILLIAMSON, D., 1986, *I.E.E.E. Trans. autom. Control*, **31**, 633.

YOULA, D. C., 1961, *I.R.E. Trans. Inf. Theory*, **7**, 172.

ŻAK, S. H., LEE, E. B., and LU, W.-S., 1986, *I.E.E.E. Trans. Circuits Syst.*, **33**, 1241.